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WELL-POSEDNESS OF DEGENERATE INTEGRO-DIFFERENTIAL EQUATIONS IN FUNCTION SPACES

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Abstract. We use operator-valued Fourier multipliers to obtain characterizations for well-posedness of a large class of degenerate integro-differential equations of second order in time in Banach spaces. We treat periodic vector-valued Lebesgue, Besov and Trieblel-Lizorkin spaces. We observe that in the Besov space context, the results are applicable to the more familiar scale of periodic vector-valued Hölder spaces. The equation under consideration are important in several applied problems in physics and material science, in particular for phenomena where memory effects are important. Several examples are presented to illustrate the results.

1. Introduction

In this article, we consider the following problem which consists in a second order degenerate integro-differential equation with infinite delay in a Banach space:

$$\begin{aligned}&(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds \\ &= \gamma u(t) + Au(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \quad 0 \le t \le 2\pi,\end{aligned} \tag{1.1}$$

and periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. Here, A, B, Λ and M are closed linear operators in a Banach space X satisfying the assumption $D(A)\cap D(B) \subset D(\Lambda)\cap D(M)$, $b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0, 2\pi]$, and γ is a constant. In case $M = 0$, the second boundary condition above becomes irrelevant and we are in the presence of a first order degenerate equation.

Equations of the form (1.1) appear in a variety of applied problems. The case where the memory effect is absent has been studied by many authors. The monograph [32] by Favini and Yagi is devoted to these problems and contains meaningful applications to concrete problems. Recently applications to inverse problems and in the context of multivalued operators have been investigated (see e.g. [31]). The book [44] by Melnikova and Filinkov also treats abstract degenerate equations.

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Evolutionary integro-differential equations arise typically in mathematical physics by constitutive laws pertaining to materials for which memory effects are important, when combined with the usual conservation laws such as balance of energy or balance of momentum. For details concerning the underlying physical principles, we refer to Coleman-Gurtin [24], Lunardi [43], Nunziato [45], and the monograph Prüss [50] (particularly Chapter II, Section 9) for work on the subject. The latter reference contains a wealth of results on general aspects of evolutionary integral equations and their relevance in concrete models from the physical sciences. Equations of first and second order in time are of interest. Typical examples for $b(\cdot)$ and $c(\cdot)$ are the completely monotonic functions $Ke^{-\omega t}t^{\mu}$ where $K \geq 0$, $\omega > 0$ and $\mu > -1$, and linear combinations thereof.

Several authors have considered particular cases of the above equation. Earlier papers: Lunardi [43], Da Prato-Lunardi [25, 26], Clement-Da Prato [21], Prüss [51], Nunziato [45], Alabau-Boussouira-Cannarsa-Sforza [1] and [53] for example, use various techniques for the solvability of problems of this type. In the case of Hilbert spaces, the results obtained by these authors are complete. This is due to the fact that Plancherel's theorem is available in Hilbert space. When X is not a Hilbert space, this is no longer the case because of Kwapien's theorem which states that the validity of Plancherel's theorem for X-valued functions requires X to be isomorphic to Hilbert space (see for example Arendt-Bu [7]). Beginning with the papers by Weis [56, 57], Arendt-Bu [7], Arendt-Batty-Bu [6], it became possible to completely characterize well-posedness of the problem in periodic vector-valued function spaces. Initially, Arendt and Bu [7] dealt with the problem $u'(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$. Maximal regularity for the evolution problem in L^p was treated earlier by Weis [56, 57] (see also [21] for a different proof of the operator-valued Mikhlin multiplier theorem using a transference principle). The study in the L^p framework (when $1 < p < \infty$) was made possible thanks to the introduction of the concept of randomized boundedness (hereafter R-boundedness, also known as Riesz-boundedness or Rademacher-boundedness). With this, necessary conditions for operator-valued Fourier multipliers were found in this context. In addition, the space X must have the UMD property. This was done initially by L. Weis [56, 57] for the evolutionary problem and then by Arendt-Bu [7] for periodic boundary conditions. For non-degenerate integro-differential equations both in the periodic and non periodic cases, operator-valued Fourier multipliers have been used by various authors to obtain well-posedness in various scales of function spaces: see [12, 15, 18, 35, 36, 37, 38, 48] and the corresponding references. The well-posedness or maximal regularity results are important in that they allow for the treatment of nonlinear problems. Earlier results on the application of operator-valued Fourier multiplier theorems to evolutionary integral equations can be found in [21]. More recent examples of second order integro-differential equations with frictional damping and memory terms have been studied in the paper [19]

