Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 87, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTI-TERM FRACTIONAL-ORDER BOUNDARY-VALUE PROBLEMS WITH NONLOCAL INTEGRAL BOUNDARY CONDITIONS

AHMED ALSAEDI, NAJLA ALGHAMDI, RAVI P. AGARWAL,
SOTIRIS K. NTOUYAS, BASHIR AHMAD

Communicated by Vicentiu D. Radulescu

Abstract. In this article, we introduce a class of multi-term fractional-order boundary-value problems involving nonlocal integral boundary conditions. Existence results for the given problem are obtained by means of standard tools of fixed point theory. The results are illustrated with the aid of examples and make a useful contribution to the existing literature on the topic.

1. Introduction

Fractional differential equations arise in the mathematical modeling of many engineering and scientific disciplines such as biophysics, bio-engineering, virology, control theory, signal and image processing, blood flow phenomena, etc. A huge amount of mathematically and physically interesting works published in recent years, including several excellent monographs, clearly reflects the overwhelming interest in the topic. For details we refer the reader the texts [8, 12, 17, 19, 20] and references cited therein.

Nonlocal boundary-value problems of fractional-order differential equations and inclusions have received significant attention. One can witness a great deal of work on the topic involving different kinds of boundary conditions in the literature, for example, see [1, 3, 7, 9, 11, 18] and the references cited therein.

There is another class of differential equations containing more than one fractional-order differential operators. Such equations appear in the modeling of the motion of a rigid plate immersed in a Newtonian fluid. Other typical examples include Bagley-Torvik [22] and Basset equation [16]. Some recent results on multi-term fractional differential equations can be found in the articles [6, 2, 4, 5, 14, 21].

In this article, we introduce and investigate the following nonlinear multi-term fractional order boundary value problem with nonlocal integral conditions:

$$(p_2{}^cD^{\delta+2} + p_1{}^cD^{\delta+1} + p_0{}^cD^{\delta})x(t) = f(t, x(t)), \quad 0 < \delta < 1,\ 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \quad x(\xi) = 0, \quad x(1) = \lambda \int_0^{\sigma} x(s)ds, \quad 0 < \sigma < \xi < 1,\ \lambda \in \mathbb{R}, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34A08, 34B10.
Key words and phrases. Caputo fractional derivative; integral boundary condition; multi-term fractional differential equations; existence; fixed point.

Submitted February 25, 2018. Published April 10, 2018.

where ${}^cD^\delta$ denote the Caputo fractional derivative of order δ, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous functions, and $p_j, j=0,1,2$ are real constants.

Existence results for problem (1.1)-(1.2) are obtained with the help of Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative, while the uniqueness result is proved via Banach contraction mapping principle. These results are presented in Section 3. Some preliminary concepts and lemmas are given in Section 2. The obtained results are well illustrated by examples.

2. Preliminary concepts and basic result

We begin this section with some definitions [12].

Definition 2.1. The Riemann-Liouville fractional integral of order $\tau>0$ for a function $h:[0,1]\to\mathbb{R}$ with $h\in L(0,1)$ is defined by

$$I^\tau h(u)=\int_0^u \frac{(u-v)^{\tau-1}}{\Gamma(\tau)}h(v)dv,\quad \text{for a.e, } u\in[0,1], \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2. The Caputo derivative of order $\tau\in(n-1,n)$ for a function $h:[0,1]\to\mathbb{R}$ with $h\in C^n[0,1]$ is defined by

$${}^cD^\tau h(u)=\frac{1}{\Gamma(n-\tau)}\int_0^u \frac{h^{(n)}(v)}{(u-v)^{\tau+1-n}}dv=I^{n-\tau}h^{(n)}(u),\quad u>0.$$

Property 2.3. *With the given notations, the following equality holds:*

$$I^\tau({}^cD^\tau h(u))=h(u)-c_0-c_1u-\dots-c_{n-1}u^{n-1},\quad u>0,\ n-1<\tau<n, \tag{2.2}$$

where c_i $(i=1,\dots,n-1)$ are arbitrary constants.

To define the solution for problem (1.1)-(1.2), we consider its linear variant in the following lemma.

Lemma 2.4. *Let p_0,p_1,p_2 be positive constants such that ${p_1}^2-4p_0p_2>0$ and $y\in C(0,1)\cap L(0,1)$. Then the solution of the linear multi-term fractional differential equation*

$$({p_2}{}^cD^{\delta+2}+{p_1}{}^cD^{\delta+1}+{p_0}{}^cD^{\delta})x(t)=y(t),\quad 0<\delta<1,\ 0<t<1, \tag{2.3}$$

supplemented with the boundary conditions (1.2) *is given by*

$$\begin{aligned} x(t) &= \frac{1}{p_2(m_2-m_1)}\Big\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}y(u)\,du\,ds\\ &+\rho_1(t)\int_0^\xi\int_0^s \Phi(\xi)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}y(u)\,du\,ds\\ &+\rho_2(t)\Big[\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}y(u)\,du\,ds\\ &-\lambda\int_0^\sigma\int_0^s\Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma-s)}-1)}{m_1}\Big)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}y(u)\,du\,ds\Big]\Big\}, \end{aligned} \tag{2.4}$$

