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REGULARITY OF TRANSMISSION PROBLEMS FOR UNIFORMLY ELLIPTIC FULLY NONLINEAR EQUATIONS

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Dedicated to our loved Anna Aloe

ABSTRACT. We investigate the regularity of transmission problems for a general class of uniformly elliptic fully non linear equations. We prove that, if the forcing term is Lipschitz, then viscosity solution are $C^{1,\gamma}$.

1. Introduction and main results

In this article we consider viscosity solutions to the transmission problem

$$\begin{aligned} \mathcal{F}^+(D^2u) &= f^+ \quad \text{in } B_1^+ := B_1(0) \cap \{x_n > 0\} \\ \mathcal{F}^-(D^2u) &= f^- \quad \text{in } B_1^- := B_1(0) \cap \{x_n < 0\} \\ a(u_n)^+ - b(u_n)^- &= 0 \quad \text{on } B_1' := B_1(0) \cap \{x_n = 0\}, \end{aligned} \tag{1.1}$$

where $(u_n)^+$ and $(u_n)^-$ denote the derivatives in the e_n direction of u restricted to the upper and lower half ball, respectively. Here $a > 0, b \geq 0$ and $\mathcal{F}^\pm$ are fully nonlinear uniformly elliptic operators, with ellipticity constants $\Lambda \geq \lambda > 0$ and $\mathcal{F}^\pm(0) = 0$. That is, for every $M, N \in \mathcal{S}^n$,

$$\mathcal{M}^-_{\lambda,\Lambda}(N) \leq \mathcal{F}^\pm(M+N) - \mathcal{F}^\pm(M) \leq \mathcal{M}^+_{\lambda,\Lambda}(N),$$

where $\mathcal{S}^n$ denotes the set of square symmetric $n \times n$ matrices and $\mathcal{M}^-_{\lambda,\Lambda}$, $\mathcal{M}^+_{\lambda,\Lambda}$ are the extremal Pucci operators defined by

$$\mathcal{M}^-_{\lambda,\Lambda}(N) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{Tr}(AN), \quad \mathcal{M}^+_{\lambda,\Lambda}(N) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{Tr}(AN),$$
$$\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}^n : \lambda I \leq A \leq \Lambda I\}.$$

In the sequel we write $\mathcal{F}^\pm(D^2u) = f^\pm$, in $B_1^\pm$ to denote both interior equations in (1.1). Now we define viscosity solution for problem (1.1).

Definition 1.1. We say that $u \in C(B_1)$ is a viscosity subsolution (supersolution) to (1.1) if

(i) $\mathcal{F}^\pm(D^2u) \geq f^\pm$ ($\leq f^\pm$) in $B_1^\pm$ in the viscosity sense;

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(ii) Let $x_0 \in \{x_n = 0\}$, $\delta > 0$ small, and $\varphi \in C^2(\overline{B}^+_\delta(x_0)) \cap C(\overline{B}^-_\delta(x_0))$. If φ touches u from above (below) at x_0, then

$$a(\varphi_n)^+(x_0) - b(\varphi_n)^-(x_0) \geq 0 \quad (\leq 0).$$

We say that u is a viscosity solution if it is both a viscosity subsolution and supersolution.

It is easily seen that condition (ii) in Definition 1.1 can be replaced by the following condition

(ii') Let

$$\psi(x) = P(x') + px_n^+ - qx_n^-,$$

where P is a quadratic polynomial. If ψ touches from above (below) u at $x_0 \in \{x_n = 0\}$, then

$$ap - bq \geq 0 \quad (\leq 0).$$

Transmission problems as (1.1) play a key role for example in the regularity theory for two-phase free boundary problems developed by the authors in [2, 3, 4]. Our purpose here is to review and extend to the case of distributed sources the regularity results about transmission problems provided in [4] for the homogeneous case. Our main result is the following one.

