Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 07, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

REGULARITY CRITERIA FOR WEAK SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS IN BOUNDED DOMAINS VIA BMO NORM

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Communicated by Jesus Ildefonso Diaz

Abstract. We study three-dimensional incompressible Navier-Stokes equations in bounded domains with smooth boundary. We present regularity criteria of weak solutions to this equation via the BMO norm.

1. Introduction

We study the three-dimensional Navier-Stokes equation

$$u_t + (u \cdot \nabla)u - \Delta u + \nabla \pi = 0, \quad \operatorname{div} u = 0 \quad \text{in } Q_T := \Omega \times (0, T), \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega \in C^2$. Here $u : Q_T \to \mathbb{R}^3$ is the flow velocity vector and $\pi : Q_T \to \mathbb{R}$ is the pressure. We consider the initial-boundary value problem of (1.1) with initial condition

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{1.2}$$

together with two types of boundary conditions: Either

$$u = 0, \tag{1.3}$$

or

$$u \cdot n = 0, \quad (\nabla \times u) \times n = 0, \tag{1.4}$$

where n is the outward unit normal vector along boundary $\partial\Omega$. The initial conditions satisfy the compatibility condition, i.e. $\nabla \cdot u_0(x) = 0$. A weak solution u of (1.1)–(1.2) with boundary conditions either

(1.3) or (1.4) is regular in Q_T provided that $\|u\|_{L^\infty(Q_T)} < \infty$. The notion of weak solutions will be introduced in Definition 2.1 of Section 2.

The initial conditions hold the compatibility condition, i.e. $\nabla \cdot u_0(x) = 0$. Since Leray [24] proved the existence of weak solutions of the Navier-Stokes equations (see also [16]), regularity question has remained open.

Definition 1.1. A weak solution u of (1.1)–(1.2) with boundary conditions (1.3) or (1.4) is regular in Q_T provided that $\|u\|_{L^\infty(Q_T)} < \infty$.

2010 *Mathematics Subject Classification.* 35Q30, 35B65, 30H35.
Key words and phrases. Navier-Stokes equation; regularity criteria; BMO space.

Submitted March 27, 2018. Published January 17, 2019.

It is known that any weak solution becomes unique and regular in Q_T, provided that the following scaling invariant conditions [3, 7, 27, 30], so called Serrin's type conditions, are satisfied:

$$
\begin{aligned}
u &\in L^q(0,T;L^p(\mathbb{R}^3)), \quad 3/p+2/q \le 1,\ 3<p\le\infty,\\
\nabla u &\in L^q(0,T;L^p(\mathbb{R}^3)), \quad 3/p+2/q \le 2,\ \frac{3}{2}<p\le\infty,\\
\pi &\in L^q(0,T;L^p(\mathbb{R}^3)), \quad 3/p+2/q \le 2,\ \frac{3}{2}<p\le\infty,\\
\nabla\pi &\in L^q(0,T;L^p(\mathbb{R}^3)), \quad 3/p+2/q \le 3,\ 1<p\le\infty.
\end{aligned}
$$

In this direction, thee are numerous contributions, see [2, 9, 11, 14, 23, 25, 26, 29]. In view of the regularity conditions in view of the BMO space, Kozono and Taniuchi proved in [20] that a weak solution u become regular if u satisfies

$$
\begin{gathered}
u \in L^2(0,T;\mathrm{BMO}(\mathbb{R}^3)),\\
w := \nabla\times u \in L^1(0,T;\mathrm{BMO}(\mathbb{R}^3)), \quad T<\infty,
\end{gathered}
$$

which is the result to the space BMO, which is larger than $L^\infty(\mathbb{R}^3)$. Also, Fan and Ozawa proved in [12] that a weak solution u become regular if u satisfies

$$
\nabla p \in L^{2/3}(0,T;\mathrm{BMO}(\mathbb{R}^3)), \quad 0<T<\infty.
$$

Our study is motivated by the works above, that is, we obtain the regularity conditions for a weak solution to 3D Naiver-Stokes equations (1.1)–(1.2) with the boundary conditions (1.3) or (1.4) in bounded domains. In particular, for bounded domains, the difficulty lies in treating the pressure. To be more precise, in the case that $\Omega=\mathbb{R}^3$, using the equation of pressure, we observe that the pressure π satisfies

$$
\|\pi\|_{L^p(\mathbb{R}^n)} \le C\|u\|^2_{L^{2p}(\mathbb{R}^3)}, \quad 1<p<\infty. \tag{1.5}
$$

