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MEROMORPHIC SOLUTIONS TO NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. We consider the non-linear differential-difference equation

$$c(z)w(z+1)+a(z)\frac{w'(z)}{w(z)}=R(z,w(z)),$$

where $R(z,w(z))$ is rational in $w(z)$ with rational coefficients, $a(z)$ and $c(z)$ are non-zero rational functions. We give necessary conditions on the degree of $R(z,w)$ for the above equation to admit a transcendental meromorphic solution of hyper-order $\rho_2(w)<1$. We also consider the admissible rational solutions of the above equation.

1. Introduction

Ablowitz, Halburd and Herbst [1] considered the existence of sufficiently many finite order meromorphic solutions of a difference equation, which could be viewed as a good difference analogue of the Painlevé property for complex difference equations. It is a landmark on the applications of Nevanlinna theory in the studies of complex difference equations. Recently, it has become an important topic to consider complex difference equations and the properties of meromorphic solutions using Nevanlinna theory. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [9]. A function $a(z)(\not\equiv 0,\infty)$ is called a small function with respect to $w(z)$, if $T(r,a)=S(r,w)$, where $S(r,w)$ denotes any quantity satisfying $S(r,w)=o(T(r,w))$ with $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. For a meromorphic function $w(z)$, the order of w is defined by

$$\rho(w)=\limsup_{r\to\infty}\frac{\log T(r,w)}{\log r}$$

and the hyper-order is defined by

$$\rho_2(w)=\limsup_{r\to\infty}\frac{\log\log T(r,w)}{\log r}.$$

Halburd and Korhonen [5] singled out a list of possible equations of the form

$$w(z+1)+w(z-1)=R(z,w(z)), \tag{1.1}$$

where $R(z,w)$ is rational in $w(z)$ with meromorphic coefficients in z, provided that $w(z)$ is assumed to have finite order but grow faster than the coefficients. It was

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later proved in [6] that the same result is also valid by replacing the assumption finite order with the hyper-order less than one.

In [7], if the difference equation

$$w(z+1)w(z-1) = R(z, w(z)) \tag{1.2}$$

has an admissible meromorphic solution of finite order, then (1.2) can be transformed by Möbius transformation in w to a list of equations, in which the difference Painlevé III equation is included, unless $w(z)$ is a solution of difference Riccati equations.

Recently, Halburd and Korhonen [8, Theorem 1.1] considered the properties of meromorphic solutions on non-linear differential-difference (delay-differential) equation

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1.3}$$

where $a(z)$ is a rational function, $P(z, w)$ is a polynomial in w having rational coefficients in z, $Q(z, w)$ is a monic polynomial in w with the roots that are non-zero rational functions of z and not the roots of $P(z, w)$. They obtained the following theorem.

Theorem 1.1. *Let $w(z)$ be a non-rational solution of* (1.3)*. If $\rho_2(w) < 1$, then*

$$\deg_w(P) = \deg_w(Q) + 1 \le 3$$

or the degree of $R(z, w)$ as a rational function in w is either 0 or 1.

A natural question to ask is what happens if (1.3) is more general, for example

$$c(z)w(z+1) - b(z)w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1.4}$$

where $a(z)$, $b(z)$, $c(z)$ are rational functions. It seems that there is no difficulty to obtain the same result as Theorem 1.1 if $b(z)$, $c(z)$ are non-zero rational functions using the same method as in [8]. However, if one of $b(z), c(z)$ vanishes, the situation is different. In the paper, we will consider this case as the supplement of Theorem 1.1 and give the details of the proof, the idea is similar as in [8]. Without loss of generality, we assume $b(z) \equiv 0$, then (1.4) reduces to

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1.5}$$

here $P(z, w)$ and $Q(z, w)$ also satisfy the conditions above Theorem 1.1. We state our result as follows.

Theorem 1.2. *Let $w(z)$ be a non-rational meromorphic solution of* (1.5)*. If $\rho_2(w) < 1$, then*

$$\deg_w(P) = \deg_w(Q) + 1 = 2 \tag{1.6}$$

or the degree of $R(z, w)$ as a rational function in w is either 0 or 1.

