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Nontrivial solutions of semilinear elliptic systems in $\mathbb{R}^N$ *

Jianfu Yang

Abstract

We establish an existence result for strongly indefinite semilinear elliptic systems in $\mathbb{R}^N$.

1 Introduction

The main objective of this paper is to establish existence results for the semilinear elliptic system

$$-\Delta u + u = g(x,v), \quad -\Delta v + v = f(x,u) \quad \text{in} \mathbb{R}^N, \tag{1.1}$$
$$u(x) \to 0 \quad \text{and} \quad v(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{1.2}$$

The existence of solutions of (1.1)-(1.2) is usually investigated by finding critical points of a related functional. Typical features of the problem are that firstly, the related functional is strongly indefinite; secondly, the growths of f in u and g in v at infinity may not be 'symmetric'; and lastly, Sobolev embeddings in general are not compact, then the Palais - Smale condition may not be satisfied. Existence results were recently obtained in [12] and [15] in bounded domains. The arguments lie in the decomposition of Sobolev spaces by eigenfunctions of Laplacian operator and a use of linking theorems. Using spectral family theory of non-compact operator, the author and Figueiredo [13] find a suitable linking structure for the functional associate to (1.1)-(1.2) and prove that problem (1.1)-(1.2) possesses at least a positive solution if f and g depend on the variable x radially. Furthermore, it is also shown in [13] that all positive solutions of problem (1.1)-(1.2) are exponentially decaying. In this paper, we establish existence results for general cases. Assume that

H1) $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are measurable in the first variable, continuous in the second variable. Both $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$ are increasing and strictly convex in t.

H2) $\lim_{t\to 0} f(x,t)/t = 0, \quad \lim_{t\to 0} g(x,t)/t = 0 \quad$ uniformly in $\quad x \in \mathbb{R}^N$.

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H3) There is a constant $c > 0$ such that $|f(x,t)| \le c(|t|^p + 1)$ and $|g(x,t)| \le c(|t|^q + 1)$, where $0 < p, q < (N+2)/(N-2)$, $N \ge 3$.

H4) There are constants $\alpha, \beta > 2$ such that $0 < \alpha F(x,t) \le tf(x,t)$ and $0 < \beta G(x,t) \le tg(x,t)$, for $t \ne 0$.

H5) $f(x,t) \to \bar{f}(t)$ and $g(x,t) \to \bar{g}(t)$ uniformly for t bounded as $|x| \to \infty$. $|f(x,t) - \bar{f}(t)| \le \epsilon(R)|t|$ and $|g(x,t) - \bar{g}(t)| \le \epsilon(R)|t|$ whenever $|x| \ge R$, $|t| \le \delta$, where $\epsilon(R) \to \infty$ as $R \to \infty$.

H6) $F(x,t) \ge \bar{F}(t)$ and $G(x,t) \ge \bar{G}(t)$, meas$\{x \in R^N : f(x,t) \not\equiv \bar{f}(t)\} > 0$ or meas$\{x \in R^N : g(x,t) \not\equiv \bar{g}(t)\} > 0$.

H7) Both $\bar{f}(t)/t$ and $\bar{g}(t)/t$ are increasing in t.

Our main result is as follows.

Theorem 1.1 *Assume (H1)-(H7). Then problem (1.1)-(1.2) possesses at least one nontrivial exponentially decaying solution.*

The restriction of exponents in (H3) is due to the fact that we only know the decaying law in the case.

We analyze the convergence of Palais-Smale sequence of associate functional to (1.1)-(1.2) in Section 3. It is shown that the energy levels of solutions of the related autonomous system

$$-\Delta u + u = \bar{g}(v), \quad -\Delta v + v = \bar{f}(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$
$$u(x) \to 0, \quad v(x) \to 0 \quad \text{as} \quad |x| \to 0\,. \tag{1.4}$$

are obstacle levels preventing strong convergence of Palais-Smale sequences of (1.1)-(1.2). The possible critical values to be found are between obstacle levels. To retain the compactness, we have to get control of critical values. It is harder to handle critical values described by linking structure than that by the Mountain Pass Theorem. We use dual variational method as [3], [4] and [11]. The method is of the advantage avoiding the indefinite character of original functional. We start with problem (1.1)-(1.2) in bounded domains. Although existence result in the case is known, it has no control of critical values. We establish in Section 2 an existence result by the Mountain Pass Theorem and bound uniformly corresponding critical values by the energy level of ground state of problem (1.3)-(1.4). Then we construct a Palais - Smale sequence for the functional associated to problem (1.1)-(1.2). Theorem 1.1 is proved in Section 4.

2 Existence results in bounded domains

Let Ω be a bounded domain. We consider the problem

$$-\Delta u + u = g(x,v), \quad -\Delta v + v = f(x,u) \quad \text{in} \quad \Omega, \tag{2.1}$$
$$u = 0, \quad v = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.2}$$

The solutions of (2.1)-(2.2) will be found by looking for critical points of associate functional. The main result in this section is as follows.

Theorem 2.1 *Assume* $(H1)-(H4)$. *Then problem (2.1)-(2.2) possesses at least a nontrivial solution.*

To prove Theorem 2.1 we will need the lemmas below. First we define the dual functional associate to (2.1)-(2.2). It is well known that the inclusions

$$i_r : W_o^{1,r}(\Omega) \to L^{p+1}(\Omega), \quad i_s : W_o^{1,s}(\Omega) \to L^{q+1}(\Omega)$$

are compact if $2 < p+1 < \frac{rN}{N-r}, N > r$ and $2 < q+1 < \frac{sN}{N-s}, N > s$. The operator $-\Delta + id : W_o^{1,r}(\Omega) \to W_o^{-1,r'}(\Omega)$ is an isomorphism, where $r' = \frac{r}{r-1}$. Hence

$$\mathcal{T} = i_2(-\Delta + id)^{-1} i_2^* : L^{1+1/q}(\Omega) \to L^{p+1}(\Omega).$$

is continuous. Denote by $X = L^{p+1}(\Omega) \times L^{q+1}(\Omega), X^* = L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega)$ and let

$$A = \begin{pmatrix} 0 & \mathcal{T} \\ \mathcal{T} & 0 \end{pmatrix}, \quad K = A^{-1} = \begin{pmatrix} 0 & \mathcal{T}^{-1} \\ \mathcal{T}^{-1} & 0 \end{pmatrix}.$$