However, it is not known yet whether or not the estimate above (1.5) holds for domains with the boundary condition. Thus, the methods of proof in a whole space $\mathbb{R}^3$ do not seem to be applicable to our case. To overcome these difficulties, we use the maximal estimates of Stokes system for both cases of slip and no-slip boundary conditions, regarding the nonlinear term as an external force (see Lemma 2.2 in section 2). Since such estimates of the Stokes system are also available for domain with boundaries, this approach allows for control of pressure and is useful for our analysis. On the other hand, to obtain the regularity condition for a vorticity vector, we consider the vorticity equations for Navier-Stokes equations to avoid the estimate of terms containing the pressure term. In this case, our proof is based on a priori estimate for the vorticity. At last, we give regularity criteria for the pressure to this equations using the maximal regularity theorem (see Lemma 2.2 in section 2). Our main results read as follows.

Theorem 1.2. *Suppose that u is a weak solution to* (1.1)–(1.2) *with initial conditions $u_0 \in H^2(\Omega)\cap W^{1,q}(\Omega)$, $q>3$ and boundary conditions* (1.3) *or* (1.4). *Assume further that u satisfies*

$$
\|u\|_{L^2(0,T;\mathrm{BMO}(\Omega))} < \infty
$$

Then, u becomes regular in $\overline{Q_T}$.

Theorem 1.3. *Suppose that u is a weak solution to* (1.1)–(1.2) *with initial conditions $u_0 \in H^2(\Omega) \cap W^{1,q}(\Omega)$, $q > 3$ and boundary conditions* (1.3) *or* (1.4)*. Assume further that $w := \nabla \times u$ satisfies*

$$\|w\|_{L^1(0,T;\mathrm{BMO}(\Omega))} < \infty$$

Then, u becomes regular in $\overline{Q_T}$.

Theorem 1.4. *Suppose that u is a weak solution to* (1.1)–(1.2) *with initial conditions $u_0 \in H^2(\Omega) \cap W^{1,q}(\Omega)$, $q > 3$ and boundary condition* (1.3)*. Assume further that u satisfies*

$$\|\pi\|_{L^2(0,T;\mathrm{BMO}(\Omega))} < \infty$$

Then, u become regular in $\overline{Q_T}$.

Theorem 1.5. *Suppose that u is a weak solution to* (1.1)–(1.2) *with initial conditions $u_0 \in H^2(\Omega) \cap W^{1,q}(\Omega)$, $q > 3$ and boundary conditions* (1.3) *or* (1.4)*. Assume further that u satisfies*

$$\|\nabla\pi\|_{L^{2/3}(0,T;\mathrm{BMO}(\Omega))} < \infty$$

Then, u becomes regular in $\overline{Q_T}$.

Remark 1.6. Theorem 1.4 can be extended to any dimension $\Omega \subset \mathbb{R}^N$, because we do not deal with the terms related to pressure. On the other hand, Theorems 1.2, 1.3 and 1.5 can be restricted to the case $n = 3, 4$ in view of [17, Remark 3.2] or [19].

Remark 1.7. Theorems 1.5 is given in [13] under the boundary condition (1.4). For the convenience of readers, we give a sketch of the proof.

This article is organized as follows. In Section 2, we recall the notion of weak solutions and review some known results. In Section 3, we present the proofs of Theorems 1.2–1.5.

2. Preliminaries

In this section, we introduce the notation and definitions used throughout this paper. We also recall some lemmas which are useful for our analysis. For $1 \le q \le \infty$ and a nonnegative integer k, $W^{k,q}(\Omega)$ indicates the standard Sobolev space with norm $\|\cdot\|_{k,q}$, i.e., $W^{k,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$. As usual, $W_0^{k,q}(\Omega)$ is defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ in $W^{k,q}(\Omega)$. When $q = 2$, we write $W^{k,q}(\Omega)$ as $H^k(\Omega)$. Let I be a finite time interval. For a function $f(x,t)$, $\Omega \subset \mathbb{R}^3$, we denote $\|f\|_{L^{p,q}_{x,t}(\Omega\times I)} = \|f\|_{L^q_t(I;L^p_x(\Omega))} = \big\|\|f\|_{L^p_x(\Omega)}\big\|_{L^q_t(I)}$. All generic constants will be denoted by C, which may vary from line to line. We recall first the definition of weak solutions.