We use the operator-valued Fourier multiplier theorems obtained by Arendt and Bu [8] on $B^s_{pq}(0, 2\pi; X)$, and Bu and Kim [17] on $F^s_{pq}(0, 2\pi; X)$ to give a characterization of well-posedness of (1.1) in these spaces in terms of operator-valued Fourier multipliers and then we derive concrete conditions that allow us to apply this characterization.

More recently, degenerate equations have attracted the attention of many authors. Both first and second order equations have been considered. The first order degenerate equation

$$(Mu)'(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \tag{1.2}$$

with periodic boundary condition $Mu(0) = Mu(2\pi)$, has been studied by Lizama and Ponce [42]; under suitable assumptions on the modified resolvent operator associated to (1.2), they gave necessary and sufficient conditions to ensure the well-posedness of (1.2) in Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$.

Recently Bu [13] studied the following second order degenerate equation

$$(Mu')'(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \tag{1.3}$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. He also obtained necessary and sufficient conditions to ensure the well-posedness of (1.3) in Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$ under some suitable conditions on the modified resolvent operator associated to (1.3). Operator-valued Fourier multiplier techniques have been used recently, most notably by Bu and Cai for handling degenerate problems in various classes of function spaces (see e.g. [14, 18].

For more references on degenerate equations and their relevance in concrete problems, we refer to the book [32] by Favini and Yagi. Other references are Barbu and Favini [32], Favaron and Favini [30] and Showalter [54, 55]. The latter author has studied extensively the class of Sobolev type equations.

When more than one unbounded operators are involved in (1.1), a strengthening of the definition of well-posedness is necessary. The resulting definition (Definition 3.4 below) which we provide, seems to be new in this context. In fact, our definition is parallel to the usual one for partial differential equations, in the sense of Hadamard, namely existence, uniqueness and continuous dependence of the solution on the data of the problem. The definition given is consistent with the previously adopted ones in the case where only one unbounded operator appears in the equation.

We study equation (1.1) in the spaces of 2π-periodic vector-valued functions, namely: Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$.

This article is organized as follows: in Section 2 we collect some preliminary results and definitions. In Section 3, we give necessary and sufficient conditions for well-posedness of the (1.1) in the Lebesgue Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin $F^s_{pq}(0, 2\pi; X)$ spaces in terms of operator-valued Fourier multipliers. In Section 4, we give concrete conditions on the data ensuring applicability of the results established in Section 3. We stress that in the L^p case, the results use the concept of R-boundedness and require the space X to be UMD (this is equivalent to the continuity of the Hilbert transform on $L^p(\mathbb{R}, X)$, $1 < p < \infty$). The the concept of R-boundedness first appeared in the context of evolution equations in the papers [56, 57] of Weis (see also the article [34]).

In the other cases (namely $B^s_{pq}(0, 2\pi; X)$ and $F^s_{pq}(0, 2\pi; X)$), these restrictions are no longer needed but one requires instead higher order boundedness conditions on the "modified resolvents" involved.

In the final Section 5, we consider some examples where the results above apply. We single out the following modified version of problem which is considered in

Favini-Yagi [32, Example 6.1]

$$
\begin{gathered}
\frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t}) - \Delta\frac{\partial u(t,x)}{\partial t} \\
= \Delta u(t,x) + \int_{-\infty}^{t} b(t-s)\Delta u(s,x)ds + f(t,x), \quad (t,x) \in [0,2\pi]\times\Omega, \\
u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0, \quad (t,x) \in [0,2\pi]\times\partial\Omega, \\
u(0,x) = u(2\pi,x), \quad m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x \in \Omega,
\end{gathered}
\tag{1.4}
$$

where $\Omega \subset \mathbb{R}^n$ is an open subset and Δ is the Laplace operator. We consider the problem in the space $X = L^r(\Omega)$, $1 < r < \infty$. This is a degenerate wave equation with memory and a damping term. We treat the problem for periodic boundary conditions. The authors of the cited papers also study the evolutionary problem as well, including asymptotic behavior of solutions. They consider only the case when $a = 0$, that is they do not incorporate the memory term in the equation. They restrict their study to the Hölder spaces. For periodic boundary conditions, we obtain complete characterization of well-posedness in the three scales of spaces: L^p, B^s_{pq}, and F^s_{pq}. We are also able to treat this problem replacing Δ with $-\Delta$ in the right hand side. The latter equation is the focus of the reference [32].