where

$$
\begin{gathered}
\Phi(\kappa) = e^{m_2(\kappa-s)} - e^{m_1(\kappa-s)} \quad \kappa = t, 1, \xi\,, \\
m_1 = \frac{-p_1 - \sqrt{p_1^2 - 4p_0p_2}}{2p_2}, \quad m_2 = \frac{-p_1 + \sqrt{p_1^2 - 4p_0p_2}}{2p_2}, \\
\rho_1(t) = \frac{\omega_4\varrho_1(t) - \omega_3\varrho_2(t)}{\mu_1}, \quad \rho_2(t) = \frac{\omega_1\varrho_2(t) - \omega_2\varrho_1(t)}{\mu_1}, \\
\varrho_1(t) = \frac{m_1(1-e^{m_2t}) - m_2(1-e^{m_1t})}{m_1m_2}, \\
\varrho_2(t) = p_2(m_2-m_1)(e^{m_2t} - e^{m_1t}), \\
\mu_1 = \omega_1\omega_4 - \omega_2\omega_3 \neq 0, \quad \omega_1 = \frac{m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})}{m_1m_2}, \qquad (2.5) \\
\omega_2 = p_2(m_2-m_1)(e^{m_1\xi} - e^{m_2\xi}), \\
\begin{aligned}
\omega_3 = \Big(& m_2\big(1 - e^{m_1} - \lambda\sigma + \lambda/m_1(e^{m_1\sigma} - 1)\big) \\
& - m_1\big(1 - e^{m_2} - \lambda\sigma + \lambda/m_2(e^{m_2\sigma} - 1)\big)\Big)/(m_1m_2), \\
\omega_4 = p_2&(m_2-m_1)\Big((e^{m_1} + \lambda/m_1(1-e^{m_1\sigma})) \\
& - (e^{m_2} + \lambda/m_2(1-e^{m_2\sigma}))\Big).
\end{aligned}
\end{gathered}
$$

Proof. Applying the operator I^δ on (2.3) and using Property (2.3), we get

$$
(p_2D^2 + p_1D + p_0)x(t) = \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} y(s)ds + c_1, \tag{2.6}
$$

where c_1 is an arbitrary constant. By the method of variation of parameters, the solution of (2.6) can be written as

$$
\begin{aligned}
x(t) = c_2e^{m_1t} + c_3e^{m_2t} &- \frac{1}{p_2(m_2-m_1)} \int_0^t e^{m_1(t-s)} \Big(\int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)du + c_1 \Big) ds \\
&+ \frac{1}{p_2(m_2-m_1)} \int_0^t e^{m_2(t-s)} \Big(\int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)du + c_1 \Big) ds,
\end{aligned} \tag{2.7}
$$

where m_1 and m_2 are given by (2.5). Using $x(0) = 0$ in (2.7), we get

$$
\begin{aligned}
x(t) &= c_1 \Big[\frac{m_2(1-e^{m_1t}) - m_1(1-e^{m_2t})}{p_2m_1m_2(m_2-m_1)} \Big] + c_2 \Big(e^{m_1t} - e^{m_2t} \Big) \\
&\quad - \frac{1}{p_2(m_2-m_1)} \Big[\int_0^t \Big(e^{m_1(t-s)} - e^{m_2(t-s)} \Big) \Big(\int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)du \Big) ds \Big],
\end{aligned} \tag{2.8}
$$

which together with the conditions $x(\xi) = 0$ and $x(1) = \lambda \int_0^\sigma x(s)ds$ yields the following system of equations in the unknown constants c_1 and c_2:

$$
c_1\omega_1 + c_2\omega_2 = V_1, \tag{2.9}
$$

$$
c_1\omega_3 + c_2\omega_4 = V_2. \tag{2.10}
$$

where

$$
V_1 = \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\,du\,ds,
$$

$$
\begin{aligned}
V_2 = &\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&- \lambda \int_0^\sigma \int_0^s \Big[\frac{(e^{m_1(\sigma-s)}-1)}{m_1} - \frac{(e^{m_2(\sigma-s)}-1)}{m_2} \Big] \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds.
\end{aligned}
$$

Solving system (2.9)-(2.10) and using (2.5), we find that

$$
c_1 = \frac{V_1\omega_4 - V_2\omega_2}{\mu_1}, \quad c_2 = \frac{V_2\omega_1 - V_1\omega_3}{\mu_1}.
$$

Substituting the value of c_1 and c_2 in (2.8), we obtain the solution (2.4). The converse of the lemma follows by direct computation. This completes the proof. □

Remark 2.5. (i) When ${p_1}^2 - 4p_0p_2 = 0$ the solution of (2.3) equipped with condition (1.2) is

$$
\begin{aligned}
x(t) = &\frac{1}{p_2}\Big\{ \int_0^t \int_0^s \Psi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&+ \chi_1(t) \int_0^\xi \int_0^s \Psi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&+ \chi_2(t) \Big[\int_0^1 \int_0^s \Psi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&- \lambda \int_0^\sigma \Big(\frac{m(\sigma-s)e^{m(\sigma-s)} - e^{m(\sigma-s)} + 1}{m^2} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \Big] \Big\},
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{gathered}
\Psi(\kappa) = (\kappa - s)e^{m(\kappa-s)} \quad \kappa = t, 1, \xi, \\
m = \frac{-p_1}{2p_2}, \\
\chi_1(t) = \frac{\varpi_3 v_2(t) - \varpi_4 v_1(t)}{\mu_2}, \quad \chi_2(t) = \frac{\varpi_2 v_1(t) - \varpi_1 v_2(t)}{\mu_2}, \\
v_1(t) = \frac{mte^{mt} - e^{mt} + 1}{m^2}, \quad v_2(t) = p_2 te^{mt}, \\
\varpi_1 = \frac{m\xi e^{m\xi} - e^{m\xi} + 1}{m^2}, \quad \varpi_2 = p_2\xi e^{m\xi}, \\
\varpi_3 = \frac{m^2e^m - me^m + m - m\sigma e^{m\sigma} + 2e^{m\sigma} - 2 - m\sigma}{m^3}, \\
\varpi_4 = p_2 \frac{m^2e^m - \lambda m\sigma e^{m\sigma} + \lambda e^{m\sigma} - \lambda}{m^2}, \\
\mu_2 = \varpi_1\varpi_4 - \varpi_2\varpi_3 \neq 0;
\end{gathered}
\tag{2.12}
$$