Corollary 1.3. *If $w(z)$ is a transcendental entire solution of* (1.5) *with $\rho_2(w) < 1$, then $\deg_w(P) = 1$ and $\deg_w(Q) = 0$ holds.*

From Theorem 1.2, we see that if $\deg_w(Q) = 1$, that is $Q(z, w) = w(z) - b(z)$ where $b(z) \not\equiv 0$, thus (1.5) implies that $w(z)$ and $w(z) - b(z)$ have finitely many zeros, which is impossible for $w(z)$ is an entire function, thus we get Corollary 1.3. The following example shows that the assertion (1.6) holds.

Example 1.4. The meromorphic function $w(z) = \frac{1}{e^z+1}$ solves

$$w(z+1) + \frac{w'(z)}{w(z)} = \frac{(1-e)w(z)^2 + 2ew(z) - e}{(1-e)w(z) + e}.$$

Here, $\deg_w(P) = \deg_w(Q) + 1 = 2$.

The following two examples show that the case of $\deg_w(R) = 1$ happens. Example 1.5 also shows that Corollary 1.3 occurs.

Example 1.5. The entire function $w(z) = e^z$ solves

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = ec(z)w(z) + a(z),$$

which implies that $\deg_w(R) = 1$.

Example 1.6. The meromorphic function $w(z) = \tan\frac{\pi}{2}z$ solves

$$w(z+1) + \frac{2}{\pi}\frac{w'(z)}{w(z)} = w(z),$$

which implies that $\deg_w(R) = 1$.

If $\deg_w(R) = 0$, that is $R(z, w(z))$ does not depend on $w(z)$ in (1.5), then (1.5) becomes

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = b(z), \tag{1.7}$$

where $a(z)$, $b(z)$ and $c(z)$ are rational functions. We obtain the following result.

Theorem 1.7. *The equation* (1.7) *has no transcendental entire solutions. If $w(z)$ is a transcendental meromorphic solution with finite order of* (1.7)*, then $\lambda(w) = \rho(w)$.*

The following example shows that the case of $\deg_w(R) = 0$ happens in Theorem 1.2 and $\lambda(w) = \rho(w)$ occurs in Theorem 1.7.

Example 1.8. The meromorphic function $w(z) = \frac{1}{e^{2i\pi z}+1}$ solves

$$w(z+1) - \frac{1}{2i\pi}\frac{w'(z)}{w(z)} = 1.$$

We continue considering the case that (1.5) admits a rational solution where all the coefficients of (1.5) are constants. There are several cases for different degrees on $P(z,w)$ and $Q(z,w)$, we mainly consider two of them and obtain the following result. It is not difficult to discuss other cases to obtain the relationship between m, n using the similar method in the proof of Theorem 1.9.

Theorem 1.9. *Let $w(z) = M(z)/N(z)$ be a non-constant rational solution of* (1.5)*, $M(z)$ and $N(z)$ be polynomials as follows*

$$M(z) = a_m z^m + \dots + a_1 z + a_0, \quad N(z) = b_n z^n + \dots + b_1 z + b_0,$$

and let $h := P(z,0)$, $g := P'(z,0)$ and $e := Q(z,0)$ in (1.5)*. We have*

1. *if $\deg_w(P) = 0$, then $m = n$;*
2. *if $\deg_w(P) = 3$ and $\deg_w(Q) = 2$, then $m \geq n$ except that $n = m+1$ provided that $h = 0$ $e \neq 0$ and $g \neq 0$.*

Example 1.10. The rational function $w(z) = 1/z$ solves the equation

$$cw(z+1) + a\frac{w'(z)}{w(z)} = \frac{w^2(z) + 2w(z)}{w(z)+1}$$

with $c = 1$, $a = -1$. In this case, we know that $\deg_w(P) = 2$,$\deg_w(Q) = 1$, $h = 0$, and $m = 0$, $n = 1$, which implies $\deg_w(P) = \deg_w(Q)+1$, $n = m+1$. The exception case in Case (2) happens.

2. Preliminaries

Lemma 2.1 ([6, Lemma 8.3])**.** *Let $T : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1-\rho_2)$, then

$$T(r+s) = T(r) + o\Big(\frac{T(r)}{r^\delta}\Big),$$

where r runs to infinity outside of a set of finite logarithmic measure.