For each x, the Legendre-Fenchel transformations $F^*(x,\cdot)$ of $F(x,\cdot)$, and $G^*(x,\cdot)$ of $G(x,\cdot)$ are defined by

$$F^*(x,s) = \sup_{t \in R}\{st - F(x,t)\}, \quad G^*(x,s) = \sup_{t \in R}\{st - G(x,t)\} \tag{2.3}$$

respectively. Equivalently, we have

$$F^*(x,s) = st - F(x,t) \quad \text{with} \quad s = f(x,t), \quad t = F_s^{*'}(x,s) \tag{2.4}$$

and

$$G^*(x,s) = st - G(x,t) \quad \text{with} \quad s = g(x,t), \quad t = G_s^{*'}(x,s). \tag{2.5}$$

In the same way, we define $\bar{F}^*, \bar{G}^*$ for $\bar{F}, \bar{G}$. By (H6) and properties of Legendre-Fenchel transformations, we have

$$F^*(x,s) \le \bar{F}^*(s), \quad G^*(x,s) \le \bar{G}^*(s). \tag{2.6}$$

We may verify following properties of F^*, G^* in Lemmas 2.2 and 2.3 as [3], [10] and [16].

Lemma 2.2 $F^*, G^* \in C^1$ *and*

$$F^*(x,s) \ge (1 - \frac{1}{\alpha}) s F^{*'}(x,s), \quad G^*(x,s) \ge (1 - \frac{1}{\beta}) s G^{*'}(x,s), \tag{2.7}$$

$$F^*(s,x) \ge C|s|^{1+1/p} - C, \quad G^*(x,s) \ge C|s|^{1+1/q} - C. \tag{2.8}$$

Lemma 2.3 *There exist* $\delta > 0, C_\delta$ *and* $C'_\delta > 0$ *such that*

$$F^*(x,s) \ge \begin{cases} C_\delta |s|^2, & \text{if} \quad |s| \le \delta \\ C'_\delta |s|^{1+\frac{1}{p}}, & \text{if} \quad |s| \ge \delta \end{cases}, \quad G^*(x,s) \ge \begin{cases} C_\delta |s|^2, & \text{if} \quad |s| \le \delta \\ C'_\delta |s|^{1+\frac{1}{q}}, & \text{if} \quad |s| \ge \delta \end{cases}.$$

We may verify that the dual functional

$$\Psi(w) = \Psi_\Omega(w) = \int_\Omega (F^*(x, w_1) + G^*(x, w_2))\, dx - \frac{1}{2} \int_\Omega \langle w, Kw \rangle\, dx,$$

is well defined and C^1 on X^*. A critical point w of Ψ satisfies

$$(-\Delta + id)^{-1} w_2 = F_s^{*'}(x, w_1) \quad \text{and} \quad (-\Delta + id)^{-1} w_1 = G_s^{*'}(x, w_2).$$

Let

$$u = (-\Delta + id)^{-1} w_2, \quad v = (-\Delta + id)^{-1} w_1.$$

Then (u, v) satisfies (2.1)-(2.2). Furthermore, denoting by

$$\Phi(z) = \int_\Omega (\nabla u \nabla v + uv)\, dx - \int_\Omega F(x, u)\, dx - \int_\Omega G(x, v)\, dx$$

the functional of (2.1) -(2.2) defined on $H_o^1(\Omega) \times H_o^1(\Omega)$, we deduce by (2.4) and (2.5) that $\Phi(z) = \Psi(w)$. Such a result is also valid for solutions of (1.1)-(1.2). Now we use the Mountain Pass Theorem to find critical points of Ψ.

Following arguments of [6], we know that (H2) implies $F^*(x, t)/t^2 \to \infty$ and $G^*(x, t)/t^2 \to \infty$. Thus 0 is a local minmum of Ψ. Precisely,

Lemma 2.4 *There exist constants $\alpha, \rho > 0$, independent of Ω, such that*

$$\Psi(w) \geq \alpha > 0 \quad \text{if} \quad \|w\|_{X^*} = \rho.$$

Lemma 2.5 *There exist $T > 0$ and $w \in E$ such that $\Psi(tw) \leq 0$ whenever $t \geq T$.*

Proof. Taking $w \in X^*, w \not\equiv 0$ such that

$$\int_\Omega \langle w, Kw \rangle\, dx > 0,$$

whence by (H4), for $t > 0$

$$\Psi(tw) \leq t^{\frac{\alpha}{\alpha-1}} \int_\Omega |w_1|^{\frac{\alpha}{\alpha-1}}\, dx + t^{\frac{\beta}{\beta-1}} \int_\Omega |w_2|^{\frac{\beta}{\beta-1}}\, dx - \frac{1}{2} t^2 \int_\Omega \langle w, Kw \rangle\, dx.$$

Since $\frac{\alpha}{\alpha-1}, \frac{\beta}{\beta-1} < 2$, the assertion follows for $t > 0$ large. $\square$

Let

$$\Gamma = \{g \in C([0,1], X^*) : g(0) = 0, g(1) = e\},$$

where $e = Tw$. We define

$$c = c_\Omega = \inf_{g\in\Gamma} \sup_{t\in[0,1]} \Psi(g(t)). \tag{2.9}$$

The Mountain Pass Theorem will imlpy that c is a critical value of Ψ if the Palais-Smale ((PS) for short) condition holds. It is known from Lemma 2.4 that corresponding critical points are nontrivial. Then the proof of Theorem 2.1 is completed.