(ii) When ${p_1}^2 - 4p_0p_2 < 0$ the solution of (2.3) equipped with condition (1.2) is

$$
\begin{aligned}
x(t) = &\frac{1}{p_2 b}\Big\{ \int_0^t \int_0^s \Omega(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&+ \tau_1(t) \int_0^\xi \int_0^s \Omega(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \\
&+ \tau_2(t) \Big[\int_0^1 \int_0^s \Omega(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda}{a^2+b^2} \int_0^\sigma \Big(b + b e^{-a(\sigma-s)} \cos b(\sigma-s) \\
&- a e^{-a(\sigma-s)} \sin b(\sigma-s) \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} y(u)\, du\, ds \Big] \Big\},
\end{aligned}
$$

where

$$
\begin{gathered}
\Omega(\kappa) = e^{-a(\kappa-s)} \sin b(\kappa-s) \quad \kappa = t, 1, \xi\,, \\
m_{1,2} = -a \pm bi, \quad a = \frac{p_1}{2p_2}, \quad b = \frac{\sqrt{4p_0p_2 - {p_1}^2}}{2p_2}, \\
\tau_1(t) = \frac{q_3\nu_2(t) - q_4\nu_1(t)}{\mu_3}, \quad \tau_2(t) = \frac{q_2\nu_1(t) - q_1\nu_2(t)}{\mu_3}, \\
\nu_1(t) = \frac{b + be^{-at}\cos bt - ae^{-at}\sin bt}{a^2+b^2}, \quad \nu_2(t) = p_2 b e^{-at} \sin bt \\
q_1 = \frac{b - be^{-a\xi}\cos b\xi - ae^{-a\xi}\sin b\xi}{a^2+b^2}, \quad q_2 = p_2 b e^{-a\xi}\sin b\xi, \qquad (2.13) \\
q_3 = \frac{1}{a^2+b^2}\Big[b - be^{-a}\cos b - ae^{-a}\sin b - b\lambda\sigma + \frac{b\lambda}{a^2+b^2}(a - ae^{-a\sigma}\cos b\sigma \\
+ be^{-a\sigma}\sin b\sigma) + \frac{a\lambda}{a^2+b^2}(b - be^{-a\sigma}\cos b\sigma - ae^{-a\sigma}\sin b\sigma)\Big], \\
q_4 = p_2 b\Big[e^{-a}\sin b - \frac{\lambda}{a^2+b^2}(b - be^{-a\sigma}\cos b\sigma - ae^{-a\sigma}\sin b\sigma)\Big], \\
\mu_3 = q_1q_4 - q_2q_3 \neq 0.
\end{gathered}
$$

3. Existence and uniqueness results

Denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. By Lemma 2.4, we can transform problem (1.1)-(1.2) into a fixed point problem as follows:

(i) For ${p_1}^2 - 4p_0p_2 > 0$, we introduce an operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ given by

$$
\begin{aligned}
(\mathcal{J}x)(t) = \frac{1}{p_2(m_2-m_1)} \Big\{ &\int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\, du\, ds \\
&+ \rho_1(t) \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\, du\, ds \\
&+ \rho_2(t) \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\, du\, ds \\
&- \lambda \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \\
&\times \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\, du\, ds \Big] \Big\},
\end{aligned} \tag{3.1}
$$

such that

$$
x = \mathcal{J}x. \tag{3.2}
$$

(ii) For ${p_1}^2 - 4p_0p_2 = 0$, we have an operator equation

$$
x = \mathcal{H}x, \tag{3.3}
$$

where the operator $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ is defined by

$$
\begin{aligned}
&(\mathcal{H}x)(t)\\
&=\frac{1}{p_2}\Big\{\int_0^t\int_0^s \Psi(t)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&+\chi_1(t)\int_0^\xi\int_0^s \Psi(\xi)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&+\chi_2(t)\Big[\int_0^1\int_0^s \Psi(1)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&-\lambda\int_0^\sigma\Big(\frac{m(\sigma-s)e^{m(\sigma-s)}-e^{m(\sigma-s)}+1}{m^2}\Big)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\Big]\Big\}.
\end{aligned}
\tag{3.4}
$$

(iii) For ${p_1}^2-4p_0p_2<0$, we have the fixed point problem:

$$x=\mathcal{K}x, \tag{3.5}$$

where the operator $\mathcal{K}:\mathcal{C}\to\mathcal{C}$ is defined by

$$
\begin{aligned}
(\mathcal{K}x)(t)=\frac{1}{p_2 b}\Big\{&\int_0^t\int_0^s \Omega(t)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&+\tau_1(t)\int_0^\xi\int_0^s \Omega(\xi)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&+\tau_2(t)\Big[\int_0^1\int_0^s \Omega(1)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\\
&-\frac{\lambda}{a^2+b^2}\int_0^\sigma\Big(b+be^{-a(\sigma-s)}\cos b(\sigma-s)ae^{-a(\sigma-s)}\sin b(\sigma-s)\Big)\\
&\times\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}f(u,x(u))\,du\,ds\Big]\Big\}.
\end{aligned}
\tag{3.6}
$$

Now we set

$$
\begin{gathered}
\widehat{\rho}_1=\max_{t\in[0,1]}|\rho_1(t)|,\quad \widehat{\rho}_2=\max_{t\in[0,1]}|\rho_2(t)|,\\
\varepsilon=\max_{t\in[0,1]}\Big|m_2(1-e^{m_1 t})-m_1(1-e^{m_2 t})\Big|,\\
\begin{aligned}
\alpha=&\frac{1}{p_2 m_1 m_2(m_2-m_1)\Gamma(\delta+1)}\Big\{\varepsilon+\xi^\delta\widehat{\rho}_1(m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi}))\\
&+\widehat{\rho}_2\Big[(m_2(1-e^{m_1})-m_1(1-e^{m_2}))\\
&+\frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma-e^{m_2\sigma}+1)-m_2^2(m_1\sigma-e^{m_1\sigma}+1))\Big]\Big\},
\end{aligned}\\
\alpha_1=\alpha-\frac{\varepsilon}{p_2 m_1 m_2(m_2-m_1)\Gamma(\delta+1)};
\end{gathered}
\tag{3.7}
$$

also we set

$$\widehat{\chi}_1 = \max_{t\in[0,1]} |\chi_1(t)|, \quad \widehat{\chi}_2 = \max_{t\in[0,1]} |\chi_2(t)|,$$

