A HYBRID SEMI-PRIMITIVE SHOCK CAPTURING SCHEME FOR CONSERVATION LAWS

RITESH KUMAR DUBEY

Abstract. A hybrid semi-primitive shock capturing scheme is presented for hyperbolic problems. Upwind based construction is done using explicit information on the wave propagation direction associated with the problem. This scheme captures the shock waves at right location but shows unphysical sonic expansion shock. This phenomena of unphysical expansion shock in the presence of expensive sonic point is not surprising and it is common for the wave based upwind schemes. A hybrid scheme approach using an iteration criteria based on sonic entropy fix is proposed to avoid such expansion shocks. Numerical results for scalar test problems are presented which show that proposed scheme captures the shock accurately.

1. Introduction

In this work we consider the hyperbolic differential equations in the following primitive (non-conservative) form,

$$\frac{\partial u}{\partial t} + \alpha(u)\frac{\partial(u)}{\partial x} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^{+}, \tag{1.1}$$

together with the appropriate boundary and initial conditions. In (1.1), the unknown variable $u \in \mathbb{R}$ can be a physically conserved variable and in such situation the characteristic speed $\alpha(u) = \frac{\partial f}{\partial u}$ where, $f = f(u)$ be the physical flux and (1.1) can be written in conservative form as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \tag{1.2}$$

It is well known that even if the initial condition is smooth enough, various kind of discontinuity e.g, shock wave, contact discontinuity and expansion fan may arise in the solution of such hyperbolic problems. Looking at literature to solve hyperbolic problems numerically one can experience an imbalance in between the number of available conservative schemes and primitive (non-conservative) schemes. The main reason for this imbalance seems to be the possible occurrence of shock waves in the solution of (1.2) and it is shown in [2] that conservative form is mandatory for capturing the shock at right location while the primitive form captures the shock

---

at wrong position and primitive schemes often converge to wrong week solution. In fact, most of the numerical schemes developed for hyperbolic problems are conservative. A state of art detail on various conservative numerical schemes for solving (1.2) can be found in [6, 7, 8, 11, 13]. On the other hand primitive or semi-primitive schemes work well for the smooth solution, discontinuities such as contact, shear waves and mild shocks [11] but these methods are simply discarded. Recently few attempts are made to construct primitive schemes. Primitive schemes based on upwinding are given in [12, 4] and centered schemes are proposed in [10].

In this work our main aim is to construct a primitive scheme which can capture the shock waves at right location. In order to do this we look for a primitive scheme which changes into conservative form around discontinuities so that it can capture shock accurately as suggested in [2]. The scheme is designed based on upwinding by using the explicit information on characteristic speed $\alpha = f'(u)$ of primitive form (1.1). Thus constructed semi-primitive scheme captures the shock wave correctly but in the presence of expensive sonic point where wave speed changes its sign, produces entropy violating solution. In order to overcome this, we use the idea of hybrid schemes and use entropy satisfying Lax-Friedrichs in the presence of sonic point. To decide upon iterations a criteria is given using an entropy fix approach.

2. Primitive scheme formulation

We define semi-primitive (semi-conservative) schemes as a scheme which remains primitive (non-conservative) for the region of smooth solution and locally corrected into conservative form around discontinuities. For the simplicity of presentation, consider the wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 \neq a \in \mathbb{R}. \tag{2.1}$$

We discretize the domain with fixed space and time step sizes Δx, Δt respectively. Integrating (2.1) over the rectangle $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$ and introducing the definitions of the spatial and temporal cell averages

$$U_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx, \quad U_{i\pm\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i\pm\frac{1}{2}},t)dt, \tag{2.2}$$

one can obtain a primitive difference scheme of the form

$$U_i^{n+1} = U_i^n - a\lambda\left(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\right) \tag{2.3}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $U_{i\pm\frac{1}{2}}$ are intermediate states defined on the cell interfaces $x_{i\pm\frac{1}{2}}$. Note that for constant wave speed a, the difference scheme (2.3) can be written in conservative form,

$$U_i^{n+1} = U_i^n - \lambda\left(H_{i+\frac{1}{2}} - H_{i-\frac{1}{2}}\right) \tag{2.4}$$

where $H_{i\pm\frac{1}{2}} = aU_{i\pm\frac{1}{2}}^n$ is the numerical flux function. It is well-known that the first order upwind approximation for (2.1) can be obtained by using the following intermediate states $U_{i\pm\frac{1}{2}}$ in (2.3)

$$U_{i+\frac{1}{2}} = \begin{cases} U_i, & a \geq 0, \\ U_{i+1}, & a \leq 0. \end{cases} \tag{2.5}$$

