International Conference on Applications of Mathematics to Nonlinear Sciences,
Electronic Journal of Differential Equations, Conference 24 (2017), pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE RESULTS FOR MULTIVALUED OPERATORS OF MONOTONE TYPE IN REFLEXIVE BANACH SPACES

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Abstract. Let X be a real reflexive Banach space and X^* its dual space. Let $T : X \supset D(T) \to 2^{X^*}$ be an operator of class $\mathcal{A}_G(S_+)$, where $G \subset X$. A result concerning the existence of pathwise connected sets in the range of T is established, and as a consequence, an open mapping theorem is proved. In addition, for certain operators T of class $\mathcal{B}_G(S_+)$, the existence of nonzero solutions of $0 \in Tx$ in $G_1 \setminus G_2$, where $G_1, G_2 \subset X$ satisfy $0 \in G_2$ and $\overline{G_2} \subset G_1$, is established. The Skrypnik's topological degree theory is used, utilizing approximating schemes for operators of classes $\mathcal{A}_G(S_+)$ and $\mathcal{B}_G(S_+)$, along with the methodology of a recent invariance of domain result by Kartsatos and the author.

1. Introduction and preliminaries

In what follows, X is a real reflexive Banach space and X^* its dual space. The norms of both X and X^* will be denoted by $\|\cdot\|$ which will be understood from the context of its use. We denote by $\langle x^*, x\rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. The symbol ∂D and $\overline{D}$ denote the strong boundary and closure of the set D, respectively. The symbol $B(x_0, r)$ denotes the open ball of radius r with center at x_0.

For a sequence $\{x_n\}$ in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence to x_0 in X by $x_n \rightharpoonup x_0$. An operator $T : X \supset D(T) \to Y$ is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y, where Y is another Banach space. The value of T at x will be denoted by either Tx or any other notation clearly understood from the context of its use. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets of Y. It is said to be "demicontinuous" if it is strong-to-weak continuous on $D(T)$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. The normalized duality mapping $J : X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \ \|x^*\| = \|x\|\}, \quad x \in X.$$

The Hahn-Banach theorem ensures that $D(J) = X$, and therefore $J : X \to 2^{X^*}$ is a multivalued mapping defined on the whole space X. By a well-known renorming

2010 *Mathematics Subject Classification.* 47H14, 47H05, 47H11.
Key words and phrases. Browder and Skrypnik degree theory; invariance of domain; nonzero solutions; bounded demicontinuous operator of type (S_+).

Published November 15, 2017.

theorem due to Trojanski [18], one can always renorm the reflexive Banach space X with an equivalent norm with respect to which both X and X^* become locally uniformly convex (therefore strictly convex). Henceforth, we assume that X is a locally uniformly convex reflexive Banach space. With this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^*.

For a multivalued operator T from X to X^*, we write $T : X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X : Tx \neq \emptyset\}$ is the effective domain of T. We denote by $Gr(T)$ the graph of T, i.e., $Gr(T) = \{(x, y) : x \in D(T),\ y \in Tx\}$.

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, y \in D(T)$ and every $u \in Tx,\ v \in Ty$ we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is said to be "maximal monotone" if $Gr(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by the set inclusion. In our setting, a monotone operator T is maximal monotone if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$.

Definition 1.1. An operator $C : X \supset D(C) \to X^*$ is said to be of type (S_+) if for every sequence $\{x_n\} \subset D(C)$ with $x_n \rightharpoonup x_0$ in X and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0,$$

we have $x_n \to x_0 \in \overline{D(C)}$ in X.

Definition 1.2. An operator $C : X \supset D(C) \to X^*$ is said to be pseudomonotone if for every sequence $\{x_n\} \subset D(C)$ with $x_n \rightharpoonup x_0$ in X and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0,$$

we have $\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle = 0$, and if $x \in D(C)$, then $Cx_n \rightharpoonup Cx_0$ in X.