Definition 2.1. Let $u_0 \in L^2(\Omega)$ with $\operatorname{div} u_0 = 0$. We say that u is a distributional solution (or weak solution) of (1.1)–(1.2) if u satisfies the following:

(1) $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and u satisfies
$$\int_0^T \int_\Omega \Big(\frac{\partial\phi}{\partial t} + (u\cdot\nabla)\phi\Big) u \,dx\,dt + \int_\Omega u_0\phi(x,0)dx = \int_0^T \int_\Omega \nabla u : \nabla\phi \,dx\,dt$$
for all $\phi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$ with $\operatorname{div}\phi = 0$.

(2) u satisfies divergence free condition; that is, $\int_\Omega u \cdot \nabla\psi dx = 0$ for any $\psi \in \mathcal{C}^\infty(\bar\Omega)$.

We consider the following Stokes system which is the linearized Navier-Stokes equations,

$$v_t - \Delta v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \text{in } Q_T := \Omega \times (0,T) \tag{2.1}$$

with initial data $v(x,0) = v_0(x)$. As in (1.3) and (1.4), boundary data of v are again assumed to be either no-slip or slip conditions, namely

$$v(x,t) = 0, \quad x \in \partial\Omega \quad \text{or} \tag{2.2}$$

$$v \cdot n = 0, \quad (\nabla \times v) \times n = 0, \quad x \in \partial\Omega. \tag{2.3}$$

Next, we recall maximal estimates of the Stokes system in terms of mixed norms (see [15, Theorem 5.1] and [28, Theorem 1.2] for no-slip and slip boundary cases, respectively).

Lemma 2.2. *Let $1 < l, m < \infty$. Suppose that $f \in L^{l,m}_{x,t}(Q_T)$ and $v_0 \in D_l^{1-\frac{1}{m},m}$. If (v,p) is the solution of the Stokes system* (2.1) *satisfying one of the boundary conditions* (2.2) *or* (2.3)*, then the following estimate is satisfied:*

$$\begin{aligned} &\|v_t\|_{L^{l,m}_{x,t}(Q_T)} + \|\nabla^2 v\|_{L^{l,m}_{x,t}(Q_T)} + \|\nabla p\|_{L^{l,m}_{x,t}(Q_T)} \\ &\le C\|f\|_{L^{l,m}_{x,t}(Q_T)} + \|v_0\|_{D_l^{1-\frac{1}{m},m}(\Omega)}. \end{aligned} \tag{2.4}$$

We note that $\|v_0\|_{D_l^{1-\frac{1}{m},m}(\Omega)} \le \|v_0\|_{W^{1,l}(\Omega)}$ because

$$D_l^{1-\frac{1}{m},m}(\Omega) := [L_l(\Omega), W^{1,l}((\Omega))]_{1-\frac{1}{m},m}$$

(see e.g., [1, Chapter 7]) and, therefore, $\|v_0\|_{D_l^{1-\frac{1}{m},m}(\Omega)}$ in (2.4) can be replaced by $\|v_0\|_{W^{1,l}(\Omega)}$.

The John-Nirenberg space or the space of the Bounded Mean Oscillation (in short BMO space) [22] consists of all functions f which are integrable on every ball $B_R(x) \subset \mathbb{R}^3$ and satisfy

$$\|f\|^2_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^3} \sup_{R>0} \frac{1}{B(x,R)} \int_{B(x,R)} |f(y) - f_{B_R}(y)| dy < \infty.$$

Here, f_{B_R} is the average of f over all ball $B_R(x)$ in $\mathbb{R}^3$. Next we recall a Gagliardo-Nirenberg inequality using BMO-norm (See [8, Theorem 2.3] and [21, Theorem 2.2]).

Lemma 2.3. *Suppose that $1 \le p < r < \infty$ and $f \in L^p(\Omega) \cap \mathrm{BMO}(\Omega)$. Then there exists a constant $C = C(n,p,r,\Omega)$ such that*

$$\|f\|_{L^r(\Omega)} \le C\|f\|^{p/r}_{L^p(\Omega)} \|f\|^{1-\frac{p}{r}}_{\mathrm{BMO}(\Omega)}.$$

Also, we recall estimates with respect to smooth vector field under the slip boundary condition. (See [4, Lemma 2.2],[5, Theorem 2.1] and [6, Lemma 2.1-2.2]).