Theorem 1.2. *Let u be a viscosity solution to (1.1) in B_1. Assume that $f^\pm \in C^{0,1}(B_1^\pm) \cap L^\infty(B_1)$. Then, for any $\rho < 1$, $u \in C^{1,\alpha}(\bar{B}^\pm_\rho)$ with norm bounded by a constant depending on n, λ, Λ, ρ, $\|u\|_\infty$, $\|f\|_\infty$ and $\|f^\pm\|_{C^{0,1}}$. In particular the transmission condition is satisfied in the classical sense.*

2. Hölder continuity

In this section we prove the Hölder continuity of a solution u to problem (1.1). Here we only need $f^\pm \in C(B_1^\pm) \cap L^\infty(B_1)$. We introduce a special class of functions, based on the extremal Pucci operators, in the spirit of [1]. Since $a > 0$ and $b \geq 0$ are defined up to a multiplicative constant and the problem is invariant under reflection with respect to $\{x_n = 0\}$, we can assume that $a = 1$, $0 \leq b \leq 1$.

We denote by $\underline{\mathcal{S}}_{\lambda,\Lambda}(f^\pm)$ the class of continuous functions u in B_1 such that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f^\pm \quad \text{in } B_1^\pm, \quad \text{and}$$
$$(u_n)^+ - b(u_n)^- \geq 0 \quad \text{on } B_1'$$

in the sense of Definition 1.1 with comparison functions touching u from above.

Analogously, we denote by $\overline{\mathcal{S}}_{\lambda,\Lambda}(f^\pm)$ the class of continuous functions u in B_1 such that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f^\pm \quad \text{in } B_1^\pm, \quad \text{and}$$
$$(u_n)^+ - b(u_n)^- \leq 0 \quad \text{on } B_1'$$

in the sense of Definition 1.1 with comparison functions touching u from below. Finally we set

$$\mathcal{S}^*_{\lambda,\Lambda}(f^\pm) = \underline{\mathcal{S}}_{\lambda,\Lambda}(-|f^\pm|) \cap \overline{\mathcal{S}}_{\lambda,\Lambda}(|f^\pm|).$$

Theorem 2.1. *Let $u \in \mathcal{S}^*_{\lambda,\Lambda}(f^\pm)$ with $f^\pm \in C(B_1^\pm) \cap L^\infty(B_1)$. Then $u \in C^\alpha(B_{1/2})$ with α and $\|u\|_{C^\alpha(B_{1/2})}$ depending on $n, \lambda, \Lambda, \|f^\pm\|_\infty$ and $\|u\|_\infty$.*

It is sufficient to show that $u \in C^\alpha(B_{\rho_0})$ with ρ_0 small depending only on $\|f^\pm\|_\infty$. Then after the rescaling $u(x) \to u(rx)$, the theorem follows easily from the iteration of the following lemma. Here constants depending on n, λ, Λ are called universal.

Lemma 2.2. *Let $u \in \mathcal{S}^*_{\lambda,\Lambda}(f^\pm)$ with $\|u\|_\infty \le 1$ and $\|f^\pm\|_\infty \le \varepsilon_0$ in B_1, ϵ_0 small universal. Assume that at $\bar{x} = \frac{1}{5}e_n$*

$$u(\bar{x}) > 0. \tag{2.1}$$

Then $u \ge -1 + c$ in $B_{1/3}$ with $0 < c < 1$ universal.

Proof. By Harnack inequality, if ε_0 is small enough depending on the Harnack constant, and by assumption (2.1), we deduce that (c_0 universal)

$$u \ge -1 + c_0 \quad \text{in } B_{1/20}(\bar{x}). \tag{2.2}$$

Let $r = |x - \bar{x}|$ and

$$w = \eta(\Gamma^\gamma(r) + \delta x_n^+) + \frac{\varepsilon_0}{2\lambda} x_n^2, \Gamma^\gamma(r) = r^{-\gamma} - (2/3)^{-\gamma}, \quad \delta = -\frac{1}{2}\Gamma^\gamma(\frac{3}{4})$$

be defined in the ring $D = B_{3/4}(\bar{x}) \backslash B_{1/20}(\bar{x})$, with $\gamma > \max\{0, \frac{\Lambda}{\lambda}(n-1) - 1\}$ and η to be chosen later. Since Γ^γ is a radial function, in an appropriate system of coordinates

$$D^2 w = \eta\gamma r^{-\gamma-2} \mathrm{diag}\{(\gamma+1), -1, \dots, -1\}.$$

Hence, in D,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) \ge \eta(\gamma r^{-\gamma-2}(\lambda(\gamma+1) - \Lambda(n-1))) + \varepsilon_0 \ge \|f^\pm\|_\infty. \tag{2.3}$$