2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let X be a complex Banach space. We denote as usual by $L^1(0,2\pi,X)$ the space of Bochner integrable functions with values in X. For a function $f \in L^1(0,2\pi;X)$, we denote by $\hat{f}(k), k \in \mathbb{Z}$ the kth Fourier coefficient of f:

$$
\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt,
$$

where $e_k(t) = e^{ikt}, t \in \mathbb{R}$.

Let $u \in L^1(0,2\pi;X)$. We denote again by u its periodic extension to $\mathbb{R}$. Let $a \in L^1(\mathbb{R}_+)$. We consider the the function

$$
F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds, \quad t \in \mathbb{R}.
$$

Since

$$
F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_0^{\infty} a(s)u(t-s)ds, \tag{2.1}
$$

we have $\|F\|_{L^1} \le \|a\|_1\|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}_+)}\|u\|_{L^1(0,\,2\pi;X)}$ and F is periodic of period $T = 2\pi$ as u. Now using Fubini's theorem and (2.1) we obtain, for $k \in \mathbb{Z}$, that

$$
\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), k \in \mathbb{Z} \tag{2.2}
$$

where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t}a(t)dt$ denotes the Laplace transform of a. This identity plays a crucial role in the paper.

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{L}(X)$.

For results on operator-valued Fourier multipliers and R-boundedness (used in the next section), as well as some applications to evolutionary partial differential equations, we refer to Amann [2], Bourgain [10, 11], Clément-de Pagter-Sukochev-Witvliet [22], Weis [56, 57], Girardi-Weis [33], [34], Kunstmann-Weis [39], Clément-Prüss [23], Arendt [4] and Arendt-Bu [7]. The scalar case is presented for example in Schmeisser-Triebel [52, Chapter 3]. This reference also considers the case where X is a Hilbert space (Chapter 6). Here, we will merely present the appropriate definitions.

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, in this section, we consider the spaces:

Lebesgue-Bochner spaces. For $1 \leq p \leq \infty$, we denote $L^p_{2\pi}(\mathbb{R}; X)$ (denoted also $L^p(0, 2\pi; X)$, $1 \leq p \leq \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable, usual modification if $p = \infty$. The space is equipped with the norm

$$\|f\|_p = \|f\|_{L^p(0,2\pi,X)} = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|_X^p dt\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t\in[0,2\pi]} \|f(t)\|_X & \text{if } p = \infty. \end{cases} \tag{2.3}$$

Besov spaces. We briefly recall the the definition of 2π-periodic Besov space in the vector-valued case introduced in [8]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(0, 2\pi)$ be the space of all infinitely differentiable functions on $[0, 2\pi]$ equipped with the locally convex topology given by the family of seminorms

$$\|f\|_\alpha = \sup_{x\in[0,2\pi]} |f^{(\alpha)}(x)|$$

for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(0, 2\pi, X) := \mathcal{L}(\mathcal{D}(0, 2\pi), X)$ be the space of all bounded linear operators from $\mathcal{D}(0, 2\pi)$ to X (X-valued distributions). In order to define the Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \leq 2\},\ I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \leq 2^k\}$$

for $k \in \mathbb{N}$. Let $\Phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi_k) \subset \overline{I}_k$ for each $k \in \mathbb{N}_0$, $\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1$ for $x \in \mathbb{R}$, and for each $\alpha \in \mathbb{N}_0$, $\sup_{x\in\mathbb{R},k\in\mathbb{N}_0} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty$. Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \Phi(\mathbb{R})$ be fixed. For $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, the X-valued 2π-periodic Besov space is denoted by $B^s_{pq}(0, 2\pi, X)$ and defined by the set

$$\Big\{f \in \mathcal{D}'(0, 2\pi; X) : \|f\|^s_{pq} := \Big(\sum_{j\geq 0} 2^{sjq}\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\|_p^q\Big)^{1/q} < \infty\Big\}$$

with the usual modification if $q = \infty$.