The Valiron-Mohon'ko identity [11, 12] is a useful tool to estimate the characteristic function of rational functions, the proof can be found easily in [10, Theorem 2.2.5].

Lemma 2.2. *Let w be a meromorphic function and $R(z,w)$ be a rational function in w and meromorphic in z. If all the coefficients of $R(z,w)$ are small compared to w, then*

$$T(r, R(z,w)) = \deg_w(R)T(r,w) + S(r,w).$$

Difference analogue lemma on the logarithmic derivative for meromorphic functions of finite order was established by Halburd and Korhonen [3, 4], and Chiang and Feng [2], independently. Let us recall the version as follows.

Lemma 2.3 ([6, Theorem 5.1])**.** *Let w be a non-constant meromorphic function and $c \in \mathbb{C}$. If w is of finite order, then*

$$m\Big(r, \frac{w(z+c)}{w(z)}\Big) = O\Big(\frac{\log r}{r} T(r,w)\Big)$$

for all r outside of a set E satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\rho_2(w) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\Big(r, \frac{w(z+c)}{w(z)}\Big) = o\Big(\frac{T(r,w)}{r^{1-\rho_2-\varepsilon}}\Big)$$

for all r outside of a set of finite logarithmic measure.

The following lemma, related to the value distribution of meromorphic solutions of a large class of differential-difference equations, is important in this article. A differential-difference polynomial in $w(z)$ is defined by

$$P(z,w)=\sum_{l\in \mathrm{L}} b_l(z)w(z)^{l_{0,0}}w(z+c_1)^{l_{1,0}}\dots w(z+c_\nu)^{l_{\nu,0}}w'(z)^{l_{0,1}}\dots w^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}}$$

where $c_1,\dots,c_\nu$ are distinct complex constants, L is a finite index set consisting of elements of the form $l=(l_{0,0},\dots,l_{\nu,\mu})$ and the coefficients $b_l(z)$ are rational functions of z for all $l\in \mathrm{L}$.

Lemma 2.4 ([8, Lemma 2.1])**.** *Let $w(z)$ be a non-rational meromorphic solution of*

$$P(z,w)=0 \tag{2.1}$$

where $P(z,w)$ is differential-difference polynomial in $w(z)$ with rational coefficients, and let $a_1,\dots,a_k$ be rational functions. If the following two conditions

1. *$P(z,a_j)\not\equiv 0$ for all $j\in\{1,\dots,k\}$;*
2. *there exist $s>0$ and $\tau\in(0,1)$ such that*

$$\sum_{j=1}^{k} n\Big(r,\frac{1}{w-a_j}\Big)\le k\tau n(r+s,w)+O(1), \tag{2.2}$$

are satisfied, then $\rho_2(w)\ge 1$.

3. Proof of Theorem 1.2

Before giving the details of the proof, we show the main idea. In the first step we prove that $\deg_w(P)\le 3$ and $\deg_w(Q)\le 3$ using Nevanlinna theory. In the second step we discuss four cases according to the numbers of the roots of $Q(z,w)$, where Lemma 2.4 plays an important part. Suppose that (1.5) has a non-rational meromorphic solution $w(z)$ with $\rho_2(w)<1$.

First step: Taking Nevanlinna characteristic function of both sides of (1.5) and applying Lemma 2.2, we have

$$T\left(r,c(z)w(z+1)+a(z)\frac{w'(z)}{w(z)}\right)=T(r,R(z,w))=\deg_w(R)T(r,w(z))+O(\log r).$$

Using Lemmas 2.1 and 2.3, and using the logarithmic derivative lemma, it follows that

$$\begin{aligned}\deg_w(R)T(r,w(z))&\le T(r,w(z+1))+T\Big(r,\frac{w'(z)}{w(z)}\Big)+O(\log r)\\&\le T(r,w)+N\Big(r,\frac{w'(z)}{w(z)}\Big)+S(r,w)\\&\le T(r,w)+\overline{N}(r,w(z))+\overline{N}\Big(r,\frac{1}{w(z)}\Big)+S(r,w)\\&\le 3T(r,w)+S(r,w).\end{aligned} \tag{3.1}$$

Therefore,

$$(\deg_w(R)-3)T(r,w(z))\le S(r,w), \tag{3.2}$$

which implies that $\deg_w R(z)\le 3$, i.e., $\deg_w(P)\le 3$ and $\deg_w(Q)\le 3$.