Now we verify the (PS) condition. By a (PS) condition for Ψ we mean that any sequence $\{w_n\} \subset X^*$ such that $|\Psi(w_n)|$ is uniformly bounded in n and $\Psi'(w_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Lemma 2.6 *The (PS) condition holds for Ψ.*

Proof. Let $\{w_n\}$ be a (PS) sequence of Ψ, that is

$$|\Psi(w_n)| \le C \quad \Psi'(w_n) \to 0 \quad \text{as} \quad n \to \infty$$

for some constant $C > 0$. This inequality and lemma 2.2 yield

$$\begin{aligned}
&\int_\Omega [F^*(x, w_n^1) + G^*(x, w_n^2)]\, dx \\
\le\ & \frac{1}{2}\int_\Omega \langle w_n, K w_n\rangle\, dx + C \\
\le\ & \frac{1}{2}\int_\Omega (F_s^{*'}(x, w_n^1) w_n^1 + G_s^{*'}(x, w_n^2) w_n^2)\, dx + o(1)\|w_n\|_{X^*} + C \\
\le\ & \frac{1}{2}\frac{\alpha}{\alpha-1}\int_\Omega F^*(x, w_n^1)\, dx + \frac{1}{2}\frac{\beta}{\beta-1}\int_\Omega G^*(x, w_n^2)\, dx + o(1)\|w_n\|_{X^*}.
\end{aligned}$$

That is

$$\int_\Omega [F^*(x, w_n^1) + G^*(x, w_n^2)]\, dx \le C + o(1)\|w_n\|_{X^*}.$$

By Lemma 2.3, we obtain

$$\|w_n^1\|_{L^{1+1/p}}^{1+1/p} + \|w_n^2\|_{L^{1+1/q}}^{1+1/q} \le C + o(1)\|w_n\|_{X^*}.$$

It implies that $\|w_n\|_{X^*}$ is bounded. We may assume $w_n \to w$ weakly in X^* as $n \to \infty$. Since the operator $(-\Delta + id)^{-1}$ is compact, it follows

$$\begin{aligned}
u_n &:= (-\Delta + id)^{-1} w_n^2 \to (-\Delta + id)^{-1} w^2 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty, \\
v_n &:= (-\Delta + id)^{-1} w_n^1 \to (-\Delta + id)^{-1} w^1 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty.
\end{aligned}$$

As a result, $w_n = (f(x, u_n), g(x, v_n)) \to w \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty$ which completes the present proof.

3 Palais-Smale sequence

In this section, we prove a global compact result for problem (1.1)-(1.2). Let $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. By our assumptions, the functional

$$\mathbf{\Phi}(z) = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\, dx - \int_{\mathbb{R}^N} (F(x, u) + G(x, v))\, dx$$

is C^1 on E. The functional $\mathbf{\Phi}^\infty$ is defined with $\bar{F}$ and $\bar{G}$ replacing F and G in $\mathbf{\Phi}$ respectively.

Proposition 3.1 *Asumme (H1)-(H6). Let* $\{z_n\}$ *be a* $(PS)_c$ *sequence of* $\mathbf{\Phi}$, *i.e.*

$$\mathbf{\Phi}(z_n) \to c \quad \text{and} \quad \mathbf{\Phi}'(z_n) \to 0 \quad \text{as} \quad n \to 0. \tag{3.1}$$

Then there exists a subsequence (still denoted by $\{z_n\}$*) for which the following holds: there exist an integer* $k \geq 0$, *sequences* $\{x_n^i\} \subset \mathbb{R}^N, |x_n^i| \to \infty$ as $n \to \infty$ *for* $1 \leq i \leq k$, *a solution* z *of (1.1)-(1.2) and solutions* $z^i(1 \leq i \leq k)$ *of (1.3)-(1.4) such that*

$$z_n \to z \quad \text{weakly} \quad \text{in} \quad E, \tag{3.2}$$

$$\mathbf{\Phi}(z_n) \to \mathbf{\Phi}(z) + \textstyle\sum_{i=1}^k \mathbf{\Phi}^\infty(z^i), \tag{3.3}$$

$$z_n - (z + \textstyle\sum_{i=1}^k z^i(x - x_n^i)) \to 0 \quad \text{in} \quad E \tag{3.4}$$

as $n \to \infty$, *where we agree that in the case* $k = 0$ *the above holds without* z^i, x_n^i.

Proof. The result can be derived from the arguments for one equation [5]. First we remark that the boundedness of $\{z_n\}$ in E can be deduced as [13] by (3.1). Therefore we may assume

$$\begin{aligned} &z_n \to z \quad \text{weakly} \quad \text{in} \quad E, \\ &z_n \to z \quad \text{strongly} \quad \text{in} \quad L^{p+1}_{loc}(\mathbb{R}^N) \times L^{q+1}_{loc}(\mathbb{R}^N), \\ &z_n \to z \quad a.e. \quad \text{in} \quad \mathbb{R}^N \end{aligned}$$

as $n \to \infty$. Denote $Q(z) = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\, dx$, we have

$$Q(z_n) = Q(z_n - z) + Q(z) + o(1). \tag{3.5}$$

It follows from Brezis & Lieb's lemma [8] that

$$\int_{\mathbb{R}^N} F(x, u_n)\, dx = \int_{\mathbb{R}^N} F(x, u_n - u)\, dx + \int_{\mathbb{R}^N} F(x, u)\, dx + o(1) \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} G(x, v_n)\, dx = \int_{\mathbb{R}^N} G(x, v_n - v)\, dx + \int_{\mathbb{R}^N} G(x, v)\, dx + o(1). \tag{3.7}$$