Since $\partial_n \Gamma^\gamma > 0$ on $\{x_n = 0\}$, the transmission condition

$$(w_n)^+ - b(w_n)^- > 0 \quad \text{on } \{x_n = 0\}, \tag{2.4}$$

is satisfied. Now choose η, ϵ_0 small universal so that

$$w \le c_0 \quad \text{on } \partial B_{1/20}(\bar{x}).$$

Then, by choosing ϵ_0 possibly smaller, we also have $w \le 0$ on $\partial B_{3/4}(\bar{x})$. Thus, in view of (2.2) we obtain that

$$w \le u + 1 \quad \text{on } \partial D.$$

From (2.3), (2.4) w is a strict classical subsolution of the transmission problem for the operator $\mathcal{M}^-_{\lambda,\Lambda}$ with right hand sides $|f^\pm|$. By the definition of viscosity solution, we conclude that it must be

$$w \le u + 1 \quad \text{in } D.$$

Since $w \ge c$ in $B_{1/3}$ the proof is complete. □

3. Upper and lower envelopes

Given a continuous function u in $B_1(0)$ and $\overline{B}_\rho \subset B_1(0)$ we define for $\varepsilon > 0$ the upper ε-envelope of u in the x'-direction,

$$u^\varepsilon(y', y_n) = \sup_{x \in \overline{B}_\rho \cap \{x_n = y_n\}} \{u(x', y_n) - \frac{1}{\varepsilon}|x' - y'|^2\} \quad y = (y', y_n) \in B_\rho.$$

Note that there is $y_\varepsilon \in \overline{B}_\rho \cap \{x_n = y_n\}$ such that

$$u^\varepsilon(y) = u(y_\varepsilon) - \frac{1}{\varepsilon}|y'_\varepsilon - y'|^2$$

with $|y'-y'_\varepsilon| \leq \sqrt{2\varepsilon\|u\|_\infty}$, since $u^\varepsilon(y) \geq u(y)$ and

$$\frac{1}{\varepsilon}|y'_\varepsilon - y'|^2 = u(y_\varepsilon) - u^\varepsilon(y) \leq u(y_\varepsilon) - u(y).$$

Lemma 3.1. *The following properties hold:*

1. *$u^\varepsilon \in C(B_\rho)$ and $u^\varepsilon \downarrow u$ uniformly in B_ρ as $\varepsilon \to 0$.*
2. *u^ε is $C^{1,1}$ in the x'-direction by below in B_ρ. Thus u^ε is pointwise second order differentiable in the x'-direction al almost every point of B_ρ.*
3. *If u is a viscosity solution to (1.1), then, in a smaller ball B_r, $r \leq \rho - 2\sqrt{\varepsilon\|u\|_\infty}$, u^ε is a viscosity subsolution to*
$$\begin{gathered} \mathcal{F}^\pm(D^2u^\epsilon) = f^\pm - \omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) \quad \text{in } B_r^\pm \\ (u_n^\varepsilon)^+ - b(u_n^\varepsilon)^- = 0 \quad \text{on } B'_r \end{gathered} \tag{3.1}$$
where $\omega_{f^\pm}$ denotes the modulus of continuity of $f^\pm$.

Proof. (1) follows from

$$|u^\varepsilon(y_0) - u^\varepsilon(y_1)| \leq \frac{6\rho}{\varepsilon}|y'_0 - y'_1|$$

and

$$0 \leq u^\varepsilon(y) - u(y) = u(y_\varepsilon) - \frac{1}{\varepsilon}|y'_\varepsilon - y'|^2 - u(y) \leq \omega_u(\sqrt{2\varepsilon\|u\|_\infty})$$

where ω_u is the modulus of continuity of u.

(2) follows from Alexandrov Theorem on concave/convex functions, since

$$u^\varepsilon(y', y_n) + \frac{1}{\varepsilon}|y'|^2 = \sup_{x\in \overline{B}_\rho \cap \{x_n = y_n\}} \big\{u(x', y_n) - \frac{1}{\varepsilon}|x'|^2 + \frac{2}{\varepsilon}\langle x', y'\rangle\big\}$$

is convex, being the supremum of a family of affine functions of y'.