Definition 1.3. The family $C(t) : X \supset D \to X^*, t \in [0, 1]$, of operators is said to be a homotopy of type (S_+) if for any sequences $\{x_n\} \subset D$ with $x_n \rightharpoonup x_0$ in X and $\{t_n\} \subset [0, 1]$ with $t_n \to t_0$ and

$$\limsup_{n\to\infty} \langle C(t_n)x_n, x_n - x_0\rangle \leq 0,$$

we have $x_n \to x_0$ in X, $x_0 \in D$ and $C(t_n)x_n \rightharpoonup C(t_0)x_0$ in X^*. A homotopy $C(t)$ of type (S_+) is bounded if the set

$$\{C(t)x \mid t \in [0, 1],\ x \in D\}$$

is bounded.

We next define the classes $\mathcal{B}_G(S_+)$ and $\mathcal{A}_G(S_+)$ of multivalued operators from X to X^*.

Definition 1.4. Let G be an open subset of X. An operator $T : X \supset D(T) \to 2^{X^*}$ is of class $\mathcal{B}_G(S_+)$ if there exists a sequence $\{T_n\}$, called an approximating sequence of T, of bounded demicontinuous mappings of type (S_+) from $\overline{G}$ to X^* with the following conditions.

- (A1) For each $C > 0$ there exists $K \geq 0$ such that $\langle T_n x, x\rangle \geq -K$ for all $x \in \overline{G}$ with $\|x\| \leq C$ and for all $n \in \mathbb{N}$.
- (A2) Let $\{t_n\} \subset [0, 1]$, $\{x_n\} \subset \overline{G}$ with $t_n \to 0$, and let $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \rightharpoonup x$ in X and $t_n T_{m_n} x_n \to z$ in X^*, then $z = 0$.

(A3) Let $\{x_n\} \subset \overline{G}$ and $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \rightharpoonup x$ in X, $T_{m_n}x_n \rightharpoonup w$ in X^* and
$$\limsup_{n\to\infty} \langle T_{m_n}x_n, x_n\rangle \le \langle w, x\rangle,$$
then $x_n \to x$ in X, $x \in D(T)$ and $w \in Tx$.

If the condition (A2) above is replaced by the following condition, the operator T is said to be of class $\mathcal{A}_G(S_+)$.

($\tilde{\text{A}}$2) Let $\{t_n\} \subset [0,1]$, $\{x_n\} \subset \overline{G}$ with $t_n \to 0$, and let $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \to x$ in X and $t_nT_{m_n}x_n \rightharpoonup z$ in X^*, then $z = 0$.

Definition 1.5. Let G be an open subset of X. An operator $T : X \supset D(T) \to 2^{X^*}$ is of class $\mathcal{B}_G(PM)$ (or $\mathcal{A}_G(PM)$) if there exists a sequence $\{T_n\}$, called an approximating sequence of T, of bounded pseudomonotone mappings from $\overline{G}$ to X^* satisfying the conditions (A1), (A2) (or (A1), ($\tilde{A}2$)) and the following condition.

(A4) Let $\{x_n\} \subset \overline{G}$ and $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \rightharpoonup x$ in X, $T_{m_n}x_n \rightharpoonup w$ in X^* and
$$\limsup_{n\to\infty} \langle T_{m_n}x_n, x_n\rangle \le \langle w, x\rangle,$$
then $\langle T_{m_n}x_n, x_n\rangle \to \langle w, x\rangle$, and if $x \in \overline{G}$, then $x \in D(T)$ and $w \in Tx$.

Remark 1.6. If $G \subset X$ is open, then the following property holds true (cf. [11, Lemma 2.2, p.9]). If $T \in \mathcal{A}_G(PM)$ and A bounded demincontinuous of type (S_+) on G, then $T + A \in \mathcal{A}_G(S_+)$. In particular, $T + J \in \mathcal{A}_G(S_+)$.