Consider the primitive form (1.2) of hyperbolic problem (1.1)

$$\frac{\partial u}{\partial t} + \alpha(u)\frac{\partial u}{\partial x} = 0, \tag{2.6}$$

On comparison with (2.1) and utilizing the explicit information of the characteristic speed $\alpha(u)$ we can construct a primitive upwind scheme so that it respects physical hyperbolicity property associated with (1.1). Analogous to first order accurate primitive difference scheme for linear equation (2.1), we propose a first order primitive upwind scheme for non-linear equation (2.6) which can be written as

$$U_i^{n+1} = U_i^n - \lambda \bar{s}_i \{U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\} \tag{2.7}$$

where $U_{i\pm\frac{1}{2}}$ are intermediate states on the cell interface at $x_{i\pm\frac{1}{2}}$ and defined similar to (2.5) as follows,

$$U_{i+\frac{1}{2}} = \begin{cases} U_i, & a_{i+\frac{1}{2}} \geq 0, \\ U_{i+1}, & a_{i+\frac{1}{2}} \leq 0, \end{cases}, \quad a_{i+\frac{1}{2}} = \begin{cases} \frac{F_{i+1}-F_i}{U_{i+1}-U_i}, & U_{i+1} \neq U_i, \\ \alpha(U_i), & U_{i+1} = U_i. \end{cases}$$

where $F_i = f(U_i)$.

Numerical wave speed $\bar{s}_i$ in (2.7) is the local linearized approximation for $\alpha(u)$ in the i_{th} cell. We define the numerical wave speed using intermediate states in such a way that the resulting scheme changes into conservative form in the vicinity of shock as follows,

$$\bar{s}_i = \begin{cases} \frac{F_{i+\frac{1}{2}}-F_{i-\frac{1}{2}}}{U_{i+\frac{1}{2}}-U_{i-\frac{1}{2}}}, & \text{if } U_{i+\frac{1}{2}} \neq U_{i-\frac{1}{2}}, \\ f'_i = f'(U_{i-\frac{1}{2}}), & \text{if } U_{i+\frac{1}{2}} = U_{i-\frac{1}{2}}, \end{cases} \tag{2.8}$$

Note that, in the vicinity of shock discontinuity, proposed scheme gives conservative approximation. For this, let shock discontinuity be locally in the i_{th} cell then $U_{i+\frac{1}{2}} \neq U_{i-\frac{1}{2}}$ and (2.7), (2.8) yield a conservative approximation

$$U_i^{n+1} = U_i^n - \lambda(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}). \tag{2.9}$$

This shows that proposed semi-primitive scheme is locally corrected by a conservative scheme, hence can capture shocks accurately [2]. Presented numerical results also validate this feature.

3. Hybrid Scheme

We have observed that the proposed semi-primitive scheme captures the shock discontinuity correctly but in presence of expansive sonic point where wave speed changes its *sign*, produces entropy violating solution which shows the less dissipative nature of semi-primitive scheme near expensive sonic point (see Fig. 3 Left). This phenomena is not new for the wave propagation based upwind schemes and it is well known in the literature that one need to use an entropy fix in the solution region where wave speed changes its sign [3, 5] . In order to apply proposed method for the problem with expensive sonic point we use hybrid schemes approach. The idea is to use entropy satisfying Lax-Friedrichs centered scheme in the region of sonic point. We propose an iteration approach based of the entropy fix formula for residual distribution scheme given in [9] as follows,