$$\begin{aligned} \beta = \frac{1}{p_2 m^2 \Gamma(\delta+1)} \Big\{ &(1+\widehat{\chi}_2)|me^m - e^m + 1| + \xi^\delta \widehat{\chi}_1 |m\xi e^{m\xi} - e^{m\xi} + 1| \\ &+ \frac{|\lambda|\sigma^\delta \widehat{\chi}_2}{|m|} |2(e^{m\sigma}-1) - m\sigma(e^{m\sigma}+1)| \Big\}, \end{aligned} \tag{3.8}$$

$$\beta_1 = \beta - \frac{|me^m - e^m + 1|}{p_2 m^2 \Gamma(\delta+1)};$$

and

$$\widehat{\tau}_1 = \max_{t\in[0,1]} |\tau_1(t)|, \quad \widehat{\tau}_2 = \max_{t\in[0,1]} |\tau_2(t)|$$

$$\begin{aligned} \gamma = \frac{1}{p_2 b(a^2+b^2)\Gamma(\delta+1)} \Big\{ &(1+\widehat{\tau}_2)|b - be^{-a}\cos b - ae^{-a}\sin b| \\ &+ \xi^\delta \widehat{\tau}_1 |b - be^{-a\xi}\cos b\xi - ae^{-a\xi}\sin b\xi| + \frac{|\lambda|\sigma^\delta \widehat{\tau}_2}{a^2+b^2} \\ &\times \left| 2ab - (a^2+b^2)b\sigma - 2abe^{-a\sigma}\cos b\sigma + (a^2-b^2)e^{-a\sigma}\sin b\sigma \right| \Big\}, \end{aligned} \tag{3.9}$$

$$\gamma_1 = \gamma - \frac{|b - be^{-a}\cos b - ae^{-a}\sin b|}{p_2 b(a^2+b^2)\Gamma(\delta+1)}.$$

Now we discus the existence and uniqueness of solutions for the problem (1.1)-(1.2) by using the standard fixed point theorems. We give the details for the case where ${p_1}^2 - 4p_0p_2 > 0$, while the details for other two cases ${p_1}^2 - 4p_0p_2 = 0$ and ${p_1}^2 - 4p_0p_2 < 0$ can be completed in a similar manner.

Our first result is based on Krasnoselskii's fixed point theorem, which is stated below.

Theorem 3.1 ([13]). *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let A_1, A_2 be the operators such that (i) $A_1x + A_2y \in M$ whenever $x, y \in Y$; (ii) A_1 is compact and continuous; and (iii) A_2 is a contraction mapping. Then there exists $w \in Y$ such that $w = A_1w + A_2w$.*

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions:*

- (A1) $|f(t,x) - f(t,y)| \le \ell |x-y|$ *for all* $t \in [0,1]$, $x, y \in \mathbb{R}$, $\ell > 0$;
- (A2) $|f(t,x)| \le \theta(t)$, *for all* $(t,x) \in [0,1]\times\mathbb{R}$ *and* $\theta \in C([0,1], \mathbb{R}^+)$.

Then problem (1.1)-(1.2) with $p_1^2 - 4p_0p_2 > 0$, has at least one solution on $[0,1]$ if

$$\ell\alpha_1 < 1, \tag{3.10}$$

where α_1 is given by (3.7).

Proof. Setting $\sup_{t\in[0,1]} |\theta(t)| = \|\theta\|$, we can fix

$$\begin{aligned} r \ge \frac{\|\theta\|}{p_2 m_1 m_2 (m_2 - m_1)\Gamma(\delta+1)} \Big\{ &\varepsilon + \xi^\delta \widehat{\rho}_1 (m_2(1-e^{m_1\xi}) + m_1(1-e^{m_2\xi})) \\ &+ \widehat{\rho}_2 \Big[(m_2(1-e^{m_1}) - m_1(1-e^{m_2})) + \frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma - e^{m_2\sigma}+1) \\ &- m_2^2(m_1\sigma - e^{m_1\sigma}+1)) \Big] \Big\}, \end{aligned} \tag{3.11}$$

and consider $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. Introduce the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ defined on B_r as follows:

$$(\mathcal{J}_1 x)(t) = \frac{1}{p_2(m_2-m_1)} \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds, \tag{3.12}$$

$$\begin{aligned}(\mathcal{J}_2 x)(t) = \frac{1}{p_2(m_2-m_1)} \Big\{ &\rho_1(t) \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \\ &+ \rho_2(t) \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \\ &- \lambda \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \\ &\times \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \Big] \Big\}. \end{aligned} \tag{3.13}$$

Observe that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$. For $x, y \in B_r$, we have

$$\begin{aligned} &\|\mathcal{J}_1 x + \mathcal{J}_2 y\| \\ &= \sup_{t\in[0,1]} |(\mathcal{J}_1 x)(t) + (\mathcal{J}_2 y)(t)| \\ &\le \frac{1}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\ &\quad + |\rho_1(t)| \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \\ &\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \\ &\quad + |\lambda| \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \\ &\quad \times \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \Big] \Big\} \\ &\le \frac{\|\theta\|}{p_2(m_2-m_1)\Gamma(\delta+1)} \sup_{t\in[0,1]} \Big\{ t^\delta \int_0^t \Big(e^{m_2(t-s)} - e^{m_1(t-s)} \Big) ds \\ &\quad + \xi^\delta |\rho_1(t)| \int_0^\xi \Big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \Big) ds \\ &\quad + |\rho_2(t)| \Big[\int_0^1 \Big(e^{m_2(1-s)} - e^{m_1(1-s)} \Big) ds \\ &\quad + |\lambda| \sigma^\delta \int_0^\sigma \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) ds \Big] \Big\} \\ &\le \frac{\|\theta\|}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \varepsilon + \widehat{\rho}_1 \xi^\delta (m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})) \\ &\quad + \widehat{\rho}_2 [(m_2(1-e^{m_1}) - m_1(1-e^{m_2})) \\ &\quad + \frac{\sigma^\delta |\lambda|}{m_1 m_2} (m_1^2(m_2\sigma - e^{m_2\sigma}+1) - m_2^2(m_1\sigma - e^{m_1\sigma}+1))] \Big\} \le r, \end{aligned}$$