Lemma 2.4. *Let Ω be a smooth domain in $\mathbb{R}^3$. Then, for each $q > 1$, regular smooth vector fields f,*

(a)

$$\begin{aligned} -\int_\Omega \Delta f \cdot f|f|^{q-2} dx &= \frac{1}{2}\int_\Omega |f|^{q-2}|\nabla f|^2 dx + \frac{4(q-2)}{q^2}\int_\Omega \big||f|^{q/2}\big|^2 dx \\ &\quad - \int_{\partial\Omega} |f|^{q-2}(n\cdot\nabla) f \cdot f dS. \end{aligned}$$

(b) *Moreover, using the vector identity,*

$$(n\cdot\nabla)f\cdot f=(f\cdot\nabla)f\cdot n+((\nabla\times f)\times n)\cdot f,$$

we deduce that

$$\begin{aligned}&-\int_\Omega \Delta f\cdot f|f|^{q-2}dx\\&=\frac12\int_\Omega |f|^{q-2}|\nabla f|^2dx+\frac{4(q-2)}{p^2}\int_\Omega \big|\nabla|f|^{q/2}\big|^2dx\\&\quad-\int_{\partial\Omega}|f|^{p-2}(f\cdot\nabla)f\cdot ndS-\int_{\partial\Omega}|f|^{p-2}((\nabla\times f)\times n)\cdot fdS.\end{aligned}$$

Lemma 2.5. *Assume that u is a regular enough satisfying the boundary condition (1.4) on $\partial\Omega$. Then, the for $w=\nabla\times u$ we have*

$$-\frac{\partial w}{\partial n}\cdot w=(\epsilon_{1jk}\epsilon_{1\beta\gamma}+\epsilon_{2jk}\epsilon_{2\beta\gamma}+\epsilon_{3jk}\epsilon_{3\beta\gamma})w_jw_\beta\partial_kn_\gamma\ \ on\partial\Omega,$$

where ϵ_{ijk} denotes the totally anti-symmetric tensor such that $(a\times b)=\epsilon_{1jk}a_jb_k$. In particular,

$$\int_\Omega \Delta w\cdot wdx\le-\int_\Omega|\nabla w|^2dx+C\int_{\partial\Omega}|w|^2dx.$$

3. Proof of main results

Proof of Theorem 1.2. Following the argument in [18, 19], it is sufficient to show the L^4-estimate of u. Suppose that T^* be the first time of singularity. Then u must satisfies for any $\delta>0$,

$$\begin{aligned}\lim_{t\nearrow T^*}\Big(\|u(\cdot,t)\|_{L^4}^4+\int_{T^*-\delta}^t\Big(\big\||\nabla u(\cdot,\tau)|\,|u(\cdot,\tau)|\big\|_{L^2}^2\\+\big\|\nabla|u(\cdot,\tau)|^2\big\|_{L^2}^2\Big)d\tau\Big)=\infty.\end{aligned}\tag{3.1}$$

In the proof, we consider only the boundary condition (1.4), since the case of (1.3) is much simpler. Multiplying the first equation of (1.1) with $|u|^2u$, and integrating over Ω, we have

$$\begin{aligned}&\frac14\frac{d}{dt}\int_\Omega|u|^4+\int_\Omega|\nabla u|^2|u|^2+\frac12\int_\Omega|\nabla|u|^2|^2\\&=-\int_\Omega\nabla\pi|u|^2u+\sum_{i,j=1}^3\int_{\partial\Omega}u_{j,x_i}u_j|u|^2n_i,\end{aligned}\tag{3.2}$$

where we used integration by parts, divergence-free conditions of u and trace theorem. Let ϵ be a sufficiently small positive number, which will be specified later. Integrating (3.2) in time over $(T^*-\epsilon,\tau)$ for any τ with $T^*-\epsilon<\tau<T^*$, we observe that

$$\begin{aligned}&\frac14\int_\Omega|u(\cdot,\tau)|^4dx-\frac14\int_\Omega|u(\cdot,T^*-\epsilon)|^4dx\\&+\int_{T^*-\epsilon}^\tau\int_\Omega|\nabla u|^2|u|^2\,dx\,dt+\frac12\int_{T^*-\epsilon}^\tau\int_\Omega\big|\nabla|u|^2\big|^2\,dx\,dt\\&\le\int_{T^*-\epsilon}^\tau\int_\Omega|\nabla\pi||u|^2|u|\,dx\,dt+\int_{T^*-\epsilon}^\tau\int_\Omega|u|^3|\nabla u|\,dx\,dt:=I_1+I_2.\end{aligned}\tag{3.3}$$