It is known that $B^s_{pq}(0, 2\pi, X)$ is independent of the choice of ϕ, and different choices of ϕ in the class $\Phi(\mathbb{R})$ lead to equivalent norms $\|\cdot\|^s_{pq}$. Equipped with the norm $\|\cdot\|^s_{pq}$, $B^s_{pq}(0, 2\pi, X)$ is a Banach space.

It is also known that is $s_1 \leq s_2$, then $B^{s_2}_{pq}(0, 2\pi, X) \subset B^{s_1}_{pq}(0, 2\pi, X)$ and the embedding is continuous [8]. When $s > 0$, it is proved in [8] that $B^s_{pq}(0, 2\pi, X) \subset L^p(0, 2\pi, X)$ and the embedding is continuous; moreover, $f \in B^{s+1}_{pq}(0, 2\pi, X)$ if and only if f is differentiable a.e on $[0, 2\pi]$ and $f' \in B^s_{pq}(0, 2\pi, X)$. In the case where

$p = q = \infty$ and $0 < s < 1$ we have that $B^s_{\infty\infty}(0, 2\pi, X)$ corresponds to the space $C^s(0, 2\pi, X)$ of Hölder continuous functions with equivalent norm

$$\|f\|_{C^s(0,2\pi;X)} = \sup_{t_1 \neq t_2} \frac{\|f(t_2) - f(t_1)\|_X}{|t_2 - t_1|^s} + \|f\|_\infty.$$

Triebel-Lizorkin spaces. Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \Phi(\mathbb{R})$ be fixed with ϕ and $\Phi(\mathbb{R})$ as above. For $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, the X-valued 2π-periodic Triebel-Lizorkin space with parameters s, p and q is denoted by $F^s_{pq}(0, 2\pi; X)$ and defined by the set

$$\Big\{ f \in \mathcal{D}'(0, 2\pi, X) : \|f\|^s_{pq} := \big\| \big(\sum_{j\ge 0} 2^{sjq} \| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \|^q_X \big)^{1/q} \big\|_p < \infty \Big\}$$

with the usual modification if $q = \infty$.

It is known that set $F^s_{pq}(0, 2\pi, X)$ is independent of the choice of ϕ, and again, different choices of ϕ lead to equivalent norms $\| \cdot \|^s_{pq}$. Equipped with the norm $\| \cdot \|^s_{pq}$, $F^s_{pq}(0, 2\pi, X)$ is a Banach space.

It is also known that if $s_1 \le s_2$, then $F^{s_2}_{pq}(0, 2\pi, X) \subset F^{s_1}_{pq}(0, 2\pi, X)$ and the embedding is continuous [17]. When $s > 0$, it is show in [17] that $F^s_{pq}(0, 2\pi, X) \subset L^p(0, 2\pi, X)$ and the embedding is continuous; moreover, $f \in F^{s+1}_{pq}(0, 2\pi, X)$ if and only if f is differentiable a.e on $[0, 2\pi]$ and $f' \in F^s_{pq}(0, 2\pi, X)$. The exceptional case $p = \infty$ will not be considered in this paper. We refer to Schmeisser-Triebel [52, Section 3.4.2] for a discussion. Note that $F^s_{pp}((0, 2\pi); X) = B^s_{pp}((0, 2\pi); X)$ by an inspection of the definitions.

We give the definition of operator-valued Fourier multipliers in each of the cases that will be of interest to us. First, in the case of Lebesgue spaces, we have: (See [7, 8, 17]).

Definition 2.1. Let X and Y be Banach spaces. For $1 \le p \le \infty$, we say that a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-multiplier, if for each $f \in L^p(0, 2\pi; X)$ there exists $u \in L^p(0, 2\pi; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

In the case of Besov spaces, we have the following concept.