The operators of class $\mathcal{A}_G(S_+)$ were introduced by Kittila in [11] and are multivalued generalizations of bounded demicontinuous operators of type (S_+). Several examples of operators of type $\mathcal{A}_G(PM)$ are given in [11, pp.36–43] in the context of elliptic equations with zeroth-order strongly nonlinear perturbations, higher-order elliptic equations with lower-order strongly nonlinear perturbations, and elliptic equations with highest-order strongly nonlinear perturbations. A topological degree theory was developed in [11] for such operators, and then the theory was applied to the study of strongly nonlinear elliptic partial differential equations in divergence form. Kittilä [11, p.13] also showed that a densely defined maximal monotone operator $T : X \supset D(T) \to 2^{X^*}$, $0 \in D(T)$, $0 \in T0$ satisfies $T \in \mathcal{A}_X(PM)$. It can be seen that the operator $T + A$ is also of class $\mathcal{B}_G(S_+)$, where A bounded demincontinuous of type (S_+) on X. In the proof of the result given below, we only include the part that is different from the one for showing $T \in \mathcal{A}_X(PM)$ in [11], and therefore $T \in \mathcal{B}_X(PM)$.

Theorem 1.7. *Let $T : X \supset D(T) \to 2^{X^*}$ be a maximal monotone operator with $0 \in D(T)$, $0 \in T0$ and $\overline{D(T)} = X$. Then $T \in \mathcal{B}_X(PM)$.*

Proof. The Yosida approximant $T_n = (T^{-1} + \frac{1}{n}J^{-1})^{-1} : X \to X^*$, where n is a positive integer, is single-valued maximal monotone and continuous operator with $T_n0 = 0$. It is well-known that $T_nx \rightharpoonup T^0x$ on $D(T)$, where T^0x is the unique element of Tx having minimal norm, i.e. $\|T^0x\| = \operatorname{dist}(0, Tx)$. We only prove that T satisfies the condition (A2). The other conditions follow from exactly the same arguments as in the proof of [11, Theorem 2.1].

To verify the condition (A2), let $\{t_n\} \subset [0,1]$, $\{x_n\} \subset \overline{G}$ be such that $t_n \to 0$ and $x_n \rightharpoonup x_0$ in X, and let $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$ such that $t_nT_{m_n}x_n \to z$

in X^*. Let $x \in D(T)$. Then $T_{m_n}x \rightharpoonup T^0x$ in X^*, and so $t_nT_{m_n}x \to 0$. Since T_{m_n} is monotone, we have

$$\langle t_nT_{m_n}x_n - t_nT_{m_n}x, x_n - x\rangle \geq 0.$$

Letting $n \to \infty$ yields

$$\langle z, x_0 - x\rangle \geq 0 \quad \text{for all } x \in D(T). \tag{1.1}$$

Let $y \in X$. Since $\overline{D(T)} = X$, there exists a sequence $\{y_j\} \subset D(T)$ such that $y_j \to x_0 - y$. Substituting y_j for x in (1.1), we get

$$\langle z, x_0 - y_j\rangle \geq 0 \quad \text{for all } j.$$

Letting $j \to \infty$ yields $\langle z, y\rangle$. Since $y \in X$ is arbitrary, we obtain $z = 0$. This verifies the condition (A2). □

The first main result of this paper is the existence of nonzero solutions of $0 \in Tx$, where $T \in \mathcal{B}_G(S_+)$. For additional facts related to the existence of nonzero solutions of nonlinear operator equations in Banach spaces, the reader is referred to Kartsatos and the author [2], and Ding and Kartsatos [8].

The second main result concerns an open mapping theorem for operators of class $\mathcal{A}_G(S_+)$, which extends the open mapping theorem of Park in [13] for bounded demicontinous operators of type (S_+). A multivalued degree for operators in $\mathcal{A}_G(S_+)$ is developed by Kittila [11] via the Skyrpnik's degree (cf. [17]). In this paper, the methodologies in [11], a recent paper of the author and Kartsatos [2], and Kartsatos and Skrypnik [10] as well as various properties of the Skrypnik's degree have been utilized. Open mapping theorems date back as far as Brouwer [5] for continuous injections in $\mathbb{R}^n$. Schauder [16] extended the Brouwer's open mapping theorem to infinite dimensional Banach spaces for operators of the form $I + C$ with C compact. Tromba [19] extended the Schauder's result to Fredholm maps of index zero. For other results concerning various continuity conditions on the main operators, the reader is referred to Berkovits [3], Deimling [6], Kartsatos [7], Nagumo [12], Petryshyn [14, 15] (for A-proper mappings), Skrypnik [17, p.59] and the references therein. For the existence of pathwise connected sets in the ranges of certain operators, the reader is referred to [8, 9] and the references therein.