(3) Let $\varphi \in C^2(B_r)$ touch from above u^ε at a point $\bar{x} \in B_r^+$. Then (ε small)

$$u^\varepsilon(\bar{x}) = u(\bar{x}_\varepsilon) - \frac{1}{\varepsilon}|\bar{x}'_\varepsilon - \bar{x}'|^2$$

with $|\bar{x}'_\varepsilon - \bar{x}'|^2 \leq 2\varepsilon\|u\|_\infty$. Consider the function

$$\Phi(y) = \varphi(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}'_\varepsilon - \bar{x}'_\varepsilon|^2.$$

With our choice of r and $y \in B_\rho^+$ close enough to $\bar{x}_\epsilon$, the point $y + \bar{x} - \bar{x}_\varepsilon \in B_\rho^+$. Thus, by the definition of u^ε,

$$u(y) \leq u^\varepsilon(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}' - \bar{x}'_\varepsilon|^2$$

and therefore

$$u(y) \leq \varphi(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}' - \bar{x}'_\varepsilon|^2.$$

with equality at $y = \bar{x}_\varepsilon$, since $\varphi(\bar{x}) = u^\varepsilon(\bar{x})$. Thus the function Φ touches from above u at $\bar{x}_\varepsilon$. Therefore

$$\mathcal{F}^+(D^2\varphi(\bar{x})) = \mathcal{F}^+(D^2\Phi(\bar{x}_\varepsilon)) \geq f^+(x_\varepsilon) \geq f^+(\bar{x}) - \omega_{f^+}big(\sqrt{2\varepsilon\|u\|_\infty}\big).$$

Similarly, we can check the transmission condition

$$(u_n^\varepsilon)^+ - b(u_n^\varepsilon)^- \geq 0 \quad \text{on } B'_r.$$

Let

$$\varphi(x) = P(x') + px_n^+ - qx_n^-$$

touch from above u^ε at a point $\bar{x} = (\bar{x}', 0) \in B'_r$, with P quadratic polynomial. Then

$$P(x') + px_n^+ - qx_n^- \geq u^\varepsilon(x)$$

near $\bar{x}$ and

$$P(\bar{x}') = u^\varepsilon(\bar{x}) = u(\bar{x}_\varepsilon) - \frac{1}{\varepsilon}|\bar{x}' - \bar{x}'_\varepsilon|^2.$$

Then, for r small and y close enough to $\bar{x}_\epsilon$

$$\begin{aligned} u(y) &\leq u^\varepsilon(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}' - \bar{x}'_\varepsilon|^2 \\ &\leq \varphi(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}'_\varepsilon - \bar{x}'_\varepsilon|^2 \end{aligned}$$

with equality at $y = \bar{x}_\varepsilon$ since $P(\bar{x}') = u^\varepsilon(\bar{x})$. Hence $\varphi(y + \bar{x} - \bar{x}_\varepsilon) + \frac{1}{\varepsilon}|\bar{x}'_\varepsilon - \bar{x}'_\varepsilon|^2$ touches from above u at $y = \bar{x}_\varepsilon$ and therefore $p - bq \geq 0$, as desired. □

Analogously we can define

$$u_\varepsilon(y', y_n) = \inf_{x \in \overline{B}_\rho \cap \{x_n = y_n\}} \left\{u(x', y_n) + \frac{1}{\varepsilon}|x' - y'|^2\right\} \quad y = (y', y_n) \in B_\rho.$$

the lower ε-envelope of u in the x'-direction. Properties (1)–(3) hold with obvious changes:

(1') $u_\varepsilon \in C(B_\rho)$ and $u_\epsilon \uparrow u$ uniformly in B_ρ as $\varepsilon \to 0$.
(2') u_ε is $C^{1,1}$ in the x'-direction by above in B_ρ. Thus u_ε is pointwise second order differentiable in the x'-direction al almost every point of B_ρ.
(3') If u is a viscosity solution to (1.1), then, in a smaller ball B_r, $r \leq \rho - 2\sqrt{\varepsilon\|u\|_\infty}$, u_ε is a viscosity supersolution to

$$\begin{gathered} \mathcal{F}^\pm(D^2 u_\epsilon) = f^\pm + \omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) \quad \text{in } B_r^\pm \\ ((u_\epsilon)_n)^+ - b((u_\epsilon)_n)^- = 0 \quad \text{on } B'_r. \end{gathered} \tag{3.2}$$

4. Proof of Theorem 1.2

For the proof of Theorem 1.2 we use the following pointwise regularity result (see [5]).