where we used (3.11). Thus $\mathcal{J}_1 x + \mathcal{J}_2 y \in B_r$. Using the assumption (A1) together with (3.10), we show that $\mathcal{J}_2$ is a contraction as follows:

$$
\begin{aligned}
&\|\mathcal{J}_2 x - \mathcal{J}_2 y\| \\
&= \sup_{t\in[0,1]} |(\mathcal{J}_2 x)(t) - (\mathcal{J}_2 y)(t)| \\
&\le \frac{1}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ |\rho_1(t)| \int_0^{\xi}\int_0^s \Phi(\xi)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \\
&\quad \times |f(u,x(u)) - f(u,y(u))|\,du\,ds \\
&\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u)) - f(u,y(u))|\,du\,ds \\
&\quad + |\lambda| \int_0^{\sigma}\int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \\
&\quad \times |f(u,x(u)) - f(u,y(u))|\,du\,ds \Big] \Big\} \\
&\le \frac{\ell}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \xi^{\delta} |\rho_1(t)| \int_0^{\xi} \Big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \Big) ds \\
&\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Big(e^{m_2(1-s)} - e^{m_1(1-s)} \Big) ds \\
&\quad + |\lambda| \sigma^{\delta} \int_0^{\sigma} \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) ds \Big] \Big\} \|x-y\| \\
&\le \frac{\ell}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \widehat{\rho}_1 \xi^{\delta} (m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})) \\
&\quad + \widehat{\rho}_2 [(m_2(1-e^{m_1}) - m_1(1-e^{m_2})) + \frac{\sigma^{\delta}|\lambda|}{m_1 m_2} (m_1^2(m_2\sigma - e^{m_2\sigma}+1) \\
&\quad - m_2^2(m_1\sigma - e^{m_1\sigma}+1))] \Big\} \|x-y\| \\
&= \ell \alpha_1 \|x-y\|.
\end{aligned}
$$

Note that continuity of f implies that the operator $\mathcal{J}_1$ is continuous. Also, $\mathcal{J}_1$ is uniformly bounded on B_r as

$$
\|\mathcal{J}_1 x\| = \sup_{t\in[0,1]} |(\mathcal{J}_1 x)(t)| \le \frac{\|\theta\|\varepsilon}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)}.
$$

Now we prove compactness of operator $\mathcal{J}_1$. We define $\sup_{(t,x)\in[0,1]\times B_r} |f(t,x)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$
\begin{aligned}
&|(\mathcal{J}_1 x)(t_2) - (\mathcal{J}_1 x)(t_1)| \\
&= \frac{1}{p_2(m_2-m_1)} \Big| \int_0^{t_1}\int_0^s \Big[\Phi(t_2) - \Phi(t_1)\Big] \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \\
&\quad + \int_{t_1}^{t_2}\int_0^s \Phi(t_2) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \Big| \\
&\le \frac{\overline{f}}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \Big(t_1^{\delta} - t_2^{\delta} \Big) \Big(m_1(1 - e^{m_2(t_2-t_1)}) \\
&\quad - m_2(1-e^{m_1(t_2-t_1)}) \Big) + t_1^{\delta} \Big(m_1(e^{m_2 t_2} - e^{m_2 t_1}) - m_2(e^{m_1 t_2} - e^{m_1 t_1}) \Big) \Big\}
\end{aligned}
$$

$$\to 0, \quad \text{as } t_1 \to t_2,$$

and is independent of x. Thus, $\mathcal{J}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{J}_1$ is compact on B_r. Thus all the assumption of Theorem (3.1) are satisfied. So by the conclusion of Theorem 3.1, the problem (1.1)-(1.2) has at least one solution [0, 1]. The proof is complete. □

Remark 3.3. In the above theorem we can interchange the roles of the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ to obtain a second result by replacing (3.10) by the following condition:

$$\frac{\ell\varepsilon}{p_2 m_1 m_2 (m_2 - m_1)\Gamma(\delta+1)} < 1.$$

Now we apply Banach's contraction mapping principle to prove existence and uniqueness of solutions for the problem (1.1)-(1.2).

Theorem 3.4. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that* (A1) *is satisfied. Then there exists a unique solution for the problem* (1.1)-(1.2) *on* $[0,1]$ *if $\ell < 1/\alpha$, where α is given by* (3.7).

Proof. Let us define $\sup_{t\in[0,1]} |f(t,0)| = M$ and select $\bar{r} \ge \frac{\alpha M}{1-\ell\alpha}$ to show that $\mathcal{J}B_{\bar{r}} \subset B_{\bar{r}}$, where $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \le \bar{r}\}$ and $\mathcal{J}$ is defined by (3.1). Using the condition (A1), we have

$$\begin{aligned} |f(t,x)| &= |f(t,x) - f(t,0) + f(t,0)| \le |f(t,x) - f(t,0)| + |f(x,0)| \\ &\le \ell\|x\| + M \le \ell\bar{r} + M. \end{aligned} \tag{3.14}$$

Then, for $x \in B_{\bar{r}}$, we obtain

$$\begin{aligned}
&\|\mathcal{J}(x)\| \\
&= \sup_{t\in[0,1]} |\mathcal{J}(x)(t)| \\
&\le \frac{1}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\
&\quad + |\rho_1(t)| \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\
&\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))| du ds \\
&\quad + |\lambda| \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \Big] \Big\} \\
&\le \frac{(\ell\bar{r}+M)}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \Big(e^{m_2(t-s)} - e^{m_1(t-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds \\
&\quad + |\rho_1(t)| \int_0^\xi \Big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds \\
&\quad + |\rho_2(t)| \Big[\int_0^1 \Big(e^{m_2(1-s)} - e^{m_1(1-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds \\
&\quad + |\lambda| \int_0^\sigma \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds \Big] \Big\} \\
&\le \frac{(\ell\bar{r}+M)}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \varepsilon + \widehat{\rho}_1 \xi^\delta (m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi}))
\end{aligned}$$