Second step: Case (1): If $Q(z,w(z))$ in (1.5) has at least two distinct non-zero rational roots for w, say $b_1(z)\not\equiv 0$ and $b_2(z)\not\equiv 0$, then (1.5) can be written as

$$c(z)w(z+1)+a(z)\frac{w'(z)}{w(z)}=\frac{P(z,w(z))}{(w(z)-b_1(z))(w(z)-b_2(z))\widetilde{Q}(z,w(z))}, \tag{3.3}$$

where $\deg_w(P)\le 3$ and $\deg_w(\widetilde{Q})\le 1$. Here, there exists the possibility that $\widetilde{Q}(z,b_1(z))\equiv 0$ or $\widetilde{Q}(z,b_2(z))\equiv 0$. We also assume that $P(z,w(z))$ and $\widetilde{Q}(z,w(z))$ do not have common roots. Obviously, neither $b_1(z)$ nor $b_2(z)$ is a solution of (3.3). Thus, the first condition of Lemma 2.4 is satisfied.

Assume that $\hat{z}\in\mathbb{C}$ is any point satisfying

$$w(\hat{z})=b_1(\hat{z}) \tag{3.4}$$

and such that none of the rational coefficients of (3.3) and their shifts have a zero or a pole at $\hat{z}$ and $P(\hat{z},w(\hat{z}))\neq 0$. Let p denote the order of the zero of $w-b_1$ at $z=\hat{z}$. Here, $\hat{z}$ is called a generic root of $w-b_1$ of order p. We will only consider generic roots, since the coefficients are rational, the contributions from the non-generic roots can be included in a bounded error term of the type $O(\log r)$. Next we discuss whether $z=\hat{z}$ is a zero or a pole of $w(z+n)$ or not.

It is easy to obtain that (3.3) implies that $w(z+1)$ has a pole at $z=\hat{z}$ of order at least p. Shifting forward (3.3), we have

$$\begin{aligned}
&c(z+1)w(z+2)+a(z+1)\frac{w'(z+1)}{w(z+1)}\\
&=\frac{P(z+1,w(z+1))}{(w(z+1)-b_1(z+1))(w(z+1)-b_2(z+1))\widetilde{Q}(z+1,w(z+1))}.
\end{aligned} \tag{3.5}$$

Subcase 1.1. Let

$$\deg_w(P)\le\deg_w(\widetilde{Q})+2. \tag{3.6}$$

Thus, $w(z+2)$ has a pole of order one at $z=\hat{z}$. Shifting forward (3.5) one more step, we have

$$\begin{aligned}
&c(z+2)w(z+3)+a(z+2)\frac{w'(z+2)}{w(z+2)}\\
&=\frac{P(z+2,w(z+2))}{(w(z+2)-b_1(z+2))(w(z+2)-b_2(z+2))\widetilde{Q}(z+2,w(z+2))}.
\end{aligned} \tag{3.7}$$

Then $w(z+3)$ also has a pole of order one at $z=\hat{z}$, $w(z+4)$ has a pole of order one at $z=\hat{z}$, and so on. Thus, in the iteration, we always can find a pole of multiplicity at least p which can be paired up with the root of $w-b_1$ at $z=\hat{z}$.

Using the same discussions for the roots of $w-b_2$ without any possible overlap in the pairing of poles with the zeros of $w-b_1$ and $w-b_2$. By adding up all points $\hat{z}$ such that (3.4) is valid, and similarly for $w(\hat{z})=b_2(\hat{z})$, it follows that

$$n\big(r,\frac{1}{w-b_1}\big)+n\big(r,\frac{1}{w-b_2}\big)\le n(r+4,w)+O(1). \tag{3.8}$$

Therefore the second condition (2.2) of Lemma 2.4 is satisfied, and so $\rho_2(w)\ge 1$, which is a contradiction with $\rho_2(w)<1$.