$$U_i^{n+1} = \begin{cases} U_i - \lambda \bar{\alpha}_i(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}), & |\bar{s}_i| \geq \delta_i, \\ \frac{(U_{i+1}+U_{i-1})}{2} - \frac{\lambda}{2}(f(U_{i+1}) - f(U_{i-1})), & |\bar{s}_i| < \delta_i. \end{cases} \tag{3.1}$$

where

$$\delta_i = \gamma \max(0, (\tilde{a}_{i+\frac{1}{2}} - \tilde{a}_{i-\frac{1}{2}})), \quad 0 \le \gamma \le 1, \quad \tilde{a}_i = f'(U_i). \tag{3.2}$$

Remarks (1) One can also use other entropy fix formulas for numerical computation [3, 5].

(2) For non-linear problem (1.2), the proposed scheme is different from the classical conservative upwind scheme though for the linear problem (2.1) it can be written as classical upwind scheme. In fact, in smooth solution region the proposed scheme (2.7) may give primitive approximation to (1.2).

4. Numerical Results

Remarks (1) In all our numerical results, we denote the results obtained using proposed semi-primitive scheme (without entropy iteration) by SP scheme and with entropy iteration that is hybrid scheme by HSP scheme. For all numerical simulation of HSP, a fixed value $\gamma = \frac{1}{2}$ is taken in (3.2).

(2)The proposed SP scheme works very well for most of the case but fails to capture centered expansion fan. Hence for safer side HSP scheme should always be used. For some tests results are given only for SP/HSP as similar results are obtained with proposed HSP/SP scheme.

(3) One can easily show that proposed scheme is stable for linear problem (2.1) under the CFL condition $|a\lambda| \le 1$ this is why we use CFL like stability condition $|\lambda\tilde{\alpha}| \le 1$ for non-linear problems.

4.1. **Convex Flux function: Inviscid Burgers Equation.** We consider the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \quad t > 0, \tag{4.1}$$

with periodic boundary conditions. It is well known that the solutions of inviscid Burgers equation may contain shocks [6]. We present three different test cases to show the accuracy and shock capturing at correct location using the proposed scheme.

4.1.1. *Case 1: Pre-/Post-Shock:* In this example, we take smooth sinusoidal initial condition,

$$u(x,0) = \sin(x), \quad x \in [0, 2\pi]. \tag{4.2}$$

In this test case the final solution beginning from smooth initial conditions contains a shock discontinuity after a breaking time $t = 1.0$.

In Fig. 1, approximate solutions obtained by proposed semi-primitive scheme SP are presented at pre-shock time $t = 0.25, 0.75$ when the solution remains smooth and at post shock time $t = 1.5$ when the shock is completely developed. Graph shows that SP gives good approximation for the smooth solution and captures the shock at right location.

4.1.2. *Case 2: Moving Shock:* Here we consider the Burgers equation (4.1) with the initial condition

$$u(x,0) = \begin{cases} 2, & \text{for } x \le 0, \\ -1, & \text{for } x > 0. \end{cases} \tag{4.3}$$

This test case has no sonic point in its solution. The initial shock discontinuity at $x = 0$ propagates without losing its initial shape with time. In Figure 2 we compare
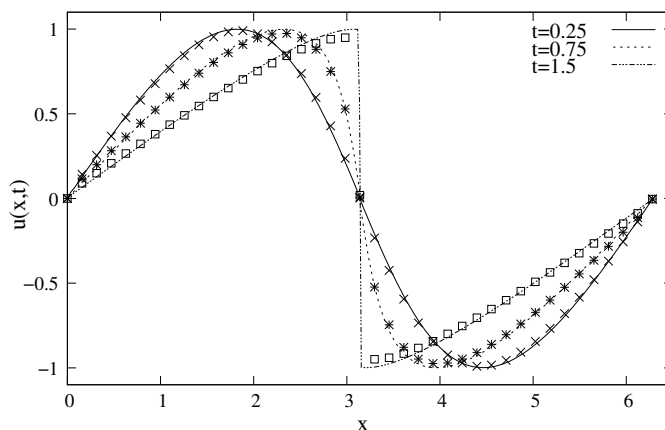
FIGURE 1. Computed solution by semi-primitive (SP) scheme for
various pre and post shock time with $CFL = 0.6$