Hence we obtain

$$\mathbf{\Phi}(z_n) = \mathbf{\Phi}(z_n - z) + \mathbf{\Phi}(z) + o(1), \quad \mathbf{\Phi}'(z_n) = \mathbf{\Phi}'(z_n - z) + \mathbf{\Phi}'(z) + o(1) \tag{3.8}$$

as $n \to \infty$. Let $z_n^1 = z_n - z$. We may deduce from (H5) as [17] and [19] that

$$\int_{\mathbb{R}^N} u_n^1 [f(x, u_n^1) - \bar{f}(u_n^1)]\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v_n^1 [g(x, v_n^1) - \bar{g}(v_n^1)]\, dx \to 0$$

as well as

$$\int_{\mathbb{R}^N} [F(x, u_n^1) - \bar{F}(u_n^1)]\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} [G(x, v_n^1) - \bar{G}(v_n^1)]\, dx \to 0$$

as $n \to \infty$. Therefore

$$\mathbf{\Phi}^\infty(z_n^1) = \mathbf{\Phi}(z_n^1) + o(1) = \mathbf{\Phi}(z_n) - \mathbf{\Phi}(z) + o(1) \tag{3.9}$$

$$\mathbf{\Phi}^{\infty'}(z_n^1) = \mathbf{\Phi}'(z_n^1) + o(1) = \mathbf{\Phi}'(z_n) - \mathbf{\Phi}'(z) + o(1). \tag{3.10}$$

Suppose $z_n^1 = z_n - z \not\to 0$ strongly in E(otherwise we shall have finished). We want to show that there exists $x_n^1 \subset \mathbb{R}^N$ such that $|x_n^1| \to +\infty$ and $z_n^1(x+x_n^1) \to z^1 \not\equiv 0$ weakly in E. We note that

$$\mathbf{\Phi}^\infty(z_n^1) \geq \alpha > 0$$

because $\|z_n^1\|_E \not\to 0$. In fact, were it not true, we would have

$$\mathbf{\Phi}^\infty(z_n^1) \to 0, \quad < \mathbf{\Phi}^{\infty'}(z_n^1), \eta >= o(1)\|\eta\|_E \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Taking $\eta = (\frac{\beta}{\alpha+\beta}u_n^1, \frac{\alpha}{\alpha+\beta}v_n^1) =: \eta_n$ in (3.11), it follows

$$\begin{aligned} o(1)\|\eta_n\|_E \quad &= \quad \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} u_n^1 \bar{f}(u_n^1)\,dx + \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} v_n^1 \bar{g}(v_n^1)\,dx \\ &\quad - \int_{\mathbb{R}^N} \bar{F}(u_n^1)\,dx - \int_{\mathbb{R}^N} \bar{G}(v_n^1)\,dx. \end{aligned} \tag{3.12}$$

Using hypothesis (H4) we obtain

$$\int_{\mathbb{R}^N} (\bar{F}(u_n^1) + \bar{G}(v_n^1))\,dx = o(1)\|\eta\|_E.$$

This and (3.12) yield

$$\int_{\mathbb{R}^N} u_n^1 \bar{f}(u_n^1)\,dx = o(1)\|\eta_n\|_E, \quad \int_{\mathbb{R}^N} v_n^1 \bar{g}(v_n^1)\,dx = o(1)\|\eta_n\|_E. \tag{3.13}$$

It follows from assumptions (H2)-(H4) that

$$|\bar{f}(t)|^2 \leq Ct\bar{f}(t) \quad \text{if} \quad |t| \leq 1, \quad |\bar{f}(t)|^{(p+1)'} \leq Ct\bar{f}(t) \quad \text{if} \quad |t| > 1. \tag{3.14}$$

Taking $\eta = (\phi, 0)$ in (3.11) and using (3.14) and Hölder's inequality, we obtain

$$\begin{aligned} &|\int_{\mathbb{R}^N} (\nabla\phi\nabla v_n^1 + \phi v_n^1)\,dx| \\ &\leq \quad |\int_{\{|u_n^1|\leq 1\}} + \int_{\{|u_n^1|>1\}} \phi\bar{f}(u_n^1)\,dx| \qquad (3.15) \\ &\leq \quad C(\int_{\mathbb{R}^N} |f(u_n^1)|^2\,dx)^{\frac{1}{2}}\|\phi\|_{L^2} + C(\int_{\mathbb{R}^N} |\bar{f}(u_n^1)|^{(p+1)'}\,dx)^{1/(p+1)'}\|\phi\|_{L^{p+1}} \\ &\leq \quad C\|\phi\|_{H^s}[(\int_{\mathbb{R}^N} u_n^1\bar{f}(u_n^1)\,dx)^{\frac{1}{2}} + C(\int_{\mathbb{R}^N} u_n^1\bar{f}(u_n^1)\,dx)^{1/(p+1)'}]. \end{aligned}$$

which with (3.13) imply that

$$\|v_n^1\|_{H^1} = o(1). \tag{3.16}$$

Similarly, we show that

$$\|u_n^1\|_{H^1} = o(1). \tag{3.17}$$

(3.16) and (3.17) yield $\|z_n^1\|_E \to 0$, we get a contradiction.

We decompose $\mathbb{R}^N$ into N-dimensional unit hypercubes Q_j with vertices having integer coordinates and put

$$d_n = \max_j(\|u_n^1\|_{L^{p+1}(Q_j)} + \|v_n^1\|_{L^{q+1}(Q_j)}).$$

We claim that there is a $\beta > 0$ such that

$$d_n \geq \beta > 0 \quad \forall n \in \mathbb{N}. \tag{3.18}$$

Suppose, by contradiction, that $d_n \to 0$ as $n \to \infty$. Since

$$\mathbf{\Phi}^{\infty'}(z_n^1) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.19}$$