Subcase 1.2: Let

$$\deg_w(P)>\deg_w(\widetilde{Q})+2.$$

Since $\deg_w(P) \le 3$ and $\deg_w(\widetilde{Q}) \le 1$, the only possibility when the inequality above holds is

$$\deg_w(P) = 3, \quad \deg_w(\widetilde{Q}) = 0.$$

In this case, we suppose again that $\hat{z}$ is a generic root of $w - b_1$ of order p. As before, it follows by (3.3) that $w(z+1)$ has a pole of order at least p at $z = \hat{z}$. If $p > 1$, by (3.5), then $w(z+2)$ has a pole of order at least p at $z = \hat{z}$, which implies $w(z+3)$ also has a pole of order at least p at $z = \hat{z}$, and so on. Identical reasoning holds also for the roots of $w - b_2$. Hence, in this case, we have

$$n\big(r, \frac{1}{w-b_1}\big) + n\big(r, \frac{1}{w-b_2}\big) \le \frac{1}{3} n(r+3, w) + O(1).$$

Lemma 2.4 therefore reads that $\rho_2(w) \ge 1$, which is a contradiction with $\rho_2(w) < 1$.

However, if $p = 1$, it may in principle be possible that the pole of the right hand of (3.5) at $z = \hat{z}$ cancels with the pole of the term

$$a(z+1)\frac{w'(z+1)}{w(z+1)}$$

at $z = \hat{z}$ in such a way that $c(\hat{z}+1)w(\hat{z}+2)$ remains finite. By the assumption that none of the rational coefficients of (3.3) and their shifts have a zero or a pole at $\hat{z}$, it yields three possible cases as follows:

(a) $w(\hat{z}+2) = 0$;
(b) $w(\hat{z}+2) \ne 0$ and $w(\hat{z}+2) \ne b_j(\hat{z}+2)$, $j \in \{1, 2\}$;
(c) $w(\hat{z}+2) = b_j(\hat{z}+2)$, $j \in \{1, 2\}$.

If the case (a) is valid, then by (3.7), it yields that $w(z)$ has a pole of order one at $\hat{z}+3$, which implies that $w(z)$ has a pole of order one at $\hat{z}+4$ or $w(\hat{z}+4)$ also is finite. Thus the following iteration is the same as before. In fact, it is a cyclic iteration. If the case (b) is valid, we obtain that $w(\hat{z}+3)$ is finite, which implies that the following iteration may be similar to the iteration from point $\hat{z}$ to $\hat{z}+3$. For the case (c), by (3.7), it follows that $w(z)$ has a pole at $z = \hat{z}+3$. Therefore, we can find a pole at least of order $p = 1$ which can be associated with the zero of $w - b_1$ at $z = \hat{z}$. By adding up all roots of $w - b_1$ and $w - b_2$, we still have the inequality

$$n\big(r, \frac{1}{w-b_1}\big) + n\big(r, \frac{1}{w-b_2}\big) \le n(r+3, w) + O(1).$$

Hence, the second condition of Lemma 2.4 is satisfied again, which yields that $\rho_2(w) \ge 1$.

Case (2): Suppose that $Q(z, w(z))$ in (1.5) has at least one non-zero rational root, say $b_1(z) \not\equiv 0$, then (1.5) can be written as

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{(w(z) - b_1(z))^n \widehat{Q}(z, w(z))}, \tag{3.9}$$

where $\deg_w(P) \le 3$ and $n + l \le 3$, $\deg_w(\widehat{Q}) = l$. Note that l may in principle be zero. Then $b_1(z)$ is not a solution of (3.9), and thus the first condition of Lemma 2.4 is satisfied for $b_1(z)$. Assume that $n \in \{2, 3\}$, and $\hat{z}$ is a generic root of $w - b_1$ of order p.

Subcase 2.1. Let

$$\deg_w(P) \le n + l. \tag{3.10}$$

Then $\hat{z}+1$ is a pole of $w(z)$ of order at least np and $\hat{z}+2$ is a pole of $w(z)$ of order one. By continuing the iteration, it follows that $\hat{z}+3$ is a pole of $w(z)$ of order one, and so on. In this case, we therefore have

$$n\big(r, \frac{1}{w-b_1}\big) \le \frac{1}{n} n(r+3, w) + O(1).$$

The second condition of Lemma 2.4 is satisfied, thus $\rho_2(w) \ge 1$ holds.