2. Main Results

The first main result is the existence of nonzero solutions of the operator inclusion $0 \in Tx$, where $T : X \supset D(T) \to 2^{X^*}$ is of the class $\mathcal{B}_G(S_+)$, $G \subset X$.

Theorem 2.1. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $T : X \supset D(T) \to 2^{X^*}$ be an operator of class $\mathcal{B}_{G_1}(S_+)$. Moreover, we assume the following conditions.*

- (H1) *There exists $v^* \in X^*$, $v^* \neq 0$, such that $\lambda v^* \notin Tx$ for every $(\lambda, x) \in \mathbb{R}_+ \times (D(T) \cap \partial G_1)$.*
- (H2) *For every $(\lambda, x) \in \mathbb{R}_+ \times (D(T) \cap \partial G_2)$, we have $0 \notin (T + \lambda J)x$.*

Then there exists $x \in D(T) \cap (G_1 \setminus G_2)$ such that $0 \in Tx$.

Proof. Since $T \in \mathcal{B}_{G_1}(S_+)$, there exists an approximating sequence $\{T_n\}$ in the sense of Definition 1.4, satisfying the conditions (A1)–(A3). Consider the approximate equation

$$T_nx = 0. \tag{2.1}$$

We first show that (2.1) has a solution $x_n \in G_1 \setminus G_2$ for sufficiently large n. To this end, we first show that there exists $\tau_0 > 0$ and n_0 such that the equation

$$T_n x = \tau v^* \tag{2.2}$$

has no solution in G_1 for every $\tau \ge \tau_0$ and for all $n \ge n_0$. Assuming the contrary implies the existence of $\{\tau_n\} \subset (0, \infty)$, $\{x_n\} \subset G_1$, and a subsequence of $\{T_n\}$ which we again denote by $\{T_n\}$, such that $\tau_n \to \infty$, $x_n \rightharpoonup x_0$, and

$$T_n x_n = \tau_n v^*. \tag{2.3}$$

Since $v^* \neq 0$, we have $\|T_n x_n\| \to \infty$, and therefore

$$\frac{T_n x_n}{\|T_n x_n\|} \to \frac{v^*}{\|v^*\|}.$$

Let $t_n = 1/\|T_n x_n\|$ and $h = v^*/\|v^*\|$. This implies that $t_n T_n x_n \to h$ and $t_n \to 0$. By the condition (A2), we get $h = 0$, which is a contradiction.

Consider the homotopy

$$H_n(s, x) := T_n x - s\tau_0 v^*, \quad (s, x) \in [0, 1] \times \overline{G_1}, \tag{2.4}$$

where $n \ge n_0$. We show that the equation $H_n(s, x) = 0$ has no solution on ∂G_1 for sufficiently large n and for all $s \in [0, 1]$. Assume the contrary and let $\{x_n\} \subset \partial G_1$ and $\{s_n\} \subset [0, 1]$ be such that $s_n \to s_0$, $x_n \rightharpoonup x_0$, and

$$T_n x_n = s_n \tau_0 v^*.$$

Since $T_n x_n \to s_0 \tau_0 v^*$, the condition (A3) yields $x_n \to x_0 \in \partial G_1$, $x_0 \in D(T)$ and $s_0 \tau_0 v^* \in T x_0$. This contradicts the hypothesis (H1). By following an argument used in the proof of [1, Theorem 1], we can show that $T_n - s\tau_0 v^*$ is a bounded demicontinuous mapping of type (S_+) for each n, and therefore $H_n(s, x)$ is an admissible homotopy for the Skrypnik's degree, d_{S_+}.

Suppose that $\mathrm{d}_{S_+}(H_{n_1}(1, \cdot), G_1, 0) \neq 0$ for a sufficiently large $n_1 \ge n_0$, then the equation

$$T_{n_1} x = \tau_0 v^*$$

has a solution $x \in G_1$; however, this contradicts our choice of τ_0. Consequently,

$$\mathrm{d}_{S_+}(T_n, G_1, 0) = \mathrm{d}_{S_+}(H_n(0, \cdot), G_1, 0) = \mathrm{d}_{S_+}(H_n(1, \cdot), G_1, 0) = 0 \tag{2.5}$$

for all $n \ge n_0$.