Definition 2.2. Let X and Y be Banach spaces. For $1 \le p, q \le \infty$, $s > 0$, we say that a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an B^s_{pq}-multiplier, if for each $f \in B^s_{pq}(0, 2\pi; X)$ there exists $u \in B^s_{pq}(0, 2\pi; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

Finally, in the case of Triebel-Lizorkin spaces, we have the following concept.

Definition 2.3. Let X and Y be Banach spaces. For $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$, we say that a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an F^s_{pq}-multiplier, if for each $f \in F^s_{pq}(0, 2\pi; X)$ there exists $u \in F^s_{pq}(0, 2\pi; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

From the uniqueness theorem of Fourier series, it follows that u is uniquely determined by f in each of the above mentioned cases.

We denote by $\mathcal{Y} = \mathcal{Y}(X)$ any of the following spaces of X-valued functions: $L^p(0,2\pi;X)$, $1 \le p \le \infty$; $B^s_{pq}(0,2\pi;X)$, $1 \le p,q \le \infty$, $s > 0$; $F^s_{pq}(0,2\pi;X)$, $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$. We define the sets

$$\mathcal{Y}^{[1]} = \{u \in \mathcal{Y} : u \text{ is almost everywhere differentiable and } u' \in \mathcal{Y}\},$$

$$\mathcal{Y}^{[1]}_{\rm per} = \{u \in \mathcal{Y} : \exists v \in \mathcal{Y}, \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}$$

In the case that $\mathcal{Y} = L^p(0,2\pi;X)$, $\mathcal{Y}^{[1]}$ is denoted by $W^{1,p}(0,2\pi;X)$ and $\mathcal{Y}^{[1]}_{\rm per}$ by $W^{1,p}_{\rm per}(0,2\pi;X)$. In the case that $\mathcal{Y} = B^s_{pq}(0,2\pi;X)$, $\mathcal{Y}^{[1]} = B^{s+1}_{pq}(0,2\pi;X)$. In the case that $\mathcal{Y} = F^s_{pq}(0,2\pi;X)$, $\mathcal{Y}^{[1]} = F^{s+1}_{pq}(0,2\pi;X)$.

Remark 2.4. Using integration by parts, the fact that $\mathcal{Y} \subset L^1(0,2\pi,X)$ and the uniqueness theorem of Fourier coefficients, we have

$$\begin{gathered} \mathcal{Y}^{[1]}_{\rm per} = \{u \in \mathcal{Y}^{[1]} : u(0) = u(2\pi)\}, \\ \mathcal{Y}^{[1]}_{\rm per} = \{u \in \mathcal{Y}^{[1]} : \widehat{u}'(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}. \end{gathered} \tag{2.4}$$

Therefore, if $u \in \mathcal{Y}^{[1]}_{\rm per}$, then u has a unique continuous representative such that $u(0) = u(2\pi)$. We always identify u with this continuous function.

Remark 2.5. It is clear from the definitions that:

(a) if $(M_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ are $\mathcal{Y}$-Fourier multipliers and α,β are constants, then $(\alpha M_k + \beta N_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is a $\mathcal{Y}$-Fourier multiplier as well.

(b) if $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ and $(N_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(Y,Z)$ are $\mathcal{Y}$-Fourier multipliers, then $(N_kM_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Z)$ is a $\mathcal{Y}$-Fourier multiplier as well. In particular, when $X = Y = Z$, if $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers, then $(N_kM_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier as well.

Proposition 2.6 ([7, Fejer's Theorem])**.** *Let $f \in L^p(0,2\pi;X))$, then one has*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with convergence in $L^p(0,2\pi;Y))$.

Remark 2.7. (a) If $(kM_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier, then $(M_k)_{k\in\mathbb{Z}}$ is also a $\mathcal{Y}$-Fourier multiplier.

(b) If $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is a $\mathcal{Y}$-Fourier multiplier, then there exists a bounded linear operator $T \in \mathcal{L}(\mathcal{Y}(X),\mathcal{Y}(Y))$ satisfying $\widehat{(Tf)}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. This implies in particular that the sequence $(M_k)_{k\in\mathbb{Z}}$ must be bounded.