For convenience, we denote $Q_\tau := \Omega \times (T^* - \epsilon, \tau)$. Using Hölder's inequality, the first term I_1 can be estimated as follows:

$$\begin{aligned} I_1 &\le \int_{T^*-\epsilon}^{\tau} \|\nabla \pi\|_{L^2} \|u\|_{L^6}^3 \le C \int_{T^*-\epsilon}^{\tau} \|\nabla \pi\|_{L^2} \|u\|_{L^4}^2 \|u\|_{\mathrm{BMO}} \\ &\le C \|\nabla \pi\|_{L^2(Q_\tau)} \|u\|_{L^2(0,\tau;\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2, \end{aligned}$$

For convenience of computations, we denote $\mathcal{C}_\epsilon := \|u(\cdot, T^* - \epsilon)\|_{W^{1,2}(\Omega)}$. Using the estimate (2.4), we continue to estimate I_1 as

$$\begin{aligned} I_1 &\le C\Big(\|(u\cdot\nabla)u\|_{L^2(Q_\tau)} + \mathcal{C}_\epsilon\Big) \|u\|_{L^2((T^*-\epsilon,\tau);\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2 \\ &\le C\|(u\cdot\nabla)u\|_{L^2(Q_\tau)} \|u\|_{L^2((T^*-\epsilon,\tau);\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2 \\ &\quad + C\mathcal{C}_\epsilon \|u\|_{L^2(0,\tau;\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2. \end{aligned} \tag{3.4}$$

On the other hand, by direct calculations, I_2 is bounded by

$$C\epsilon^{1/2} \big\||u||\nabla u|\big\|_{L^2(Q_\tau)} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2.$$

Summing the estimates of I_1 and I_2 with using Young's inequality, we obtain

$$\begin{aligned} &\frac{1}{4}\int_\Omega |u(\cdot,\tau)|^4 dx - \frac{1}{4}\int_\Omega |u(\cdot,T^*-\epsilon)|^4 dx \\ &+ \int_{T^*-\epsilon}^{\tau}\int_\Omega |\nabla u|^2 |u|^2 \,dx\,dt + \frac{1}{2}\int_{T^*-\epsilon}^{\tau}\int_\Omega \big|\nabla |u|^2\big|^2 \,dx\,dt \\ &\le C\|\,|u||\nabla u|\,\|_{L^2(Q_\tau)} \|u\|_{L^2(0,\tau;\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2 \\ &\quad + CC_\epsilon \|u\|_{L^2(0,\tau;\mathrm{BMO}(\Omega))} \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^2 \\ &\le \frac{1}{2}\|\,|u||\nabla u|\,\|_{L^2(Q_\tau)}^2 + CC_\epsilon^2 + C(\|u\|_{L^2(0,\tau;\mathrm{BMO}(\Omega))}^2 + \epsilon) \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^4. \end{aligned}$$

Since the above estimate holds for all t with $T^* - \epsilon < t < \tau$, we obtain

$$\begin{aligned} &\sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^4 + \int_{T^*-\epsilon}^{\tau}\int_\Omega |\nabla u|^2 |u|^2| \,dx\,dt + \frac{1}{2}\int_{T^*-\epsilon}^{\tau}\int_\Omega \big|\nabla |u|^2\big|^2 \,dx\,dt \\ &\le \int_\Omega |u(\cdot,T^*-\epsilon)|^4 dx + CC_\epsilon^2 \\ &\quad + C\Big(\|u\|_{L^2((T^*-\epsilon,\tau);\mathrm{BMO}(\Omega))}^2 + \epsilon\Big) \sup_{T^*-\epsilon<t<\tau} \|u(\cdot,t)\|_{L^4}^4. \end{aligned}$$