the solution obtained with SP at different time level which shows that the scheme
is able to capture the shock at right location within 2 grid point. Similar results
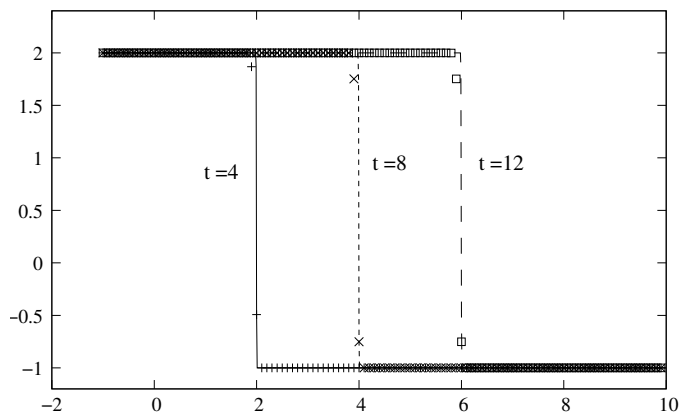are obtained with HSP.

FIGURE 2. Shock capturing feature of proposed SP scheme for
$C = 0.5$, $\Delta x = 0.1$ at various time level

4.1.3. *Case 3: Sonic expansion fan in presence of sonic point:* Consider the Burgers
equation (4.1) with the initial condition

$$u(x,0) = \begin{cases} -1, & \text{for } x < 0, \\ 2, & \text{for } x \geq 0. \end{cases} \tag{4.4}$$

The exact solution is an expansion fan emanating from the sonic point at $x = 0$. In
Figure 3, numerical solutions are compared with exact reference solution at time
$t = 0.4$ on a uniform grid with 200 points. The solution computed by proposed SP
scheme shows an expansion shock while the entropy fix based HSP scheme removes
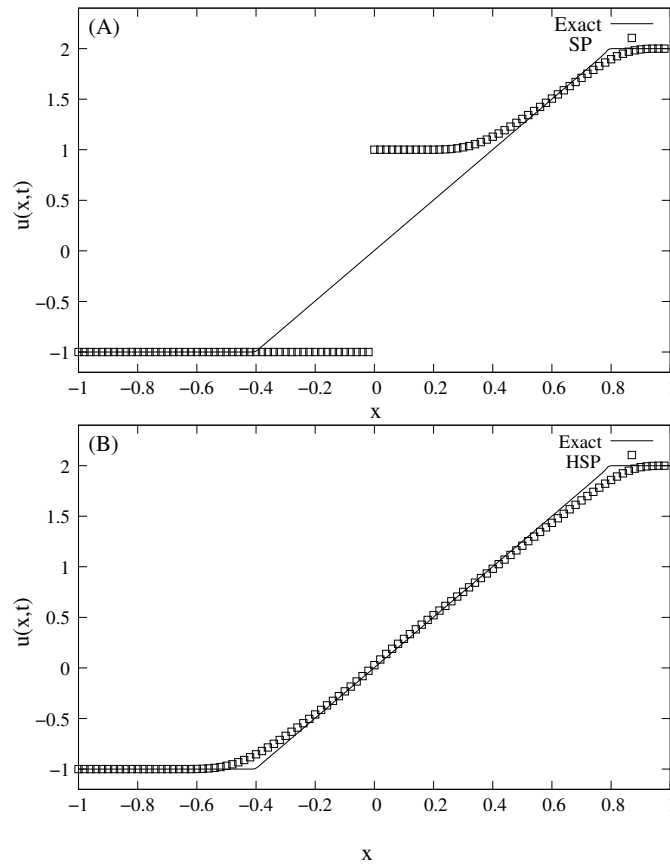the expansion shock very satisfactorily.