noting that $\|z_n^1\|_E$ is bounded, we have by (H2) and (H3) that

$$\begin{aligned}
0 \quad &\leq \quad \mathbf{\Phi}^{\infty}(z_n^1) \leq \int_{\mathbb{R}^N} u_n^1 \bar{f}(u_n^1)\, dx + \int_{\mathbb{R}^N} v_n^1 \bar{g}(v_n^1)\, dx + o(1) \\
&\leq \quad C_\epsilon(\|u_n^1\|^{p+1}_{L^{p+1}(R^N)} + \|v_n^1\|^{q+1}_{L^{q+1}(R^N)}) + \epsilon(\|u_n^1\|^2_{L^2(R^N)} + \|v_n^1\|^2_{L^2(R^N)}) \\
&\leq \quad C_\epsilon \sum_j (\|u_n^1\|^{p+1}_{L^{p+1}(Q_j)} + \|v_n^1\|^{q+1}_{L^{q+1}(Q_j)}) + \epsilon(\|u_n^1\|^2_{L^2(R^N)} + \|v_n^1\|^2_{L^2(R^N)}) \\
&\leq \quad C_\epsilon d_n^{p-1} \sum_j \|u_n^1\|^2_{L^{p+1}(Q_j)} + C_\epsilon d_n^{q-1} \sum_j \|v_n^1\|^2_{L^{q+1}(Q_j)} + \epsilon C \\
&\leq \quad C_\epsilon d_n^{p-1} \sum_j \|u_n^1\|^2_{H^1(Q_j)} + C_\epsilon d_n^{q-1} \sum_j \|v_n^1\|^2_{H^1(Q_j)} + \epsilon C \\
&\leq \quad C_\epsilon d_n^{p-1} \|u_n^1\|^2_{H^1} + C_\epsilon d_n^{q-1} \|v_n^1\|^2_{H^1} + \epsilon C.
\end{aligned}$$

Let $n \to \infty$ and then $\epsilon \to 0$, we obtain $\Phi^\infty(z_n^1) \to 0 \quad \text{as} \quad n \to \infty$. This and (3.19) imply as above that $\|z_n^1\|_E \to 0$ as $n \to \infty$, a contradiction, hence we have (3.18).

Let $\{x_n^1\}$ be the center of a hypercube Q_j in which

$$d_n = \|u_n^1\|_{L^{p+1}(Q_j)} + \|v_n^1\|_{L^{q+1}(Q_j)}.$$

Now we show that

$$|x_n^1| \to \infty \quad \text{as} \quad n \to \infty. \tag{3.20}$$

If $\{x_n^1\}$ were bounded, by passing to a subsequence if necessary we should find that x_n^1 would be in the same Q_j and so they should coincide. Therefore in that Q_j, for every $n > n_o$, n_o fixed and large enough, we should have

$$\begin{aligned}
\mathbf{\Phi}^\infty|_{E(Q_j)}(\bar{z}_n^1) \quad &= \quad \int_{Q_j} (\nabla \bar{u}_n^1 \nabla \bar{v}_n^1 + \bar{u}_n^1 \bar{v}_n^1)\, dx - \int_{Q_j} (\bar{F}(\bar{u}_n^1) + \bar{G}(\bar{v}_n^1))\, dx + o(1) \\
&\geq \quad (\alpha - 1) \int_{R^N} \bar{F}(\bar{u}_n^1)\, dx + (\beta - 1) \int_{R^N} \bar{G}(\bar{v}_n^1)\, dx + o(1) \\
&\geq \quad C(\|\bar{u}_n^1\|^\alpha_{L^\alpha(Q_j)} + \|\bar{v}_n^1\|^\beta_{L^\beta(Q_j)}) + o(1) \\
&\geq \quad C(\|\bar{u}_n^1\|^\alpha_{L^{p+1}(Q_j)} + \|\bar{v}_n^1\|^\beta_{L^{q+1}(Q_j)}) + o(1),
\end{aligned}$$

and

$$\mathbf{\Phi}^{\infty'}(\bar{z}_n^1) \to 0 \quad \text{as} \quad n \to 0,$$

where

$$\bar{z}_n^1(x) = \begin{cases} z_n^1(x) & z \in Q_j \\ 0 & x \in \mathbb{R}^N \backslash Q_j. \end{cases}$$

Hence $\bar{z}_n^1$ should converge strongly in $E(Q_j)$ to a nonzero function, contradicting to $z_n^1 \to 0 \quad \text{weakly} \quad \text{in} \quad E$, so we have (3.20). Let

$$z_n^1(\cdot + x_n^1) \to z^1 \quad \text{weakly} \quad \text{in} \quad E.$$

Denote by $\bar{Q}$ the unit hypercube centered at the origin, we have $\|z_n^1\|_{E(\bar{Q})} \geq \beta > 0$, thus $z^1 \not\equiv 0$ and

$$\langle \mathbf{\Phi}^{\infty'}(z^1), \eta \rangle = 0, \quad \forall \eta \in E. \tag{3.21}$$

Iterating the procedure, we obtain sequences $x_n^l, |x_n^l| \to \infty$ and

$$\begin{gathered} z_n^l(x) = z_n^{l-1}(x + x_m) - z^{l-1}(x), \quad j \geq 2 \\ z_n^l(x + x_n^l) \to z^l(x) \quad \text{weakly} \quad \text{in} \quad E \end{gathered}$$

as $n \to 0$, where each z^l satisfying (3.21) and by induction

$$\|z_n^l\|_E = \|z_n^{l-1}\|_E^2 - \|z^{l-1}\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 - \sum_{i=1}^{l-1} \|z^i\|_E^2 + o(1).$$

$$\mathbf{\Phi}^\infty(z_n^l) = \mathbf{\Phi}^\infty(z_n^{l-1}) - \mathbf{\Phi}^\infty(z^{l-1}) + o(1) = \mathbf{\Phi}(z_n) - \mathbf{\Phi}(z) - \sum_{i=1}^{l-1} \mathbf{\Phi}(z^i) + o(1).$$

Since z^l is a solution of (1.3)-(1.4) and $z^l \not\equiv 0$, we may prove as Lemma 4.1 below that $\|z^l\|_E \geq C > 0$. Thus the iteration will terminate at some index $k \geq 0$. The assertion follows.