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0,1]$, i.e. $r_j(t) = sgn(\sin(2^j\pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of R-bounded for a given family of bounded linear operators is defined as follows.

Definition 2.8. A family $\mathbf{T} \subset \mathcal{L}(X,Y)$ is called R-bounded if there exists $c_q \ge 0$ such that

$$\|\sum_{j=1}^{n} r_j \otimes T_jx_j\|_{L^q(0,1;X)} \le c_q\|\sum_{j=1}^{n} r_j \otimes x_j\|_{L^q(0,1;X)} \tag{2.5}$$

for all $T_1,\dots,T_n \in \mathbf{T}, x_1,\dots,x_n \in X$ and $n \in \mathbb{N}$, where $1 \le q < \infty$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (2.5) holds.

Remark 2.9. Several useful properties of R-bounded families can be found in the monograph of Denk-Hieber-Prüss [28, Section 3], see also [4, 7, 22, 47, 39]. We collect some of them here for later use.

(a) Any finite subset of $\mathcal{L}(X)$ is is R-bounded.
(b) If $\mathbf{S} \subset \mathbf{T} \subset \mathcal{L}(X)$ and $\mathbf{T}$ is R-bounded, then $\mathbf{S}$ is R-bounded and $R_p(\mathbf{S}) \leq R_p(\mathbf{T})$.
(c) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S} \cdot \mathbf{T} := \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded and
$$R_p(\mathbf{S} \cdot \mathbf{T}) \leq R_p(\mathbf{S}) \cdot R_p(\mathbf{T}).$$
(d) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S}+\mathbf{T} := \{S+T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R- bounded and
$$R_p(\mathbf{S} + \mathbf{T}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$
(e) If $\mathbf{T} \subset \mathcal{L}(X)$ is R- bounded, then $\mathbf{T} \cup \{0\}$ is R-bounded and $R_p(\mathbf{T} \cup \{0\}) = R_p(\mathbf{T})$.
(f) If $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R- bounded, then $\mathbf{T} \cup \mathbf{S}$ is R-bounded and
$$R_p(\mathbf{T} \cup \mathbf{S}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$
(g) Also, each subset $M \subset \mathcal{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded (I denotes the identity operator on X).

The proofs of (a), (e), (f), and (g) rely on Kahane's contraction principle.

We sketch a proof of (f). Since we assume that $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R-bounded, it follows from (e) (which is a consequence of Kahane's contraction principle) that $\mathbf{S} \cup \{0\}$ and $\mathbf{T} \cup \{0\}$ are R-bounded. We now observe that $\mathbf{S} \cup \mathbf{T} \subset \mathbf{S} \cup \{0\} + \mathbf{T} \cup \{0\}$. Then using (d) and (b) we conclude that $\mathbf{S} \cup \mathbf{T}$ is R-bounded.

We make the following general observation which will be valid throughout the paper, notably in Section 4. Whenever we wish to establish R-boundedness of a family of operators $(M_k)_{k \in \mathbb{Z}}$, if at some point we make an exception such as $(k \neq 0)$, $(k \notin \{-1, 0\})$ and so on, then later we recover the property for the entire family using items (a), (c) and (f) of the foregoing remark. The corresponding observation for boundedness is clear.

Remark 2.10. If $X = Y$ is a UMD space and $M_k = m_k I$ with $m_k \in \mathbb{C}$, then the Marcinkiewicz condition $\sup_k |m_k| + \sup_k |k(m_{k+1} - m_k)| < \infty$ implies that the set $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p-multiplier. (see [7] or [2, Theorem 4.4.3]).

Another important notion in Banach space theory is that of Fourier type for a Banach space. Conditions for Fourier multipliers are simplified when the Banach spaces involved satisfy this condition. The Hausdorff-Young inequality states that for $1 \leq p \leq 2$, the Fourier transform maps $L^p(\mathbb{R}) := L^p(\mathbb{R}; \mathbb{C})$ continuously into $L^{p'}(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention that $p' = \infty$ when $p = 1$. In particular, when $p = 2$, Plancherel's theorem holds. When X is a Banach space and one considers $L^p(\mathbb{R}; X)$, the situation is no longer the same. It is known that Plancherel's theorem (here we mean L^2-continuity of the $X-$valued Fourier transform) holds if and only if X is isomorphic to a Hilbert space (see e.g. [2, 6, 7, 34]). For every Banach space, the Hausdorff-Young theorem holds with $p = 1$. A Banach space is said to have non-trivial Fourier type if the Hausdorff-Young theorem holds true for some $p \in (1, 2]$. By a result of Bourgain [10, 11], UMD spaces are examples of spaces with nontrivial Fourier type (see [34, 5]). More generally,