With sufficiently small ϵ so that $\big(\|u\|_{L^2((T^*-\epsilon,\tau);\mathrm{BMO}(\Omega))}^2 + \epsilon\big) \le \frac{1}{2C}$ with a constant $C > 0$ in the above estimate, we have

$$\begin{aligned} &\|u(\cdot,t)\|_{L^{4,\infty}_{x,t}(Q_\tau)}^4 + \frac{1}{2}\|\,|\nabla u||u|\,\|_{L^2(Q_\tau)}^2 + \frac{1}{2}\|\nabla |u|^2\|_{L^2(Q_\tau)}^2 \\ &\le 2\|u(\cdot, T-\epsilon)\|_{L^4_x(\Omega)}^4 + CC_\epsilon^2. \end{aligned}$$

For simplicity, we denote $Q_\epsilon = \Omega \times (T^* - \epsilon, T^*)$. Since τ is arbitrary with $\tau < T^*$, we obtain

$$\|u(\cdot,t)\|_{L^{4,\infty}_{x,t}(Q_\epsilon)}^4 + \frac{1}{2}\big\||\nabla u||u|\big\|_{L^2(Q_\epsilon)}^2 + \frac{1}{2}\|\nabla |u|^2\|_{L^2(Q_\epsilon)}^2 \le C,$$

where C is a constant depending on $\|u(\cdot, T^*-\epsilon)\|_{W^{1,2}(\Omega)}$. This is contrary to the hypothesis of (3.1). Therefore, T^* cannot be a maximal time of existence less than or equal to T. This completes the proof. $\square$

Proof of Theorem 1.3. First, we consider the vorticity equation

$$w_t - \Delta w + (u\cdot\nabla)w - (u\cdot\nabla)w = 0. \tag{3.5}$$

Multiplying the first equation of (3.5) by w, integrating over Ω, and adding them, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2 + \int_\Omega |\nabla w|^2 \le \int_\Omega |w||\nabla u||w| + \int_{\partial\Omega} \big|\frac{\partial w}{\partial n}\cdot w\big| := II_1 + II_2,$$

where we use Lemmas 2.4 and 2.5. Using Hölder inequality and Lemma 2.3, the term II_1 is estimated as follows:

$$II_1 \le \|\nabla u\|_{L^3(\Omega)}\|w\|^2_{L^3(\Omega)} \le C\|w\|^3_{L^3(\Omega)} \le C\|w\|^2_{L^2(\Omega)}\|w\|_{\mathrm{BMO}(\Omega)},$$

Next, we can easily estimate II_2. Indeed, we use the Trace theorem (see e.g., [10, pp 257-258]) and smoothness of boundary to find

$$II_2 \le \int_{\partial\Omega}\big|\frac{\partial w}{\partial n}\cdot w\big| \le C\int_\Omega |w|^2,$$

Summing the estimates II_1 and II_2, we obtain

$$\frac{d}{dt}\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} \le C(1+\|w\|_{\mathrm{BMO}(\Omega)})\|w\|^2_{L^2}. \tag{3.6}$$

Applying the Gronwall's inequality to (3.6),

$$\sup_{0<t<T}\|w(t)\|^2_{L^2} + \int_0^T \|\nabla w\|^2_{L^2} \le C\|w_0\|^2_{L^2},$$

which is the desired result. $\square$

Proof of Theorem 1.4. First, we note that, without loss of generality, the mean value of the pressure π is assumed to be zero, namely $\int_\Omega \pi(\cdot,t)dx = 0$ for each time $t\in[0,T)$. We get π satisfies

$$\|\pi\|_{L^2(\Omega)} \le C\|\nabla\pi\|_{L^2(\Omega)}.$$

The proof of Theorem 1.4 is similar to that of Theorem 1.2. Indeed, from (3.3), we note that

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |u(\cdot,\tau)|^4 dx - \frac{1}{4}\int_\Omega |u(\cdot,T^*-\epsilon)|^4 dx\\
&+\int_{T^*-\epsilon}^\tau\int_\Omega |\nabla u|^2|u|^2\,dx\,dt + \frac{1}{2}\int_{T^*-\epsilon}^\tau\int_\Omega \big|\nabla|u|^2\big|^2\,dx\,dt\\
&\le \int_{T^*-\epsilon}^\tau\int_\Omega |\pi|\,|u|\,|u||\nabla u|\ dx\,dt + \int_{T^*-\epsilon}^\tau\int_\Omega |u|^3|\nabla u|\,dx\,dt := III_1 + III_2.
\end{aligned}$$