4 Uniform bounds and proof of Theorem 1.1

We shall bound critical values defined in (2.9) by the energy of the ground state of problem (1.3)-(1.4). By a ground state of problem (1.3)-(1.4) we mean a minimizer of the variational problem

$$\Phi^\infty = \inf\{\Phi^\infty(u, v) : (u, v) \in E \text{ is a solution of (1.3)-(1.4)}, (u, v) \not\equiv (0, 0)\}. \tag{4.1}$$

It is shown in [13] that problem (1.3)-(1.4) has a positive radially decaying solution, so the variational problem (4.1) is well defined.

Lemma 4.1 *Variational problem (4.1) is assumed by a nontrivial solution of (1.3)-(1.4).*

Proof. Let $z_n = (u_n, v_n)$ be a minimizing sequence of Φ^∞. It is obvious that $\{z_n\}$ is a (PS) sequence of Φ^∞. We deduce by Proposition 3.1 that

$$\Phi^\infty = \Phi(z_n) + o(1) = \sum_{j=1}^{k} \Phi^\infty(z_j) + o(1),$$

where z_j is a solution of (1.3)-(1.4). By the definition of Φ^∞, $k = 1$. The proof will be completed if we show $z_1 \neq 0$. To this end, we bound solutions of (1.3)-(1.4) in H^1 norm below by a positive constant.

Suppose $z = (u, v)$ is a solution of (1.3)-(1.4), we have

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^N} u\bar{g}(v)\,dx,\ \ \|v\|_{H^1}^2 = \int_{\mathbb{R}^N} v\bar{f}(u)\,dx, \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,dx = \int_{\mathbb{R}^N} v\bar{g}(v)\,dx = \int_{\mathbb{R}^N} u\bar{f}(u)\,dx. \tag{4.3}$$

By assumptions (H2), (H3) and (H5), we obtain

$$\bar{f}(u) \le C_\epsilon |u|^{\frac{N+2}{N-2}} + \epsilon u, \quad \bar{g}(v) \le C_\epsilon |v|^{\frac{N+2}{N-2}} + \epsilon v. \tag{4.4}$$

We deduce by (4.2)-(4.4) and Hölder's inequality that

$$\|u\|_{H^1}^2 \le C_\epsilon \|v\|_{L^{2^*}}^{2^*-1} \|u\|_{L^{2^*}} + \epsilon \|u\|_{L^2} \|v\|_{L^2},$$

where $2^* = \frac{2N}{N-2}$. Using Young's inequality and Sobolev embedding, we obtain

$$\|u\|_{H^1}^2 \le C_\epsilon (\|u\|_{H^1}^{2^*} + \|v\|_{H^1}^{2^*}) + \epsilon \|v\|_{H^1}^2.$$

Similarly,

$$\|v\|_{H^1}^2 \le C_\epsilon (\|u\|_{H^1}^{2^*} + \|v\|_{H^1}^{2^*}) + \epsilon \|u\|_{H^1}^2.$$

So for ϵ small, we have

$$\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \le C (\|u\|_{H^1}^{2^*} + \|v\|_{H^1}^{2^*}).$$

It yields that $\|u\|_{H^1}$ or $\|v\|_{H^1} \ge C > 0$, uniformly for solutions of (1.3)-(1.4), and where $C > 0$ is independent of $z = (u, v)$. Consequently, $z_1 = (u_1, v_1) \not\equiv 0$. □

Let $R_n \to \infty, B_n = B_{R_n}(0)$. Taking $\Omega = B_n$ in problem (2.1)-(2.2), we infer from Theorem 2.1 that there exists a solution z_n of problem (2.1)- (2.2) defined on B_n for each n. Moreover,

$$\Phi(z_n) = \Psi(w_n) = c_n \ge \alpha > 0, \tag{4.5}$$

where $z_n = K w_n$, $\Phi = \Phi_{R^N}$ and $\Psi = \Psi_{R^N}$. We have extended z_n to $\mathbb{R}^N$ by letting $z_n = 0$ outside B_n.

Proposition 4.1 *$c_n < \Phi^\infty$ for n large.*

Proof. Since each element w in $X_n^* = L^{1+1/p}(B_n) \times L^{1+1/q}(B_n)$ can be extended to an element of X^* by letting $w = 0$ outside B_n, we shall denote Ψ_{B_n} as Ψ in brief. By Lemma 4.1, Φ^∞ is assumed. Let $z_o = (u_o, v_o)$ be a minimizer of Φ^∞. Choosing

$$w_1^o = \bar{f}(u_o), \quad w_2^o = \bar{g}(v_o)$$

and using (H4)-(H5) and equations (1.3)-(1.4), one has $\int_{\mathbb{R}^N} < w_o, Kw_o > dx > 0$, where $w_o = (w_1^o, w_2^o)$. Moreover, we know as Lemma 2.5 that there are $t_1, t_2 > 0$ such that

$$\max_{t \geq 0} \Psi(tw_o) = \max_{t_1 \leq t \leq t_2} \Psi(tw_o).$$

Suppose that $t_o \in (t_1, t_2)$ and

$$\Psi(t_o w_o) = \max_{t_1 \leq t \leq t_2} \Psi(tw_o).$$

Because $F(x,t) \geq \bar{F}(t)$ and $G(x,t) \geq \bar{G}(t)$, one has $F^*(x,s) \leq \bar{F}^*(s)$ and $G^*(x,s) \leq \bar{G}(s)$. By the assumption (H6),

$$\Psi(t_o w_o) < \Psi^\infty(t_o w_o),$$

it follows

$$\sup_{t \geq 0} \Psi(tw_o) < \sup_{t \geq 0} \Psi^\infty(tw_o). \tag{4.6}$$

The density of real number field implies that there exists $\epsilon > 0$ such that

$$\sup_{t \geq 0} \Psi(tw_o) + 2\epsilon < \sup_{t \geq 0} \Psi^\infty(tw_o). \tag{4.7}$$