FIGURE 3. Sonic expansion: semi-primitive scheme (A) and hybrid semi-primitive scheme (B) for data $C = 0.8, \Delta x = 0.02$ at time $t = 0.4$

4.2. **Convex-Concave flux: Buckley Leverett Equation.** A more demanding test example for the scalar one-dimensional problem is the Buckley-Leverett equation. This equation models the two phase flows that arise oil-recovery problems and physically represents a mixture of oil and water through the porous medium. In conservative form (1.2) the flux function for this problem is given by a convex-concave (S-shaped) function

$$f(u) = \frac{u^2}{u^2 + \nu(1 - u)^2}. \tag{4.5}$$

Here ν is viscosity ratio and u represents the saturation of water and lies between 0 and 1.

4.2.1. *One moving shock* [13]. Consider the flux (4.5) with $\nu = \frac{1}{2}$ and initial condition

$$u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{4.6}$$

The solution involves one single moving shock followed by a rarefaction wave. Numerical results using proposed hybrid semi-primitive scheme (*HSP*) are presented in Fig. 4. *HSP* sharply captures the moving shock at right location and rarefaction wave accurately.
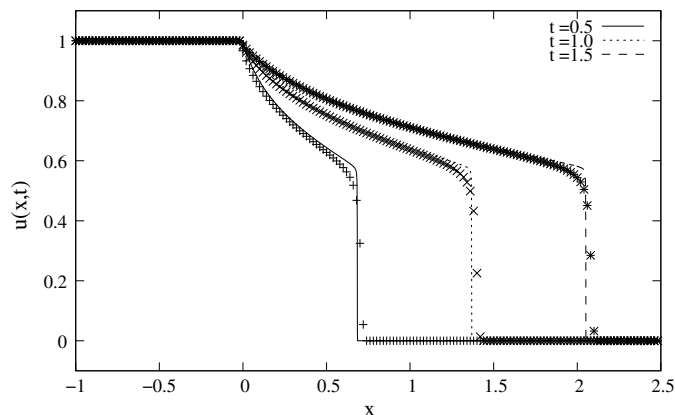

FIGURE 4. Solution profile for data $C = 0.4$, $\Delta x = 0.02$ at time $t = 0.5$, 1.0, 1.5

4.2.2. *Two moving shock* [1]. Consider the flux (4.5) with $\nu = \frac{1}{4}$ and subject to initial condition

$$u(x,\, 0) = \begin{cases} 1, & -0.5 \leq x \leq 0, \\ 0, & elsewhere. \end{cases} \qquad (4.7)$$

The solution involves two moving shocks, each followed by an rarefaction wave. Numerical result is given in Fig. 5 which shows that the *HSP* sharply captures both the fast and slow shocks. The rarefaction waves are also accurately approximated by *HSP*.
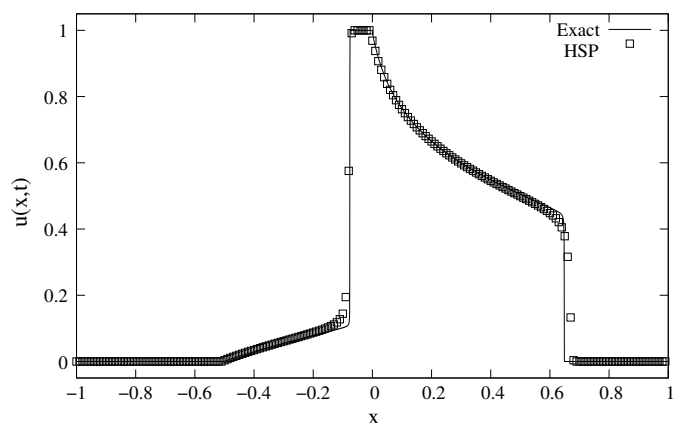

FIGURE 5. Solution profile using $C = 0.4$, $\Delta x = 0.01$ at time $t = 0.5$

4.3. **Flux Sine problem: Moving and Stationary Shock.** We consider third scalar Riemann problem with non-convex sinusoidal flux function which is introduced by Leveque in [8]. It is given by

$$\frac{\partial u}{\partial t} + \frac{\partial(\sin(u))}{\partial x} = 0 \tag{4.8}$$

with the initial condition

$$u(x,0) = \begin{cases} \frac{\pi}{4}, & x < 0, \\ \frac{15\pi}{4}, & x > 0. \end{cases} \tag{4.9}$$