Consider the homotopy

$$\mathcal{H}_n(s, x) = sT_n x + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}.$$

We show that there exists $n_1 \ge n_0$ such that the equation $\mathcal{H}_n(s, x) = 0$ has no solution on ∂G_2 for any $s \in [0, 1]$ and for any $n \ge n_1$. Let us assume the contrary and choose sequences $\{x_n\} \subset \partial G_2$, $\{s_n\} \subset [0, 1]$, and a subsequence of $\{T_n\}$ denoted again by itself, such that $x_n \rightharpoonup x_0$, $s_n \to s_0$, and

$$s_n T_n x_n + (1 - s_n) J x_n = 0. \tag{2.6}$$

Since $J0 = 0$ and J is injective, we must have $s_n > 0$ for all n. Also, if $s_n = 1$ for all large n, then $T_n x_n = 0$, and the condition (A3) yields $x_n \to x_0 \in \partial G_2$, $x_0 \in D(T)$, and $0 \in T x_0$. This is a contradiction to (H2). Suppose now that $s_n \to 0$. Then

$$\langle T_n x_n, x_n \rangle = -\big(\frac{1}{s_n} - 1\big)\langle J x_n, x_n \rangle \to -\infty.$$

This is a contradiction to the condition (A1) because the boundedness $\{x_n\}$ implies the existence of $K \ge 0$ such that $\langle T_n x_n, x_n\rangle \ge -K$ for all n. Thus, $s_0 \in (0,1]$ and (2.6) implies

$$T_n x_n \rightharpoonup -\big(\frac{1}{s_0}-1\big)j^* =: w,$$

where $j^* \in X^*$ satisfies $Jx_n \rightharpoonup j^*$. Since J is monotone, we have

$$\begin{aligned}\langle T_n x_n, x_n - x_0\rangle &= -\big(\frac{1}{s_n}-1\big)\langle Jx_n, x_n - x_0\rangle\\ &= -\big(\frac{1}{s_n}-1\big)[\langle Jx_n - Jx_0, x_n - x_0\rangle + \langle Jx_0, x_n - x_0\rangle]\\ &\le -\big(\frac{1}{s_n}-1\big)\langle Jx_0, x_n - x_0\rangle.\end{aligned}$$

Since X is reflexive, $\langle Jx_0, x_n - x_0\rangle \to 0$. This implies that

$$\limsup_{n\to\infty}\langle T_n x_n, x_n - x_0\rangle \le 0.$$

This along with

$$\limsup_{n\to\infty}\langle T_n x_n, x_n\rangle\ le \limsup_{n\to\infty}\langle T_n x_n, x_n - x_0\rangle + \limsup_{n\to\infty}\langle T_n x_n, x_0\rangle$$

implies

$$\limsup_{n\to\infty}\langle T_n x_n, x_n\rangle \le \langle w, x_0\rangle.$$

The condition (A3) yields $x_n \to x_0 \in \partial G_2$, $x_0 \in D(T)$ and $w \in Tx_0$. The continuity of J implies $Jx_n \to Jx_0 = j^*$, so that

$$w = -\big(\frac{1}{s_0}-1\big)Jx_0 \in Tx_0,$$

i.e.

$$0 \in Tx_0 + \big(\frac{1}{s_0}-1\big)Jx_0,$$

which contradicts (H2). For the sake of convenience, we assume that n_0 is sufficiently large so that we make take $n_1 = n_0$.