Let $\phi \in C_o^\infty(\mathbb{R}^N), 0 \leq \phi \leq 1$ and $\phi \equiv 1$ if $|x| \leq \frac{1}{2}$; $\phi \equiv 0$ if $|x| > 1$; $\phi_n(x) = \phi(\frac{x}{R_n})$. Then $z_n := (\phi_n u_o, \phi_n v_o)$ converges to (u_o, v_o) in E. Let

$$w_1^n = \bar{f}(\phi_n u_o), \quad w_2^n = \bar{g}(\phi_n v_o).$$

We also have $w_n \to w_o$ in X^*. Suppose

$$\Psi(t_n w_n) = \sup_{t \geq 0} \Psi(tw_n),$$

then $\{t_n\}$ is bounded. Indeed, if $t_n \to \infty$, arguments in Lemma 2.5 would yield $\sup_{t \geq 0} \Psi(tw_n) \to -\infty$. It is not possible because the value is not negative. Suppose $t_n \to \bar{t}_o$, the continuity of the functional Ψ gives

$$\Psi(t_n w_n) \to \Psi(\bar{t}_o w_o).$$

We claim that $\Psi(\bar{t}_o w_o) = \sup_{t \geq 0} \Psi(tw_o)$. In fact, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Psi(t_o w_o) - \epsilon \leq \Psi(tw_o)$$

whenever $|t - t_o| < \delta$. By the continuity of Ψ, we may find $n_o > 0$ such that if $n \geq n_o$

$$\Psi(tw_o) \leq \Psi(tw_n) + \epsilon, \quad \Psi(t_n w_n) \leq \Psi(\bar{t}_o w_o) + \epsilon.$$

Therefore if $n \geq n_o$ we have

$$\Psi(t_o w_o) - \epsilon \leq \Psi(t_n w_n) + \epsilon \leq \Psi(\bar{t}_o w_o) + 2\epsilon \leq \Psi(t_o w_o) + 2\epsilon.$$

Since ϵ is arbitrary, the conclusion holds. By the same arguments, we find that there exists s_n such that $s_n \to \bar{s}_o$ and

$$\Psi^\infty(s_n w_n) = \sup_{t\geq 0} \Psi^\infty(t w_n) \to \Psi^\infty(\bar{s}_o w_o) = \sup_{t\geq 0} \Psi^\infty(t w_o)$$

as $n \to \infty$. By (4.7), we obtain $\Psi(t_n w_n) + \epsilon < \Psi^\infty(s_n w_n)$ for n large enough. We may assume $s_n > 0$, and then

$$\frac{d\Psi^\infty(t w_n)}{dt} \mid_{t=s_n} = 0,$$

that is

$$\int_{\mathbb{R}^N} [\bar{F}_s^{*'}(s_n w_1^n) w_1^n + \bar{G}_s^{*'}(s_n w_2^n) w_2^n] \, dx - s_n \int_{\mathbb{R}^N} < w_n, K w_n > \, dx = 0. \quad (4.8)$$

By the definition of Legendre - Fenchel transformation, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} [\bar{F}^*(s_n w_1^n) + \bar{G}^*(s_n w_2^n)] \, dx \\
&= \int_{\mathbb{R}^N} [\bar{F}_s^{*'}(s_n w_1^n) s_n w_1^n + \bar{G}_s^{*'}(s_n w_2^n) s_n w_2^n] \, dx \\
&\quad - \int_{\mathbb{R}^N} [\bar{F}(\bar{f}^{-1}(s_n w_1^n)) + \bar{G}(\bar{g}^{-1}(s_n w_2^n))] \, dx \qquad (4.9) \\
&= s_n^2 \int_{\mathbb{R}^N} < w_n, K w_n > \, dx - \int_{\mathbb{R}^N} [\bar{F}(\bar{f}^{-1}(s_n w_1^n)) + \bar{G}(\bar{g}^{-1}(s_n w_2^n))] \, dx.
\end{aligned}$$

Consider

$$(-\Delta + id)^{-1} w_2^n = u_o + \sigma_n, \quad (-\Delta + id)^{-1} w_1^n = v_o + \mu_n \quad \text{in} \quad \mathbb{R}^N,$$

we obtain

$$(-\Delta + id)\sigma_n = \bar{g}(\phi_n v_o) - \bar{g}(v_o), \quad (-\Delta + id)\mu_n = \bar{f}(\phi_n u_o) - \bar{f}(u_o).$$

In terms of L^p-estimates, $\sigma_n \to 0$ and $\mu_n \to 0$ in $H^{2,2}$ as $n \to \infty$. Furthermore, we infer from (4.8) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} s_n (w_1^n)^2 [\frac{\bar{f}^{-1}(s_n w_1^n)}{s_n w_1^n} - \frac{\bar{f}^{-1}(w_1^n)}{w_1^n}] \, dx \\
&+ \int_{\mathbb{R}^N} s_n (w_2^n)^2 [\frac{\bar{g}^{-1}(s_n w_2^n)}{s_n w_2^n} - \frac{\bar{g}^{-1}(w_2^n)}{w_2^n}] \, dx \\
&= \int_{\mathbb{R}^N} [w_1^n \sigma_n + w_2^n \mu_n + (1 - \phi_n)(w_1^n + w_2^n)] \, dx = o(1)
\end{aligned}$$

as $n \to \infty$. The equality and assumption (H7) imply $s_n \to 1$ as $n \to \infty$. hence we deduce by (4.8) and (4.9) that