Theorem 4.1. *Let $\mathcal{F}$ be a uniformly elliptic operator and u be a viscosity solution to the Dirichlet problem*

$$\mathcal{F}(D^2 u) = f \text{ in } B_1^+, \quad u(x', 0) = g(x') \text{ on } B'_1.$$

Assume that g is pointwise $C^{1,\alpha}$ at 0 and $f \in L^\infty(B_1^+)$. Then u is poinwise $C^{1,\alpha}$ at 0, that is there exists a linear function L_u such that for all r small

$$|u - L_u| \leq Cr^{1+\alpha} \quad \text{in } \bar{B}_r^+$$

with C depending only on $n, \lambda, \Lambda, \|f\|_\infty$ and the pointwise $C^{1,\alpha}$ bound of g at 0.

Our main lemma reads as follows.

Lemma 4.2. *Let $f^\pm \in C^{0,1}(B_1^\pm)$ and u be a viscosity solution to (1.1) in B_1. Then, for any $\sigma > 0$ small, and any unit vector e' in the x' direction,*

$$u(x + \sigma e') - u(x) \in S^*_{\lambda,\Lambda}(\omega_{f^\pm}(\sigma)).$$

Proof. Let $v = u(x+\sigma e')$, $w = v-u$. Following the proof of [1, Theorem 5.3], with minor modification, it follows that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) \geq -\omega_{f^\pm}(\sigma) \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) \leq \omega_{f^\pm}(\sigma)$$

in $B_1^\pm$.

To show that the free boundary condition is satisfied in the viscosity sense, we slightly modify the technique in [4] for the homogeneous case.

$$\varphi(x) = P(x') + px_n^+ - qx_n^- - Cx_n^2$$

touch w from above at $x_0 = (x_0', 0)$, where P is a quadratic polynomial. Choosing suitably C and possibly restricting the neighborhood around x_0, we can assume that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi) \leq -2\omega_{f^\pm}(\sigma) - 2\omega_{f^\pm}(\sqrt{2\|u\|_\infty}). \tag{4.1}$$

Assume by contradiction that

$$p - bq < 0.$$

Let $\delta > 0$ small, and consider the ring $A_\delta(x_0) = \bar{B}_{2\delta}(x_0)\backslash B_\delta(x_0)$. Without loss of generality we can assume that φ touches w strictly from above and therefore that $\varphi - w \geq \eta > 0$ on $A_\delta(x_0)$. Since $w_\varepsilon = u^\varepsilon - v_\varepsilon$ converges uniformly to $u - v$, for ε small enough (up to adding a small constant), we have that φ touches w_ε from above at some point x_ε and $\varphi - w_\varepsilon \geq \eta/2$ on $\partial B_\delta(x_\varepsilon)$. In view of (4.1), we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w_\varepsilon) \geq -\omega_{f^\pm}(\sigma) - 2\omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) > \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi)$$

and therefore $x_\varepsilon \in \{x_n = 0\}$. Call

$$\psi = \varphi - w_\varepsilon - \eta/2.$$

Since $\psi \geq 0$ on $\partial B_\delta(x_\varepsilon)$, $\psi(x_\varepsilon) < 0$ and φ touches w_ε from above at x_ε, from ABP estimates ([1, Lemma 3.5]) it follows that the set of points in $B_\delta(x_\varepsilon) \cap \{x_n = 0\}$ where ψ admits a touching plane $l(x')$ from below, in the x' direction, of slope less than some arbitrary small number has positive measure. We choose the slope of l small enough to have that $\bar{\varphi} = \varphi - l - \eta/2 \geq w_\varepsilon$ on $\partial B_\delta(x_\varepsilon)$ and hence in the interior. By property (2) in Lemma 3.1 we can deduce that $\bar{\varphi}(y_\varepsilon) = w_\varepsilon(y_\varepsilon)$ for some $y_\varepsilon = (y_\varepsilon', 0)$ where both v^ε and u_ε are twice pointwise differentiable in the x' direction.