Since an affine homotopy of bounded demicontinuous (S_+) mappings is an admissible homotopy for the Skrynik's degree, d_{S_+}, we have

$$\begin{aligned}\mathrm{d}_{S_+}(T_n, G_2, 0) &= \mathrm{d}_{S_+}(\mathcal{H}_n(1,\cdot), G_2, 0)\\ &= \mathrm{d}_{S_+}(\mathcal{H}_n(0,\cdot), G_2, 0)\\ &= \mathrm{d}_{S_+}(J, G_2, 0) = 1\end{aligned}$$

for all $n \ge n_0$. Thus, for all $n \ge n_0$, we have

$$\mathrm{d}_{S_+}(T_n, G_1, 0) \ne \mathrm{d}_{S_+}(T_n, G_2, 0).$$

By the excision property of the Skrypnik's degree, for each $n \ge n_0$ there exists a solution $x_n \in G_1 \setminus G_2$ of $T_n x_n = 0$. We may assume that $x_n \rightharpoonup x_0$ in X and the condition (A3) implies that $x_n \to x_0 \in \overline{G_1 \setminus G_2}$, $x_0 \in D(T)$ and $0 \in Tx_0$. Note that

$$\overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.$$

By the conditions (H2) and (H2), $x_0 \notin \partial G_1 \cup \partial G_2$. Thus, $x_0 \in D(T) \cap (G_1 \setminus G_2)$. $\square$

We proceed to prove a result about placing pathwise connected sets in the ranges of certain operators of class $\mathcal{A}_G(S_+)$. As a consequence, we obtain an open mapping theorem for such operators.

Proposition 2.2. *Let $T : X \supset D(T) \to 2^{X^*}$ be of class $\mathcal{A}_G(S_+)$ with an approximating sequence $\{T_n\}$, where $G \subset X$ is open and bounded. Assume that $T + \epsilon J(\cdot - x_0)$ is injective on G for each $\epsilon > 0$ and for every $x_0 \in D(T)$. Moreover, assume that for each $x_0 \in D(T)$, there exists a bounded $\phi_{x_0} : \mathbb{R}^+ \to \mathbb{R}$ such that $\langle T_n x, x_0\rangle \le \phi_{x_0}(\|x\|)$ for all $x \in \partial G$ and for all large n. For a pathwise connected set $M \subset X^*$, assume that $T(D(T)\cap G)\cap M \neq \emptyset$ and $T(D(T)\cap\partial G)\cap M = \emptyset$. Then $M \subset T(D(T)\cap G)$.*

Proof. Let $y_0 \in T(D(T)\cap G)\cap M$. Then there exists $x_0 \in D(T)\cap G$ such that $y_0 \in Tx_0$. Let $p \in M$. Take $f : [0,1] \to M$ be a path in M such that $f(0) = y_0$ and $f(1) = p$. We now claim that there exist $n_0 \in \mathbb{N}$ such that

$$T_n x + \frac{1}{n} J(x - x_0) = f(t) \tag{2.7}$$

has no solution $x \in \partial G$ for any $t \in [0,1]$ and for all $n \ge n_0$. Assuming the contrary and without loosing the generality, let $\{x_n\} \subset \partial G$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ be such that

$$T_n x_n + \frac{1}{n} J(x_n - x_0) = f(t_n).$$

This implies that $T_n x_n \to f(t_0)$. Since $x_n \rightharpoonup x$, we have

$$\limsup_{n\to\infty} \langle T_n x_n, x_n\rangle \le \langle f(t_0), x\rangle,$$

and then by the condition (A3) of Definition 1.1, we have $x_n \to x$, $x \in D(T)$ and $f(t_0) \in Tx$. Since $f(t_0) \in M$ and $x \in D(T)\cap\partial G$, we have a contradiction to $T(D(T)\cap\partial G)\cap M = \emptyset$.

Consider the homotopy equation

$$H_n(x,t) \equiv T_n x + \frac{1}{n} J(x - x_0) - f(t) = 0. \tag{2.8}$$

We have already established that this equation has no solution on ∂G for sufficiently large n and for any $t \in [0,1]$, and therefore this is an admissible homotopy of type (S_+).