$$\begin{aligned}
&\sup_{t\geq 0} \Psi^\infty(tw_n) \\
&\quad \leq \frac{1}{2}\int_{\mathbb{R}^N} (u_o\bar{f}(u_o) + v_o\bar{g}(v_o))\,dx - \int_{\mathbb{R}^N} (\bar{F}(u_o) + \bar{G}(v_o))\,dx + \epsilon_n \\
&\quad = \Psi^\infty + \epsilon_n,
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_n &= \frac{1}{2}(s_n^2 - 1)\int_{\mathbb{R}^N} (u_o\bar{f}(u_o) + v_o\bar{g}(v_o))\,dx \\
&\quad - \int_{\mathbb{R}^N} [(\bar{F}(\phi_n u_o) - \bar{F}(u_o)) + (\bar{G}(\phi_n v_o) - \bar{G}(v_o))]\,dx \\
&\quad + \int_{\mathbb{R}^N} [(\bar{F}(\phi_n u_o) - \bar{F}(\bar{f}^{-1}(s_n w_1^n)) + (\bar{G}(\phi_n v_o) - \bar{G}(\bar{g}^{-1}(s_n w_2^n))]\,dx.
\end{aligned}$$

The above estimates imply $\epsilon_n = o(1)$ as $n \to \infty$. Therefore

$$\sup_{t\geq 0} \Psi(tw_n) < \sup_{t\geq 0} \Psi(tw_n)^\infty - \epsilon \leq \Psi^\infty - \epsilon + o(1),$$

the assertion follows for n large. $\square$

Lemma 4.2 *z_n is a (PS) sequence of Φ in E.*

Proof. It is readily to verify that $c_n = \Phi(z_n) \leq c_{n-1} = \Phi(z_{n-1})$, thus by Proposition 4.2

$$\alpha \leq c_n \leq c_1 < \Phi^\infty, \tag{4.10}$$

we obtain

$$c_n = \Phi(z_n) \to c, \quad \alpha \leq c \leq c_1 < \Phi^\infty. \tag{4.11}$$

Now we show that

$$\Phi'(z_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.12}$$

In fact, $\forall(\phi,\psi) \in C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$, there is $n_o > 0$ such that $\text{supp}\phi, \text{supp}\psi \subset B_n$ whenever $n \geq n_o$ and

$$\Phi'(z_n)(\phi,\psi) = 0, \quad \text{if} \quad n \geq n_o.$$

This implies that $\Phi'(z_n)z \to 0$ as $n \to \infty$ for all $z \in C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$. Hence (4.12) follows because $C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. $\square$

Completion of the proof of Theorem 1.1 We may prove that the (PS) sequence z_n of Φ is bounded in E as [13], and assume $z_n \to z_o$ weakly in E. Obviously, z_o solves (1.1)-(1.2). We claim that z_o is nontrivial. In fact, Lemma 2.4, Proposition 3.1 and Proposition 4.2 give that

$$\alpha \leq \Phi(z_n) = \Phi(z_o) + \sum_j \Phi^\infty(z_j) + o(1) < \Phi^\infty.$$

If $j = 0, \Phi(z_o) \geq \alpha > 0, z_o$ is a nontrivial solution; if $j \geq 1$, then $\Phi(z_o) < 0$, also implying $z_o \not\equiv 0$. The decaying law of z_o at infinity was proved in [13]. □

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References

[1] R.A. Adams, *Sobolev Spaces* Academic Press 1975.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349 - 381.

[3] A. Ambrosetti and P.N.Srikanth, Superlinear elliptic problems and the dual principle in critical point theory, *J. Math. & Phys.*, **18** (1984), 441-451.

[4] A. Ambrosetti and M. Struwe, A note on the problem $-\Delta u = \lambda u + u|u|^{2^*-1}$, *Manus. Math.*, **54** (1986), 373-379.

[5] V. Benci and G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, *Arch. Rational Mech.Anal.*, **99** (1987), 283-300.

[6] V. Benci and D. Fortunato, The dual method in critical point theory: multiplicity results for indefinite functional, *Ann. Mat. Pura Appl.*, **134** (1982), 215-242.

[7] Berestycki and P.L. Lions, Nonlinear scalar field equations, I and II, *Arch. Rational Mech. Anal.*, **82** (1983), 313-376.

[8] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math.Soc.*, **88** (1983), 486-490.

[9] D.G. Costa, On a class of elliptic systems in $\mathbb{R}^N$, *Eleton. J. Diff. Eqn.* 1994 (1994), No. 7, 1-14. (http://ejde.math.swt.edu)

[10] K.C. Chang, *Critical Point Theory and Its Applications*, Shanghai Sci. & Tech. Press, 1986.

[11] Ph. Clément and R.C.A.M.van der Vorst, On a semilinear elliptic system, *Diff.& Int.Equas.*, **8** (1995),1317-1329.

[12] D.G. de Figueiredo and P.L. Felmer, On supequadratic elliptic systems, *Trans. Amer. Math. Soc.*, **343** (1994), 99-116.

[13] D.G. de Figueiredo and Yang Jianfu, Decay, symmetry and existence of solutions of semilinear elliptic systems *Nonlinear Anal. TMA*, **33** (1998), 211-234.

[14] D. Gilbarg and N.S. Trudinger *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[15] J. Hulshof and R. van der Vorst, Differential systems with strongly indefinite variational structure, *J.Funct.Anal.*, **114** (1993), 32-58.

[16] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, 1993.

[17] P.L. Lions, The concentration-compactness principle in the calculus of variations, *Ann.I.H.Anal.Nonli.*, **1** (1984), 109-283.

[18] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf.Ser. in Math.,No.65, Amer. Math.Soc., Providence, R.I., 1986.

[19] Jianfu Yang and Xiping Zhu, On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded domains, *Acta Math.Sci.*, **7** (1987), 341-359.

Jianfu Yang
Department of Mathematics, IMECC-Unicamp
Caixa Postal 6065
Campinas 13081-970, SP, Brazil
e-mail: jfyang@ime.unicamp.br
and
Department of Mathematics, Nanchang University
Nanchang, Jiangxi 330047
P. R. of China