Now we call $\bar{u}^\varepsilon$ the solution to

$$\mathcal{F}^\pm(D^2\bar{u}^\epsilon(x)) = f^\pm(x) - \omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) \quad \text{in } B_\delta^\pm(x_\varepsilon)$$

with $w = u^\varepsilon$ on $\partial B_\delta(x_\varepsilon)$, and similarly let $\bar{v}_\epsilon$ the solution to

$$\mathcal{F}^\pm(D^2\bar{v}_\epsilon(x)) = f^\pm(x+\sigma e') + \omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) \quad \text{in } B_\delta^\pm(x_\varepsilon)$$

with $w = v_\varepsilon$ on $\partial B_\delta(x_\varepsilon)$, Set

$$\bar{w}_\varepsilon = \bar{u}^\varepsilon - \bar{v}_\varepsilon.$$

Then $\mathcal{M}^+_{\lambda,\Lambda}(D^2\bar{w}_\varepsilon) \geq -\omega_{f^\pm}(\sigma) - 2\omega_{f^\pm}(\sqrt{2\varepsilon\|u\|_\infty}) > \mathcal{M}^+_{\lambda,\Lambda}(D^2\bar{\varphi})$ in $B_\delta(x_\varepsilon)$ with $\bar{\varphi} \geq w_\varepsilon$ on the boundary. Thus, by comparison, we deduce that $\bar{\varphi} \geq \bar{w}_\varepsilon \geq w_\varepsilon$ also in the interior with contact from above at y_ε.

Also note that, by comparison, the replacements $\bar{u}^\varepsilon$ and $\bar{v}_\varepsilon$ are sub and super solution of the transmission condition on $\{x_n = 0\}$, respectively.

By the pointwise $C^{1,\alpha}$ differentiability at y_ε of the boundary data, we conclude the $C^{1,\alpha}$ differentiability of $\bar{v}^\varepsilon$ and $\bar{u}_\varepsilon$ up to y_ε. Thus there are linear functions L_v, L_u such that, for all small r,

$$|\bar{v}_\varepsilon - L_v| \le Cr^{1+\alpha},$$
$$|\bar{u}^\varepsilon - L_u| \le Cr^{1+\alpha}$$

in $B_r^+(y_\varepsilon)$. Since $\bar{\varphi}$ touches $\bar{w}_\varepsilon$ from above, we get $p \ge p_u^+ - p_v^+$, where

$$p_u^+ = (\bar{u}^\varepsilon)_n^+(y_\varepsilon), \quad p_v^+ = (\bar{v}_\varepsilon)_n^+(y_\varepsilon).$$

Arguing similarly in $B_r^-(y_\varepsilon)$ we infer $q \le q_u^- - q_v^-$, where

$$q_u^- = (\bar{u}^\varepsilon)_n^-(y_\varepsilon), \quad q_v^- = (\bar{v}_\varepsilon)_n^-(y_\varepsilon).$$

In the next lemma we show that

$$p_u^+ - p_v^+ - b(q_u^- - q_v^-) \ge 0$$

thus reaching a contradiction. □

Lemma 4.3. *Let* $g^\pm \in C(B_1^\pm)$ *and* u *be a viscosity solution to* $(0 \le b \le 1)$

$$\begin{gathered} \mathcal{F}^\pm(D^2u) = g^\pm \quad \text{in } B_1^\pm \\ (u_n)^+ - b(u_n)^- \ge 0 \quad (\le 0) \quad \text{on } B_1'. \end{gathered} \tag{4.2}$$

Assume that u *is twice differentiable at* $x = 0$ *in the* x'*-direction. Then* u *is differentiable at* 0 *and*

$$u_n^+(0) - bu_n^-(0) \ge 0 \quad (\le 0).$$

Proof. From Theorem 4.1 there exists a linear function L_u such that, for all small r,

$$|u - L_u| \le Cr^{1+\alpha} \quad \text{in } \overline{B}_r^+.$$

Without loss of generality, by subtracting a linear function, we may assume that

$$L_u = (u_n)^+(0)x_n := d^+x_n.$$

Let w be the solution to

$$\mathcal{F}^+(D^2w) = g^+ \text{ in } B_r^+, \quad w = \varphi_r \text{ on } \partial B_r^+$$

where

$$\varphi_r = \begin{cases} 2C|x|^{1+\alpha} & \text{on } \partial B_r^+ \cap \{x_n > 0\} \\ 2Cr^{\alpha-1}|x'|^2 & \text{on } B_r'. \end{cases}$$

Since u is twice pointwise differentiable at 0, we get, for r small enough,

$$u - d^+x_n \le \varphi_r \quad \text{on } \partial B_r^+$$

and, by comparison,

$$u - d^+x_n \le w \quad \text{in } B_r^+. \tag{4.3}$$

Now, the rescaling $W_r(x) = r^{-1-\alpha}w(rx)$ solves

$$\mathcal{G}(D^2W_r) = r^{1-\alpha}g^+(rx) \quad \text{in } B_1^+, \quad W_r(x') = 2C|x'|^2 \text{ on } B_1'$$

where $\mathcal{G}(M) = r^{1-\alpha}\mathcal{F}^+(r^{\alpha-1}M)$ has the same ellipticity constants of $\mathcal{F}^+$.