The solution consists one stationary shock at the origin, one moving shock followed by an expansion wave and one moving expansion wave. In Fig 6 solution obtained by *HSP* is shown for time $t = 1.0$. Figure 6 shows that *HSP* capture the stationary shock as well moving shocks nicely with a nice resolution for the two expansion waves.
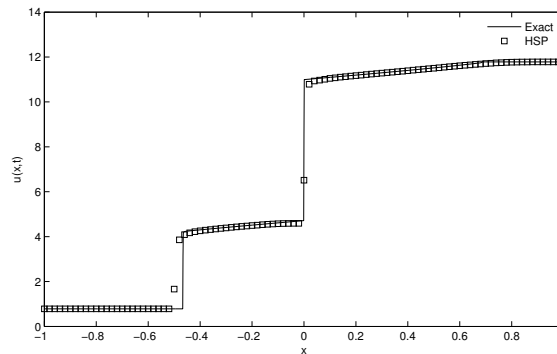

FIGURE 6. Solution profile using $C = 0.6$, $\Delta x = 0.02$ at time $t = 1.0$

Conclusion and future work. In this work, we define the semi-primitive scheme by considering the primitive form of conservation law. Primitive scheme is constructed in such a way that it becomes conservative in the vicinity of discontinuity. Proposed scheme captures the shock with high accuracy but yields rarefaction shocks in the presence of sonic point. A hybrid scheme approach is proposed based on entropy fix iteration. Presented numerical examples show that hybrid scheme captures shock at right location and resolves the expansion fan. Extension for the hyperbolic system and high order accurate hybrid semi-primitive scheme is under investigation.

REFERENCES

[1] Xin, J.; Flaherty, J. E.; Viscous stabilization of discontinuous Galerkin solutions of hyperbolic conservation laws, App. Num. Math. 56 (2006), 444-458.
[2] Hou, T.Y.; LeFloch, P.G.; Why non-conservative schemes converge to wrong solutions: error analysis, Mathematics of Computation, 62(1994), 497-530.
[3] Harten, A.; Hyman, J. M.; Self-Adjusting Grid Methods for One-Dimensional Hyperbolic Conservation Laws, J. of Comput. Physics, 50(1983), 235-269.

[4] Karni, S.; Multicomponent flow calculations using a consistent primitive algorithm. J. Comp. Phys. 112(1994), 31-43.

[5] Kermani, M. J.; Plett, E. G.; Modified Entropy Correction Formula for the Roe Scheme, AIAA 2001- 0083.

[6] Laney, C. B.; Computational Gas-dynamics (Ist edn). Cambridge University press: Cambridge, 1998.

[7] Leveque, R. J.; Numerical methods for conservation Laws, Lectures in Mathematics, ETH Zurich, Birkhauser, 1999.

[8] Leveque, R. J.; Finite volume methods for hyperbolic problems, Cambridge University Press, Cambridge, 2002.

[9] Sermeus, K.; Deconinck, H.; An entropy fix for multi dimensional upwind residual distribution schemes, Computers and Fluids, 34(2005), 617-640.

[10] Toro, E. F.; Silviglia, A.; PRICE: primitive centered schemes for hyperbolic systems, Int. J. Num. Meth. Fluids, 42(2003), 1263-1291.

[11] Toro, E. F.; Riemann solvers and numerical methods for fluid dynamics, A practical introduction 2nd edition, Springer, 1999.

[12] Toro, E. F.; Primitive, conservative and adaptive schemes for hyperbolic conservation laws. In Numerical Methods for Wave Propagation. Toro E. F., Clarke J. F. (eds). Kluwer Academic Publishers: Dordrecht, 1998; 323-385.

[13] Wesseling, P.; Principles of computational fluid dynamics, Springer 2001.

Ritesh Kumar Dubey

Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse Cedex 9, Fance

E-mail address: riteshkd@gmail.com