$$
\begin{aligned}
&+ \widehat{\rho}_2[(m_2(1-e^{m_1}) - m_1(1-e^{m_2}))\\
&+ \frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma - e^{m_2\sigma}+1) - m_2^2(m_1\sigma - e^{m_1\sigma}+1))]\Big\}\\
&= (\ell\bar{r}+M)\alpha \le \bar{r},
\end{aligned}
$$

which clearly shows that $\mathcal{J}x \in B_{\bar{r}}$ for any $x \in B_{\bar{r}}$. Thus $\mathcal{J}B_{\bar{r}} \subset B_{\bar{r}}$. Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we have

$$
\begin{aligned}
&\|(\mathcal{J}x) - (\mathcal{J}y)\|\\
&\le \frac{1}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u)) - f(u,y(u))|\,du\,ds\\
&\quad + |\rho_1(t)| \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u)) - f(u,y(u))|\,du\,ds\\
&\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u)) - f(u,y(u))|\,du\,ds\\
&\quad + |\lambda| \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)}\\
&\quad \times |f(u,x(u)) - f(u,y(u))|\,du\,ds \Big] \Big\}\\
&\le \frac{\ell}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \Big(e^{m_2(t-s)} - e^{m_1(t-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds\\
&\quad + |\rho_1(t)| \int_0^\xi \Big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds\\
&\quad + |\rho_2(t)| \Big[\int_0^1 \Big(e^{m_2(1-s)} - e^{m_1(1-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds\\
&\quad + |\lambda| \int_0^\sigma \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds \Big] \Big\} \|x-y\|\\
&\le \frac{\ell}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \varepsilon + \widehat{\rho}_1 \xi^\delta (m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi}))\\
&\quad + \widehat{\rho}_2[(m_2(1-e^{m_1}) - m_1(1-e^{m_2}))\\
&\quad + \frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma - e^{m_2\sigma}+1) - m_2^2(m_1\sigma - e^{m_1\sigma}+1))] \Big\} \|x-y\|\\
&= \ell\alpha\|x-y\|,
\end{aligned}
$$

where α is given by (3.7) and depends only on the parameters involved in the problem. In view of the condition $\ell < 1/\alpha$, it follows that $\mathcal{J}$ is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), the problem (1.1) and (1.2) has a unique solution on $[0,1]$. This completes the proof. □

The next existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.5 (Nonlinear alternative for single valued maps [10]). *Let C be a closed, convex subset of a Banach space E and U be an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either (i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$ (the boundary of U in C) and $\epsilon \in (0,1)$ such that $u = \epsilon F u$.*

Theorem 3.6. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the conditions:*

- (A3) *There exist a function $g\in C([0,1],\mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+\to\ \mathbb{R}^+$ such that $|f(t,y)|\le g(t)\psi(\|y\|)$ for all $(t,y)\in[0,1]\times\mathbb{R}$;*
- (A4) *There exists a constant $K>0$ such that*
$$\frac{K}{\|g\|\psi(K)\alpha}>1.$$

Then the problem (1.1)-(1.2) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{J}:\mathcal{C}\to\mathcal{C}$ defined by (3.1). We show that $\mathcal{J}$ maps bounded sets into bounded sets in $\mathcal{C}=C([0,1],\mathbb{R})$. For a positive number ζ, let $\mathcal{B}_\zeta=\{x\in\mathcal{C}:\|x\|\le\zeta\}$ be a bounded set in $\mathcal{C}$. Then we have

$$\begin{aligned}
&\|\mathcal{J}(x)\|\\
&= \sup_{t\in[0,1]}|\mathcal{J}(x)(t)|\\
&\le \frac{1}{p_2(m_2-m_1)}\sup_{t\in[0,1]}\Big\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}|f(u,x(u))|\,du\,ds\\
&\quad+|\rho_1|(t)\int_0^\xi\int_0^s\Phi(\xi)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}|f(u,x(u))|\,du\,ds\\
&\quad+|\rho_2(t)|\Big[\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}|f(u,x(u))|\,du\,ds\\
&\quad+|\lambda|\int_0^\sigma\int_0^s\Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma-s)}-1)}{m_1}\Big)\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}|f(u,x(u))|\,du\,ds\Big]\Big\}\\
&\le \frac{\|g\|\psi(\zeta)}{p_2(m_2-m_1)}\sup_{t\in[0,1]}\Big\{\int_0^t\Big(e^{m_2(t-s)}-e^{m_1(t-s)}\Big)\frac{s^\delta}{\Gamma(\delta+1)}ds\\
&\quad+|\rho_1(t)|\int_0^\xi\Big(e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\Big)\frac{s^\delta}{\Gamma(\delta+1)}ds\\
&\quad+|\rho_2(t)|\Big[\int_0^1\Big(e^{m_2(1-s)}-e^{m_1(1-s)}\Big)\frac{s^\delta}{\Gamma(\delta+1)}ds\\
&\quad+|\lambda|\int_0^\sigma\Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma-s)}-1)}{m_1}\Big)\frac{s^\delta}{\Gamma(\delta+1)}ds\Big]\Big\}\\
&\le \frac{\|g\|\psi(\zeta)}{p_2m_1m_2(m_2-m_1)\Gamma(\delta+1)}\big\{\varepsilon+\widehat{\rho}_1\xi^\delta(m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi}))\\
&\quad+\widehat{\rho}_2[(m_2(1-e^{m_1})-m_1(1-e^{m_2}))\\
&\quad+\frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma-e^{m_2\sigma}+1)-m_2^2(m_1\sigma-e^{m_1\sigma}+1))]\big\},
\end{aligned}$$

which yields

$$\begin{aligned}
\|\mathcal{J}x\|\le\ &\frac{\|g\|\psi(\zeta)}{p_2m_1m_2(m_2-m_1)\Gamma(\delta+1)}\Big\{\varepsilon+\widehat{\rho}_1\xi^\delta(m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi}))\\
&+\widehat{\rho}_2[(m_2(1-e^{m_1})-m_1(1-e^{m_2}))\\
&+\frac{\sigma^\delta|\lambda|}{m_1m_2}(m_1^2(m_2\sigma-e^{m_2\sigma}+1)-m_2^2(m_1\sigma-e^{m_1\sigma}+1))]\Big\}.
\end{aligned}$$