Subcase 2.2. Let

$$\deg_w(P) \ge n+l+1.$$

The case $n=1, l=1$ means that $Q(z,w)$ has at least two zeros which has been considered in Case (1). We consider now that $\deg_w(P) = 3$ and $n=2, l=0$. Suppose once more that $\hat{z}$ is a generic root of $w-b_1$ of order p. Similar as before, $\hat{z}+1$ is a pole of $w(z)$ of order $2p$. Shifting forward (3.9), it follows that $\hat{z}+2$ is a pole of order $2p$ of $w(z)$ and $\hat{z}+3$ is also a pole of order $2p$ of $w(z)$. In this case, we have found at least $6p$ poles, taking into account multiplicities, which can be paired up with p roots of $w-b_1$. We can go through all roots of $w-b_1$ in this way. Thus

$$n\big(r, \frac{1}{w-b_1}\big) \le \frac{1}{6} n(r+3, w) + O(1).$$

Lemma 2.4 implies that $\rho_2(w) \ge 1$.

Case (3): Suppose now that $Q(z, w(z))$ in (1.5) has only one simple root, say $b_1(z) \not\equiv 0$. Then (1.5) can be written as

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{w(z) - b_1(z)}. \tag{3.11}$$

Subcase 3.1. Assume that

$$\deg_w(P) = 3.$$

Let $\hat{z}$ be a generic root of $w(z) - b_1(z)$ of order p. Then by (3.11), it follows that $\hat{z}+1$ is a pole of $w(z)$ of order p. By continuing the iteration, it follows that $\hat{z}+2$ is a pole of $w(z)$ of order $2p$, and $\hat{z}+3$ is a pole of $w(z)$ of order $4p$, then $\hat{z}+4$ is a pole of $w(z)$ of order $8p$, and so on. Therefore, we have found $15p$ poles, taking into account multiplicities, that correspond uniquely to p roots of $w-b_1$. In this case, we have

$$n\big(r, \frac{1}{w-b_1}\big) \le \frac{1}{15} n(r+4, w) + O(1),$$

Lemma 2.4 thus implies that $\rho_2(w) \ge 1$.

Subcase 3.2. Assume that

$$\deg_w(P) \le 2.$$

If $\deg_w(P) = 2$, then $\deg_w(P) = \deg_w(Q) + 1$. Thus, the assertion (1.6) holds. If $\deg_w(P) = 1$, then $\deg_w(R) = 1$, which implies that the assertion except that (1.6) of Theorem 1.2 also holds.

Case (4): The final remaining case is the one that $R(z, w(z))$ is a polynomial in $w(z)$. Then we write (1.5) as follows

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = P(z, w(z)), \tag{3.12}$$

where $\deg_w(P) \le 3$. If $\deg_w(P) = 0$ or $\deg_w(P) = 1$, the assertion except that (1.6) of Theorem 1.2 holds. Therefore we assume that

$$\deg_w(P) = p \ge 2,$$

and $w(z)$ has either infinitely many zeros or poles (or both).

Suppose that there is a pole or a zero of $w(z)$ at $z = \hat{z}$. Then either there is a cancelation with one of the coefficients or $w(z)$ has a pole of order at least one at $z = \hat{z} + 1$. Since the coefficients of (3.12) are rational, we can always choose a pole or a zero of $w(z)$ such that there is no cancelation with the coefficients. By shifting forward (3.12), it follows that $w(z)$ has a pole of order p at $z = \hat{z} + 2$, and has a pole of order p^2 at $z = \hat{z} + 3$, and so on. The only way that this string of poles with exponential growth in the multiplicity can terminate, or there exist a drop in the orders of poles, is that there is a cancelation with a suitable zero of a coefficient of (3.12). But since the coefficients are rational and thus have finitely many zeros or poles, $w(z)$ has infinitely many zeros or poles, we always can choose the staring point $\hat{z}$ of the iteration from outside a sufficiently large disc in such way that no cancelation occurs. Thus

$$n(d + |\hat{z}|, w) \geq p^d \tag{3.13}$$

for all $d \in \mathbb{N}$, and so

$$\begin{aligned} \lambda_2\big(\frac{1}{w}\big) &= \limsup_{r\to\infty} \frac{\log\log n(r,w)}{\log r} \\ &\geq \limsup_{d\to\infty} \frac{\log\log n(d+|\hat{z}|,w)}{\log(d+|\hat{z}|)} \\ &\geq \limsup_{d\to\infty} \frac{\log\log p^d}{\log(d+|\hat{z}|)} = 1. \end{aligned}$$

Thus, $\rho_2(w) \geq \lambda_2(\frac{1}{w}) \geq 1$.