Using Hölder's inequality, the first term III_1 can be estimated as

$$\begin{aligned}
III_1 &\le \int_{T^*-\epsilon}^\tau \|\pi\|_{L^4}\|u\|_{L^4}\big\||u||\nabla u|\big\|_{L^2} \le C\int_{T^*-\epsilon}^\tau \|\pi\|_{L^4}\|u\|_{L^4}\big\||u||\nabla u|\big\|_{L^2}\\
&\le \int_{T^*-\epsilon}^\tau C\|\pi\|^{1/2}_{L^2}\|\pi\|^{1/2}_{\mathrm{BMO}}\|u\|_{L^4}\big\||u||\nabla u|\big\|_{L^2}
\end{aligned}$$

$$\le C\|\nabla\pi\|^{1/2}_{L^2(Q_\tau)}\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}$$

For convenience of computations, we denote $\mathcal{C}_\epsilon := \|u(\cdot,T^*-\epsilon)\|_{W^{1,2}(\Omega)}$. Using the estimate (2.4), we obtain

$$\begin{aligned}
III_1 &\le C\Big(\|(u\cdot\nabla)u\|^{1/2}_{L^2(Q_\tau)}+\mathcal{C}_\epsilon\Big)\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)},\\
&\le C\|(u\cdot\nabla)u\|^{1/2}_{L^2(Q_\tau)}\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\\
&\quad + C\mathcal{C}_\epsilon\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}.
\end{aligned}\tag{3.7}$$

Following similar computations as in I_2, we obtain

$$III_2 \le C\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|^2_{L^4}.\tag{3.8}$$

Summing (3.7)-(3.8) and using Young's inequality, we obtain

$$\begin{aligned}
&\frac14\int_\Omega|u(\cdot,\tau)|^4dx-\frac14\int_\Omega|u(\cdot,T^*-\epsilon)|^4dx\\
&+\int_{T^*-\epsilon}^{\tau}\int_\Omega|\nabla u|^2|u|^2\,dx\,dt+\frac12\int_{T^*-\epsilon}^{\tau}\int_\Omega\big|\nabla|u|^2\big|^2\,dx\,dt\\
&\le C\|(u\cdot\nabla)u\|^{1/2}_{L^2(Q_\tau)}\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\\
&\quad+C\mathcal{C}_\epsilon\Big(\int_{T^*-\epsilon}^{\tau}\|\pi\|^2_{\mathrm{BMO}}\|u\|^4_{L^4}\,dt\Big)^{1/4}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\\
&\quad+C\epsilon^{\frac12}\big||u||\nabla u|\big|_{L^2(Q_\tau)}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|^2_{L^4}\\
&\le\frac12\big\||u||\nabla u|\big\|^2_{L^2(Q_\tau)}+C\mathcal{C}^4_\epsilon+C\Big[\int_{T^*-\epsilon}^{\tau}\|\pi(\cdot,t)\|^2_{\mathrm{BMO}}\,dt+\epsilon\Big]\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|^4_{L^4}.
\end{aligned}$$

With sufficiently small ϵ so that

$$\Big(\int_{T^*-\epsilon}^{\tau}\|\pi(\cdot,t)\|^2_{\mathrm{BMO}}\,dt+\epsilon\Big)\le\frac{1}{2C}$$

with a constant C in the above estimate, we have

$$\begin{aligned}
&\|u(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\tau)}+\frac12\|\,|\nabla u||u|\|^2_{L^2(Q_\tau)}+\frac12\,\|\nabla|u|^2\|^2_{L^2(Q_\tau)}\\
&\le 2\|u(\cdot,T-\epsilon)\|^4_{L^4_x(\Omega)}+CC^4_\epsilon.
\end{aligned}$$

By the same argument as in the proof of Theorem 1.2, we finally obtain

$$\|u(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\epsilon)}+\frac12\big\||\nabla u||u|\big\|^2_{L^2(Q_\epsilon)}+\frac12\|\nabla|u|^2\|^2_{L^2(Q_\epsilon)}\le C,$$

where C is a constant depending on $\|u(\cdot,T^*-\epsilon)\|_{W^{1,2}(\Omega)}$. □