Next we show that $\mathcal{J}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_\zeta$, where $\mathcal{B}_\zeta$ is a bounded set of $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
&|(\mathcal{J}x)(t_2) - (\mathcal{J}x)(t_1)| \\
&\le \frac{1}{p_2(m_2-m_1)} \Big\{ \Big| \int_0^{t_1} \int_0^s \Big[\Phi(t_2) - \Phi(t_1)\Big] \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \\
&\quad + \int_{t_1}^{t_2} \int_0^s \Phi(t_2) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f(u,x(u))\,du\,ds \Big| \\
&\quad + |\rho_1(t_2) - \rho_1(t_1)| \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \\
&\quad + |\rho_2(t_2) - \rho_2(t_1)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \\
&\quad + |\lambda| \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,y(u))|\,du\,ds \Big] \Big\} \\
&\le \frac{\overline{f}}{p_2 m_1 m_2 (m_2-m_1)\Gamma(\delta+1)} \Big\{ \Big(t_1^\delta - t_2^\delta\Big) \Big(m_1(1-e^{m_2(t_2-t_1)}) \\
&\quad - m_2(1-e^{m_1(t_2-t_1)}) \Big) + t_1^\delta \Big(m_1(e^{m_2t_2} - e^{m_2t_1}) - m_2(e^{m_1t_2} - e^{m_1t_1}) \Big) \\
&\quad + |\rho_1(t_2) - \rho_1(t_1)| \xi^\delta (m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})) \\
&\quad + |\rho_2(t_2) - \rho_2(t_1)| [(m_2(1-e^{m_1}) - m_1(1-e^{m_2})) \\
&\quad + \frac{\sigma^\delta|\lambda|}{m_1m_2} (m_1^2(m_2\sigma - e^{m_2\sigma}+1) - m_2^2(m_1\sigma - e^{m_1\sigma}+1))] \Big\},
\end{aligned}
$$

which tends to zero independently of $x \in \mathcal{B}_\zeta$ as $t_2 - t_1 \to 0$. As $\mathcal{J}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $x = \vartheta \mathcal{J} x$ is bounded for $\vartheta \in [0,1]$. For that, let x be a solution of $x = \vartheta\mathcal{J}x$ for $\vartheta \in [0,1]$. Then, for $t \in [0,1]$, we have

$$
\begin{aligned}
&|x(t)| \\
&= |\vartheta \mathcal{J} x(t)| \\
&\le \frac{1}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\
&\quad + |\rho_1|(t) \int_0^\xi \int_0^s \Phi(\xi) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\
&\quad + |\rho_2(t)| \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \\
&\quad + |\lambda| \int_0^\sigma \int_0^s \Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1} \Big) \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} |f(u,x(u))|\,du\,ds \Big] \Big\} \\
&\le \frac{\|g\|\psi(\|x\|)}{p_2(m_2-m_1)} \sup_{t\in[0,1]} \Big\{ \int_0^t \Big(e^{m_2(t-s)} - e^{m_1(t-s)} \Big) \frac{s^\delta}{\Gamma(\delta+1)} ds
\end{aligned}
$$

$$
\begin{aligned}
&+|\rho_1(t)|\int_0^{\xi}\Big(e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\Big)\frac{s^{\delta}}{\Gamma(\delta+1)}ds\\
&+|\rho_2(t)|\Big[\int_0^{1}\Big(e^{m_2(1-s)}-e^{m_1(1-s)}\Big)\frac{s^{\delta}}{\Gamma(\delta+1)}ds\\
&+|\lambda|\int_0^{\sigma}\Big(\frac{(e^{m_2(\sigma-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma-s)}-1)}{m_1}\Big)\frac{s^{\delta}}{\Gamma(\delta+1)}ds\Big]\Big\}\\
\le\ &\frac{\|g\|\psi(\|x\|)}{p_2m_1m_2(m_2-m_1)\Gamma(\delta+1)}\Big\{\varepsilon+\widehat{\rho}_1\xi^{\delta}(m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi}))\\
&+\widehat{\rho}_2[(m_2(1-e^{m_1})-m_1(1-e^{m_2}))\\
&+\frac{\sigma^{\delta}|\lambda|}{m_1m_2}(m_1^2(m_2\sigma-e^{m_2\sigma}+1)-m_2^2(m_1\sigma-e^{m_1\sigma}+1))]\Big\}\\
=\ &\|g\|\psi(\|x\|)\alpha,
\end{aligned}
$$

which implies

$$
\frac{\|x\|}{\|g\|\psi(\|x\|)\alpha}\le 1.
$$

In view of (A4), there is no solution x such that $\|x\|\neq K$. Let us set

$$
U=\{x\in\mathcal{C}:\|x\|<K\}.
$$

The operator $\mathcal{J}:\overline{U}\to\mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u=\vartheta\mathcal{J}(u)$ for some $\vartheta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [10], we deduce that $\mathcal{J}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

Example 3.7. Consider the boundary-value problem

$$
({}^cD^{5/2}+3{}^cD^{3/2}+2{}^cD^{1/2})x(t)=\frac{A}{\sqrt{t^2+49}}\Big(\cos x+\tan^{-1}t\Big),\quad 0<t<1, \tag{3.15}
$$

$$
x(0)=0,\quad x(1/3)=0,\quad x(1)=\int_0^{1/5}x(s)ds. \tag{3.16}
$$

Here, $\delta=1/2$, $\sigma=3/5$, $\xi=1/3$, $p_2=1$, $p_1=3$, $p_0=2$, $\lambda=1$, A is a positive constant and

$$
f(t,x)=\frac{A}{\sqrt{t^2+49}}\Big(\cos x+\tan^{-1}t\Big).
$$

Clearly the constants p_2,p_1, and p_0 satisfy the condition of Lemma 2.4, and

$$
|f(t,x)-f(t,y)|\le A|x-y|/7,
$$

where $\ell=A/7$. Using the given values, we find $\alpha\approx 0.44269$ and $\alpha_1\approx 0.21725$, It is easy to check that $|f(t,x)|\le\frac{A(2+\pi)}{2\sqrt{t^2+49}}=\theta(t)$ and $\ell\alpha_1<1$ when $A<32.22094$. As all the condition of Theorem 3.2 are satisfied the problem (3.15)-(3.16) has at least one solution on $[0,1]$. On the other hand, $\ell\alpha<1$ whenever $A<15.81242$ and thus there exists a unique solution for the problem (3.15)-(3.16) on $[0,1]$ by Theorem 3.4.