B-convex spaces, in particular superreflexive Banach spaces have nontrivial Fourier type ([11, Proposition 3]). However, there exist non reflexive Banach spaces with nontrivial Fourier type. The implications of the property of having non trivial Fourier type are studied in Giradi-Weis [34].

For Banach spaces with non trivial Fourier type, in particular for UMD spaces, the conditions for the validity of operator-valued Fourier multiplier theorems are greatly simplified.

3. Characterization in terms of Fourier multipliers

In this section, we characterize the well-posedness of the problem

$$\begin{gathered} (Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds \\ = \gamma u(t) + Au(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \quad 0 \le t \le 2\pi, \\ u(0) = u(2\pi) \quad \text{and} \quad (Mu)'(0) = (Mu)'(2\pi) \end{gathered} \tag{3.1}$$

in the vector-valued Lebesgue, Besov, and Triebel-Lizorkin spaces. Here A, B, Λ and M are closed linear operators in a Banach space X satisfying $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$, $b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0, 2\pi]$, and γ is a constant. The results are in terms of operator-valued Fourier multipliers.

Let b, c be complex valued functions and γ a constant. We define the M, Λ-resolvent set of A and B, $\rho_{\Lambda,M,\tilde b,\tilde c}(A, B)$, associated to (3.1) by

$$\{\lambda \in \mathbb{C} | \mathcal{M}(\lambda) : D(A) \cap D(B) \to X \text{ is bijective and } [\mathcal{M}(\lambda)]^{-1} \in \mathcal{L}(X)\}$$

where $\mathcal{M}(\lambda) = \lambda^2 M - A - \tilde b(\lambda)B - \lambda\Lambda - \lambda\tilde c(\lambda)I - \gamma I$. Thus, $\lambda \in \rho_{\Lambda,M,\tilde a,\tilde b,\tilde c}(A, B)$ if and only if $[\mathcal{M}(\lambda)]^{-1}$ is a linear continuous isomorphism from X onto $D(A) \cap D(B)$. Here we consider $D(A)$, $D(B)$, $D(\Lambda)$ and $D(M)$ as normed spaces equipped with their respective graph norms. These are Banach space since all the operators are closed. For $a \in L^1(\mathbb{R}_+)$, $u \in \mathcal{Y}$, we denote by $a * u$ the function

$$(a * u)(t) := \int_{-\infty}^{t} a(t-s)u(s)ds \tag{3.2}$$

Since $\mathcal{Y} \subset L^1(0, 2\pi; X)$, it follows that $a*u \in L^1(0, 2\pi; X)$ and $(a*u)(0) = (a*u)(2\pi)$ by (2.1). With this notation we may rewrite (1.1) in the following way:

$$(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}(c * u)(t) = \gamma u(t) + Au(t) + (b * Bu)(t) + f(t), \quad 0 \le t \le 2\pi.$$

If b, $c \in L^1(\mathbb{R}_+)$ and $u \in L^1(0, 2\pi; D(A)) \cap L^1(0, 2\pi; D(B))$, then $c * u$, $b * Bu \in L^1(0, 2\pi; X)$ by (2.1) and $\widehat{(c * u)}(k) = \tilde c(ik)\hat u(k)$, $\widehat{(a * Au)}(k) = \tilde a(ik)A\hat u(k)$ and $\widehat{(b * Bu)}(k) = \tilde b(ik)B\hat u(k)$ by (2.2). If additionally we have that $\frac{d}{dt}(c * u) \in L^1(0, 2\pi; X)$, then $c * u \in W^{1,1}(0, 2\pi; X)$ and $(c * u)(0) = (c * u)(2\pi)$. Then $\widehat{\frac{d}{dt}(c * u)}(k) = ik\tilde c(ik)\hat u(k)$ by (2.4).