Proof of Theorem 1.5. This proof is similar to that of Theorem 1.2. Indeed, from From (3.3), we note that

$$\frac14\int_\Omega|u(\cdot,\tau)|^4dx-\frac14\int_\Omega|u(\cdot,T^*-\epsilon)|^4dx$$

$$
\begin{aligned}
&+\int_{T^*-\epsilon}^{\tau}\int_{\Omega}|\nabla u|^2|u|^2\,dx\,dt+\frac{1}{2}\int_{T^*-\epsilon}^{\tau}\int_{\Omega}\big|\nabla|u|^2\big|^2\,dx\,dt\\
&\le\int_{T^*-\epsilon}^{\tau}\int_{\Omega}|\nabla\pi|\,|u|^2|u|\,dx\,dt+\int_{T^*-\epsilon}^{\tau}\int_{\Omega}|u|^3|\nabla u|\,dx\,dt:=IV_1+IV_2.
\end{aligned}
$$

Using Hölder's inequality, the first term IV_1 can be estimated as

$$
\begin{aligned}
IV_1&\le\int_{T^*-\epsilon}^{\tau}\|\nabla\pi\|_{L^4}\|u\|_{L^4}^3\le C\int_{T^*-\epsilon}^{\tau}\|\nabla\pi\|_{L^2}^{1/2}\|\nabla\pi\|_{\mathrm{BMO}}^{1/2}\|u\|_{L^4}^3\\
&\le C\|\nabla\pi\|_{L^2(Q_\tau)}^{1/2}\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3,
\end{aligned}
$$

For convenience of computations, we denote $\mathcal{C}_\epsilon:=\|u(\cdot,T^*-\epsilon)\|_{W^{1,2}(\Omega)}$. Using the estimate (2.4), we obtain

$$
\begin{aligned}
IV_1&\le C\Big(\|(u\cdot\nabla)u\|_{L^2(Q_\tau)}^{1/2}+\mathcal{C}_\epsilon\Big)\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\\
&\quad\times\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3,\\
&\le C\|(u\cdot\nabla)u\|_{L^2(Q_\tau)}^{1/2}\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3\\
&\quad+C\mathcal{C}_\epsilon\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3.
\end{aligned}\tag{3.9}
$$

Following similar computations as in I_2, we obtain

$$
IV_2\le C\epsilon^{1/2}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^2.\tag{3.10}
$$

Summing (3.7)-(3.8) and using Young's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{4}\int_{\Omega}|u(\cdot,\tau)|^4dx-\frac{1}{4}\int_{\Omega}|u(\cdot,T^*-\epsilon)|^4dx\\
&+\int_{T^*-\epsilon}^{\tau}\int_{\Omega}|\nabla u|^2|u|^2\,dx\,dt+\frac{1}{2}\int_{T^*-\epsilon}^{\tau}\int_{\Omega}\big|\nabla|u|^2\big|^2\,dx\,dt\\
&\le C\|(u\cdot\nabla)u\|_{L^2(Q_\tau)}^{1/2}\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3\\
&\quad+C\mathcal{C}_\epsilon\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt\Big)^{3/4}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^3\\
&\quad+C\epsilon^{1/2}\big\||u||\nabla u|\big\|_{L^2(Q_\tau)}\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^2\\
&\le\frac{1}{2}\|\,|u||\nabla u|\,\|_{L^2(Q_\tau)}^2+CC_\epsilon^4+C\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt+\epsilon\Big)\\
&\quad\times\sup_{T^*-\epsilon<t<\tau}\|u(\cdot,t)\|_{L^4}^4.
\end{aligned}
$$

With sufficiently small ϵ so that

$$
\Big(\int_{T^*-\epsilon}^{\tau}\|\nabla\pi(\cdot,t)\|_{\mathrm{BMO}}^{2/3}\,dt+\epsilon\Big)\le\frac{1}{2C}
$$

with a constant C in the above estimate, we have

$$
\|u(\cdot,t)\|_{L^{4,\infty}_{x,t}(Q_\tau)}^4+\frac{1}{2}\|\,|\nabla u||u|\,\|_{L^2(Q_\tau)}^2+\frac{1}{2}\|\nabla|u|^2\|_{L^2(Q_\tau)}^2
$$

$$\leq 2\|u(\cdot, T-\epsilon)\|^4_{L^4_x(\Omega)} + CC^4_\epsilon.$$

By the same argument from the proof of Theorem 1.2, we finally obtain the desired result. □

Remark 3.1. The arguments of Theorems 1.2–1.4 also hold for a whole space $\mathbb{R}^n$ because Lemma 2.2 also established for these cases.

Acknowledgments. I would like to thank the anonymous referee for the useful comments. Jae-Myoung Kim was supported by grants NRF-20151009350 and NRF-2016R1D1A1B03930422.

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