Example 3.8. Consider the boundary-value problem

$$
({}^cD^{5/2}+3{}^cD^{3/2}+2{}^cD^{1/2})x(t)=\frac{1}{4\pi}\sin(2\pi x)+\frac{|x|^2}{1+|x|^2},\quad 0<t<1, \tag{3.17}
$$

$$x(0) = 0, \quad x(1/3) = 0, \quad x(1) = \int_0^{1/5} x(s)ds. \tag{3.18}$$

Here, $\delta = 1/2$, $\sigma = 3/5$, $\xi = 1/3$, $p_1^2 - 4p_2p_0 = 1 > 0$, $\lambda = 1$, and

$$f(t,x) = \frac{1}{4\pi}\sin(2\pi x) + \frac{|x|^2}{1+|x|^2}.$$

Clearly

$$|f(t,x)| \le |\frac{1}{4\pi}\sin(2\pi x) + \frac{|x|^2}{1+|x|^2}| \le \frac{1}{2}\|x\| + 1,$$

where $g(t) = 1$, $\psi(\|x\|) = \frac{1}{2}\|x\| + 1$.

Then by using the condition (A4), we find that $K > 0.56853$ (we have used $\alpha = 0.44269$). Thus, the conclusion of Theorem 3.6 applies to problem (3.17)-(3.18).

Acknowledgements. This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-11-130-39). The authors, therefore, acknowledge with gratitude the DSR technical and financial support.

References

[1] R. P. Agarwal, B. Ahmad, A. Alsaedi; Fractional-order differential equations with anti-periodic boundary conditions: a survey, *Bound. Value Probl.* (2017), 2017:173.

[2] B. Ahmad, A. Alsaedi, S. Aljoudi, S. K. Ntouyas; On a coupled system of sequential fractional differential equations with variable coefficients and coupled integral boundary conditions, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, **60(108)** (2017), 3-18.

[3] B. Ahmad, A. Alsaedi, S. K. Ntouyas, J. Tariboon; *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities.* Springer, Cham, 2017.

[4] B. Ahmad, M. M. Matar, O. M. El-Salmy; Existence of solutions and Ulam stability for Caputo type sequential fractional differential equations of order $\alpha \in (2,3)$, *Inter. J. Anal. Appl.*, **15** (2017), 86-101.

[5] B. Ahmad, S. K. Ntouyas; A higher-order nonlocal three-point boundary value problem of sequential fractional differential equations, *Miscolc Math. Notes* **15** (2014), No. 2, pp. 265-278.

[6] A. Alsaedi, S. K. Ntouyas, R. P. Agarwal, B. Ahmad; On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, *Adv. Difference Equ.* (2015), 2015:33, 12 pp.

[7] Z. B. Bai, W. Sun; Existence and multiplicity of positive solutions for singular fractional boundary value problems, *Comput. Math. Appl.*, **63** (2012) 1369-1381.

[8] K. Diethelm; *The Analysis of Fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics 2004*, Springer-Verlag, Berlin, 2010.

[9] J. R. Graef, L. Kong, Q. Kong; Application of the mixed monotone operator method to fractional boundary value problems, *Fract. Differ. Calc.*, **2** (2011), 554-567.

[10] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[11] J. Henderson, R. Luca; Existence of nonnegative solutions for a fractional integro-differential equation, *Results Math.* **72** (2017), 747-763.

[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[13] M. A. Krasnoselskii; Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk*, **10** (1955), 123-127.

[14] C.-G. Li, M. Kostic, M. Li, Abstract multi-term fractional differential equations, *Kragujevac J. Math.* **38** (2014), 51-71.

[15] Y. Liu; Boundary value problems of singular multi-term fractional differential equations with impulse effects, *Math. Nachr.*, **289** (2016), 1526-1547.
[16] F. Mainardi; *Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Berlin, 1997, pp. 291-348.
[17] K. S. Miller, B.Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, NewYork, 1993.
[18] S. K. Ntouyas, J. Tariboon, W. Sudsutad; Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions, *Mediter. J. Math.*, **13** (2016), 939-954.
[19] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[20] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[21] S. Stanek; Periodic problem for two-term fractional differential equations, *Fract. Calc. Appl. Anal.*, **20** (2017), 662-678.
[22] P. J. Torvik, R. L. Bagley; On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.* **51** (1984), 294-298.

Addendum posted by the editor on May 2, 2018

A reader informed us that the second part of Lemma 2.4 is incorrect:

> "The converse of the lemma follows by direct computation" is not valid since solutions of (2.4) are found in the space $C[0,1]$ and it has to be shown that such a solution of (2.4) is $(2+\delta)$-Caputo differentiable for all $t \in (0,1)$ (or almost all).
>
> The authors should (probably) use the alternative definition of Caputo differential operator as given in K. Diethelm, The analysis of fractional differential equations. Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010.

The fifth author tried to prove the part needed, but instead decided to write "The converse of Lemma 2.4 remains an open problem under the current definition of fractional derivative"

End of addendum.

Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: aalsaedi@hotmail.com

Najla Alghamdi
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Department of Mathematics, Faculty of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia
E-mail address: njl-ghamdi@hotmail.com

Ravi P. Agarwal
Department of Mathematics, Texas A&M University, Kingsville, TX 78363-8202, USA.
Distinguished University Professor of Mathematics, Florida Institute of Technology, 150 West University Boulevard, Melbourne, FL 32901, USA
E-mail address: Ravi.Agarwal@tamuk.edu

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: sntouyas@uoi.gr

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: bashirahmad_qau@yahoo.com