Consider the homotopy equation

$$G_n(x,t) \equiv (1-t)\Big(T_n x + \frac{1}{n} J(x - x_0) - y_0\Big) + tJ(x - x_0) = 0. \tag{2.9}$$

We show that (2.9) has no solution on ∂G for any $t \in [0,1]$ and for all $n \ge n_0$. If not, let $\{x_n\} \subset \partial G$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ such that

$$(1-t_n)\Big(T_n x_n + \frac{1}{n} J(x_n - x_0) - y_0\Big) + t_n J(x_n - x_0) = 0. \tag{2.10}$$

Since (2.7) has no solution on ∂G for any $n \ge n_0$ and $t \in [0,1]$, we see that $t_n = 0$ is impossible for all large n. Since J is injective, $t_n = 1$ is also impossible. Suppose $t_0 = 1$. The equation (2.10) implies

$$(1-t_n)\langle T_n x_n, x_n - x_0\rangle + a_n\|x_n - x_0\|^2 = (1-t_n)\langle y_0, x_n - x_0\rangle, \tag{2.11}$$

where

$$a_n = \frac{1-t_n}{n} + t_n.$$

Since G is bounded, there exists $C > 0$ such that $\|x_n\| \le C$ for all n. By the condition (A_1), there exists a $K > 0$ such that $\langle T_n x_n, x_n\rangle \ge -K$ for all n. Also, by the hypothesis, there exists a bounded function $\phi_{x_0} : \mathbb{R}_+ \to \mathbb{R}$ such that $\langle T_n x_n, x_0\rangle \le \phi(\|x_n\|)$ for all n. Then, in view of(2.11), we have

$$-(1-t_n)K - (1-t_n)\phi_{x_0}(\|x_n\|) + a_n\|x_n - x_0\|^2 \le (1-t_n)\langle y_0, x_n - x_0\rangle. \quad (2.12)$$

Since $t_n \to 1$, $a_n \to 1$ and ϕ_{x_0} is bounded, letting $n \to \infty$ in (2.12) yields $x_n \to x_0 \in \partial G$, which is a contradiction.

Next, we assume that $t_0 \in [0,1)$. If $t_0 = 0$, define $\alpha_n = \frac{t_n}{1-t_n}$. Then $\alpha_n \downarrow 0$ and

$$T_n x_n + \big(\frac{1}{n} + \alpha_n\big)J(x_n - x_0) = y_0. \quad (2.13)$$

This equation is like (2.8) for which we have already proved the impossibility of solutions on ∂G with $f(t) \equiv y_0$. For the remaining case, $t_0 \in (0,1)$, we define

$$\beta_n = \frac{1}{n} + \frac{t_n}{1-t_n}.$$

Then $\beta_n \to \beta_0 := \frac{t_0}{1-t_0} > 0$. Then the equation becomes

$$T_n x_n + \beta_n J(x_n - x_0) = y_0. \quad (2.14)$$

If

$$\limsup_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0,$$

then, by passing to a subsequence, let

$$q := \lim_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0.$$

In view of (2.14), this yields

$$\limsup_{n\to\infty}\langle \beta_n J(x_n - x_0), (x_n - x_0) - (x - x_0)\rangle = -q < 0.$$

Since $\beta_n \to \beta_0 > 0$ and J is of type (S_+), we obtain $x_n \to x \in \partial G$. From this and (2.14), we get $T_n x_n \to w := -\beta_0 J(x - x_0) + y_0$. By the condition (A3), we obtain $x \in D(T)$ and $w \in Tx$, i.e. $y_0 \in Tx + \beta_0 J(x - x_0)$. This leads to a contradiction to the injectivity of $T + \epsilon J(\cdot - x_0)$ because $x \ne x_0$.

Thus, $H_n(x,t)$ and $G_n(x,t)$ are admissible homotopies for the Skrypnik's degree, d_{S_+}, for the mappings of type (S_+). By the invariance of the degree under these homotopies, we have

$$\begin{aligned}
\mathrm{d}_{S_+}(T_n + \frac{1}{n}J(\cdot - x_0) - p, G, 0) &= \mathrm{d}_{S_+}(H_n(\cdot,1), G, 0)\\
&= \mathrm{d}_{S_+}(H_n(\cdot,0), G, 0)\\
&= \mathrm{d}_{S_+}(G_n(\cdot,0), G, 0)\\
&= \mathrm{d}_{S_+}(G_n(\cdot,1), G, 0)\\
&= \mathrm{d}_{S_+}(J(\cdot - x_0), G, 0) = 1.
\end{aligned}$$