Suppose now that $w(z)$ has finitely many poles and zeros and $\rho_2(w) < 1$. Since $\deg_w(P) \geq 2$ in (3.12), using the difference analogue of Clunie Lemma [3], then $m(r,w) = S(r,w)$, so $T(r,w) = S(r,w)$ follows, which is impossible. Thus $\rho_2(w) \geq 1$, which is a contradiction with our assumptions. The proof of Theorem 1.2 is thus complete.

4. Proof of Theorem 1.7

Suppose that $w(z)$ is a transcendental entire solution of (1.7). We rewrite (1.7) as

$$c(z)w(z)w(z+1) = b(z)w(z) - a(z)w'(z). \tag{4.1}$$

We affirm that $w(z)$ has at most finitely many zeros. Otherwise, we assume that $w(z)$ have infinitely many zeros. Obviously, $w(z)$ can not have infinitely many multiple zeros, thus $w(z)$ has infinitely many simple zeros. In this case, we can choose a zero z_0 such that

$$b(z_0)w(z_0) - a(z_0)w'(z_0) \neq 0,$$

but the left-hand side of (4.1) is equal to zero at z_0, a contradiction. Therefore, by the Hadamard factorization theorem we assume that

$$w(z) = p(z)e^{g(z)},$$

where $p(z)$ is non-zero polynomial and $g(z)$ is an entire function. Substituting the above into (4.1), we have

$$c(z)p(z+1)e^{g(z+1)} = b(z) - a(z)\Big[\frac{p'(z)}{p(z)} + g'(z)\Big]. \tag{4.2}$$

From (4.2), then we see that the order of growth of the left-hand side is always greater than the right-hand side. It is a contradiction. So (1.7) has no transcendental entire solutions.

Let $w(z)$ be a transcendental meromorphic solution of (1.7). Rewrite (1.7) as

$$c(z)w(z) = \Big(b(z) - a(z)\frac{w'(z)}{w(z)}\Big)\frac{w(z+1)}{w(z)}. \tag{4.3}$$

Taking proximity functions from both sides of (4.3) and using the logarithmic derivative lemma, Lemma 2.3, yields $m(r,w) = S(r,w)$. Thus, we have

$$N(r,w) + S(r,w) = T(r,w),$$

which implies that $\lambda(w) = \rho(w)$.

5. Proof of Theorem 1.9

By proof of Theorem 1.2, we see that

$$\deg_w(P) \le 3, \quad \deg_w(Q) \le 3.$$

We will discuss three following cases.

Case (1): $\deg_w(P) = 0$ and $\deg_w(Q) = 2$. We rewrite (1.5) as

$$cw(z+1) + a\frac{w'(z)}{w(z)} = \frac{h}{w^2(z) + bw(z) + e}, \tag{5.1}$$

where a, b, c, e, h are constants. Substituting $w(z) = \frac{M(z)}{N(z)}$ into (5.1), we obtain

$$\frac{cM\overline{M}N + aM'N\overline{N} - aMN'\overline{N}}{MN\overline{N}} = \frac{hN^2}{M^2 + bMN + eN^2}.$$

According to the above equation, it follows that

$$\begin{aligned} &cM^3\overline{M}N + aM^2M'N\overline{N} - aM^3N'\overline{N} \\ &+ cbM^2\overline{M}N^2 + abMM'N^2\overline{N} - abM^2NN'\overline{N} \\ &+ ceM\overline{M}N^3 + aeM'N^3\overline{N} - aeMN^2N'\overline{N} \\ &= hMN^3\overline{N}. \end{aligned} \tag{5.2}$$

There are nine terms related to $M(z)$ and $N(z)$ on the left-hand side of (5.2) with the degree is $4m+n$, $3m+2n-1$, $3m+2n-1$, $3m+2n$, $2m+3n-1$, $2m+3n-1$, $2m+3n$, $m+4n-1$, $m+4n-1$, respectively. Moreover, the coefficients of maximum degree terms are different and there is no cancelation occurring in these terms. And it is easy to see that the degree of the right-hand side of (5.2) is $m+4n$. In the following, we will deduce that $m = n$.