By boundary $C^{1,\alpha}$ estimates we obtain that

$$\|W_r\|_{C^{1,\alpha}(B_{1/2}^+)} \le \bar{C}$$

for a universal $\bar{C}$. In particular

$$W_r(x) \leq 2C|x'|^2 + \bar{C}x_n \quad \text{in } \bar{B}^+_{1/2}.$$

Rescaling back we get

$$w(x) \leq 2Cr^{\alpha-1}|x'|^2 + \bar{C}r^\alpha x_n \quad \text{in } \bar{B}^+_{r/2}.$$

From (4.3),

$$u \leq 2Cr^{\alpha-1}|x'|^2 + \bar{C}r^\alpha x_n + \ d^+ x_n \quad \text{in } B^+_{r/2}.$$

Arguing similarly in B^-_r we find

$$u \leq 2Cr^{\alpha-1}|x'|^2 + \bar{C}r^\alpha x_n + \ d^- x_n \quad \text{in } B^-_{r/2}$$

with $d^- = (u_n)^+(0)$. Thus

$$\varphi(x) = 2Cr^{\alpha-1}|x'|^2 + (\bar{C}r^\alpha + \ d^+)x_n^+ - (-\bar{C}r^\alpha + \ d^-)x_n^-$$

touches u by above at zero. Therefore

$$(\bar{C}r^\alpha + \ d^+) - b(-\bar{C}r^\alpha + d^-) \geq 0$$

for all small r so that $d^+ - bd^- \geq 0$. □

From Theorem 4.1 and the arguments in Chapter 5 in [1] we deduce the following result.

Corollary 4.4. *Let u be a viscosity solution to* (1.1)*. Then $u \in C^{1,\alpha}$ in the x'-direction in $B_{3/4}$ with norm bounded by a constant depending on n, λ, Λ, $\|u\|_\infty$ and $\|f^\pm\|_{C^{0,1}}$.*

We are now ready to give the proof of our main Theorem 1.2.

Proof of Theorem 1.2. Let, say $\rho = 1/2$. The $C^{1,\alpha}$ regularity and the bounds on $\|u\|_{C^{1,\alpha}(B_{1/2})}$ follow from Corollary 4.4 and the regularity theory for fully nonlinear uniformly elliptic equations in [5] or [6]. It remains to show that the transmission condition is satisfied in the classical sense. Let us prove that at $x = 0$, $ap - bq \leq 0$ where $p = (u_n)^+(0), q = (u_n)^-(0)$. By the $C^{1,\alpha}$ regularity of u, after possibly subtracting the linear function $u(0) + \nabla_{x'}u(0) \cdot x'$, we can write

$$|u(x) - (px_n^+ - qx_n^-)| \leq Cr^{1+\alpha} \quad |x| \leq r. \tag{4.4}$$

For r small, define

$$w_r(x) = Cr^{\alpha-1}(-|x|^2 + Kx_n^2) - 2r^\alpha CK|x_n| + px_n^+ - qx_n^-.$$

Choose K large to have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w_r) \geq C(2\lambda(K-1) - 2\Lambda n) > \|f^\pm\|_\infty. \tag{4.5}$$

Using (4.4), we get $w_r < u$ on ∂B_r. Let

$$m = \min_{\overline{B}_r}(u - w_r) = (u - w_r)(x_0).$$

Since $\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) \leq \mathcal{F}^\pm(D^2 u) = f^\pm$, from (4.5) we deduce that $x_0 \notin B^\pm_r$. Also, since $(u - w_r)(0) = 0$ it follows that $m \leq 0$, hence $x_0 \notin \partial B_r$. Thus $x_0 \in B'_r$ and $w_r + m$ touches u at x_0 from below. By definition it follows that

$$a(p - 2r^\alpha CK) - b(q + 2r^\alpha CK) \leq 0.$$

Letting $r \to 0$ we get $ap - bq \leq 0$. □

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