Here, the last equality follows by considering the (S_+)-homotopy

$$Q(x,t) = (1-t)J(x - x_0) + tJx$$

with a continuous curve $y(t) = tJx_0$ so that

$$\begin{aligned}
\mathrm{d}_{S_+}(J(\cdot - x_0), G, 0) &= \mathrm{d}_{S_+}(Q(\cdot,0), G, 0)\\
&= \mathrm{d}_{S_+}(Q(\cdot,1), G, Jx_0)
\end{aligned}$$

$$= \mathrm{d}_{S_+}(J, G, Jx_0) = 1.$$

Therefore, for every n, there exists $x_n \in G$ such that

$$H_n(x_n, 1) = 0,$$

i.e.

$$T_n x_n + \frac{1}{n} J(x_n - x_0) = p,$$

which implies $T_n x_n \to p$. By the condition (A3), we deduce that $x_n \to x \in \overline{G}$, $x \in D(T)$, and $p \in Tx$. Since $T(D(T) \cap \partial G) \cap M = \emptyset$, we can only have $x \in G$. Since p was an arbitrary point in M, we obtain $M \subset T(D(T) \cap G)$. □

We use Proposition 2.2 to prove the following open mapping theorem for operators of class $\mathcal{A}_G(S_+)$.

Theorem 2.3 (Open Mapping). *Let $T : X \supset D(T) \to 2^{X^*}$ be of class $\mathcal{A}_G(S_+)$ with an approximating sequence T_n, where $G \subset X$ is open. Assume that $T + \epsilon J(\cdot - x_0)$ is locally injective on G for each $\epsilon \geq 0$ and for every $x_0 \in D(T)$. Moreover, assume that for each $x_0 \in D(T)$ and for each $r > 0$, there exists a bounded $\phi_{x_0} : \mathbb{R}^+ \to \mathbb{R}$ such that $\langle T_n x, x_0 \rangle \leq \phi_{x_0}(\|x\|)$ for all $x \in \overline{G} \cap \partial B(x_0, r)$ and for all large n. Then $T(D(T) \cap G)$ is open.*

Proof. Let $y_0 \in T(D(T) \cap G)$. Then there exists $x_0 \in G$ such that $y_0 \in Tx_0$. Since T is locally injective on G, there is $r > 0$ such that T is injective on $\overline{B(x_0, r)} \cap D(T)$, where $\overline{B(x_0, r)} \subset G$. It is then clear that $y_0 \notin T(D(T) \cap \partial B(x_0, r))$.

We claim that there exists $\delta > 0$ such that $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$. Assume the contrary, and let $y_n \in B(y_0, 1/n) \cap T(D(T) \cap \partial B(x_0, r))$. Then $y_n \to y_0$ and $y_n \in Tx_n$ with $x_n \in \partial B(x_0, r)$, and therefore the condition (A3) applies because $x_n \rightharpoonup x$ (up to subsequence) and $y_n \to y_0$. We get $x_n \to x \in \partial B(x_0, r)$, $x \in D(T)$ and $y_0 \in Tx$. This contradicts $y_0 \notin T(D(T) \cap \partial B(x_0, r))$.

Since $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$, $y_0 \in T(D(T) \cap B(x_0, r))$ and the ball $B(y_0, \delta)$ is pathwise connected, we can apply Proposition 2.2 to obtain $B(y_0, \delta) \subset T(D(T) \cap B(x_0, r))$. Since y_0 was arbitrary, $T(D(T) \cap G)$ is open. □

It would be interesting to establish analogous results via degree theories for operators of the form $A + T$, where $A : X \supset D(A) \to 2^{X^*}$ is maximal monotone and T is of class $\mathcal{A}_G(S_+)$. Similar results are also expected for the sum $L + A + T$ in the spirit of results in [2], where L is linear maximal monotone operator (densely defined) and T is of class $\mathcal{A}_G(S_+)$ with respect to $D(L)$. The class $\mathcal{A}_G(S_+)$ with respect to $D(L)$ can be defined in a fashion similar to operators of type (S_+) with respect to $D(L)$ as considered by Berkovits-Mustonen in [4].

Acknowledgments. The author is grateful to the College of Science and Mathematics at Kennesaw State University for supporting this research through the 2017 Research Stimulus Program. The author is also thankful to the anonymous referees for providing valuable suggestions.

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