If $m > n$, then

$$4m + n > 3m + 2n > 3m + 2n - 1 \ge 2m + 3n > 2m + 3n - 1 > m + 4n - 1.$$

Therefore, the maximum degree of left-hand side of (5.2) is $4m+n$ and the degree of right-hand side of (5.2) is $m+4n$, then $m = n$, a contradiction.

If $m < n$, we have

$$4m + n \le 3m + 2n - 1 < 3m + 2n \le 2m + 3n - 1 < 2m + 3n \le m + 4n - 1.$$

If $e \ne 0$, then the degree of left-hand side of (5.2) is $m + 4n - 1$ and the degree of right-hand side of (5.2) is $m + 4n$, thus $m + 4n - 1 = m + 4n$, which is impossible. If $e = 0$ and $b \ne 0$, the degree of left-hand side of (5.2) is $2m + 3n - 1$, then

$m = n+1$, which is a contradiction with the assumption $m < n$. If $e = b = 0$, then $2m = 2n + 1$, a contradiction. Anyway, $m < n$ is impossible.

Thus, we have proved that if $\deg_w(P) = 0$ and $\deg_w(Q) = 2$, then $m = n$. Furthermore, we can prove that when $\deg_w(P) = 0$ and Q satisfies other cases, $m = n$ also holds by the same method.

Case (2) If $\deg_w(P) = 3$ and $\deg_w(Q) = 2$, then (1.7) can be written as

$$cw(z+1) + a\frac{w'(z)}{w(z)} = \frac{fw^3(z) + tw^2(z) + gw(z) + h}{w^2(z) + bw(z) + e}, \tag{5.3}$$

where b, e, h, g, t are constants and a, c, f are non-zero constants. Substituting $w(z) = \frac{M(z)}{N(z)}$ into (5.3), we have

$$\begin{aligned} &cM^3\overline{M}N^2 + aM^2M'N^2\overline{N} - aM^3NN'\overline{N} \\ &+ cbM^2\overline{M}N^3 + abMM'N^3\overline{N} - abM^2N^2N'\overline{N} \\ &+ ceM\overline{M}N^4 + aeM'N^4\overline{N} - aeMN^3N'\overline{N} \\ &= fM^4N\overline{N} + tM^3N^2\overline{N} + gM^2N^3\overline{N} + hMN^4\overline{N}. \end{aligned} \tag{5.4}$$

Thus we see that the possible degrees of left-hand side of (5.4) are $4m+2n$, $3m+3n-1$, $3m+3n$, $2m+4n-1$, $2m+4n$ or $m+5n-1$ and the possible degrees of right-hand side of (5.4) are $4m+2n$, $3m+3n$, $2m+4n$ or $m+5n$. We know that h and e can not vanish at the same time, otherwise, $P(z, w)$ and $Q(z, w)$ have common roots. If $m \geq n$, since c, f are nonzero constants, the degrees of two hand sides are equal. This case may happen. We assume that $m < n$ in the following. If $h \neq 0$, the degree of left-hand side of (5.4) is at most $m + 5n - 1$ and the degree of right-hand side of (5.4) is $m + 5n$, a contradiction. We consider the case when $h = 0$ and $e \neq 0$ in the following. If $g \neq 0$, comparing with the degrees of two hand sides of (5.4), we have

$$m + 5n - 1 = 2m + 4n.$$

Thus $n = m + 1$. If $g = 0$ and $t \neq 0$, by comparing the degrees of the two hand sides of (5.4), it yields $2n = 2m + 1$, which is also a contradiction. If $g = t = 0$, similarly, we have $3n = 3m + 1$, a contradiction again. In a word, if $h = 0$, $e \neq 0$ and $g \neq 0$, we have $n = m + 1$, which is the exceptional case. Now the proof is complete.

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