In what follows, we adopt the following notation:

$$b_k := \tilde b(ik), c_k := \tilde c(ik) \tag{3.3}$$

Remark 3.1. By the Riemann-Lebesgue lemma, the sequences $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$ so defined are bounded. In fact $\lim_{|k|\to\infty} b_k = 0$, and similarly for $(c_k)_{k\in\mathbb{Z}}$. Moreover, $(b_kI)_{k\in\mathbb{Z}}$ and $(c_kI)_{k\in\mathbb{Z}}$ define a $\mathcal{Y}$-Fourier multiplier.

We now give the definition of solutions of (3.1) in our relevant cases.

Definition 3.2. A function $u \in \mathcal{Y}$ is called a *strong $\mathcal{Y}$-solution* of (3.1) if $u \in \mathcal{Y}(D(A)) \cap \mathcal{Y}(D(B)) \cap \mathcal{Y}_{\rm per}^{[1]}$, $u' \in \mathcal{Y}(D(\Lambda)) \cap \mathcal{Y}(D(M))$, $Mu' \in \mathcal{Y}_{\rm per}^{[1]}$, and equation (1.1) holds for almost all $t \in [0, 2\pi]$.

Lemma 3.3. *Let X be a Banach space, and A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that γ is a constant, $b, c \in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in* (3.3). *Assume that u is a strong $\mathcal{Y}$-solution of* (3.1). *Then*

$$[-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I]\hat{u}(k) = \hat{f}(k).$$

for all $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. Since u is a strong $\mathcal{Y}$-solution of (3.1), $u \in \mathcal{Y}(D(A)) \cap \mathcal{Y}(D(B)) \cap \mathcal{Y}_{\rm per}^{[1]}$, $u' \in \mathcal{Y}(D(\Lambda)) \cap \mathcal{Y}(D(M))$, $Mu' \in \mathcal{Y}_{\rm per}^{[1]}$ and

$$\begin{aligned}
&(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}(c * u)(t) \\
&= \gamma u(t) + Au(t) + (b * Bu)(t) + f(t), \quad \text{for a.e } t \in [0, 2\pi].
\end{aligned}$$

Since $u \in \mathcal{Y}(D(A)) \cap \mathcal{Y}(D(B))$, we have

$$\hat{u}(k) \in D(A) \cap D(B) \quad \text{and} \quad \widehat{Au}(k) = A\hat{u}(k), \hat{B}u(k) = B\hat{u}(k).$$

by [7, Lemma 3.1]. Since $u \in \mathcal{Y}_{\rm per}^{[1]}$, we have $\widehat{u}'(k) = ik\hat{u}(k)$ by (2.4). Since $u' \in \mathcal{Y}(D(\Lambda)) \cap \mathcal{Y}(D(M))$, it follows that $\widehat{(\Lambda u')} = \Lambda\widehat{u}'(k) = ik\Lambda\hat{u}(k)$, $\widehat{Mu'} = M\widehat{u}'(k) = ikM\hat{u}(k)$ by [7, Lemma 3.1]. Since $Mu' \in \mathcal{Y}_{\rm per}^{[1]}$, it follows that $\widehat{(Mu')}' = ik\widehat{Mu}'(k) = -k^2M\hat{u}(k)$ by (2.4). Since $u \in \mathcal{Y}(D(A)) \subset L^1(0, 2\pi; D(A))$, $u \in \mathcal{Y}(D(B)) \subset L^1(0, 2\pi; D(B))$ and b, $c \in L^1(\mathbb{R}_+)$, it follows that $c * u$, $b * Bu \in L^1(0, 2\pi; X)$, $(c*u)(0) = (c*u)(2\pi)$ by (2.1) and $\widehat{(c * u)}(k) = \tilde{c}(ik)\hat{u}(k)$, $\widehat{(b * Bu)}(k) = \tilde{b}(ik)B\hat{u}(k)$ by (2.2). Since $\mathcal{Y} \subset L^1(0, 2\pi; X)$, we have u, $\Lambda u'$, $(Mu')'$ and $f \in L^1(0, 2\pi; X)$. So u, Au, Bu, $b * Bu$, $\Lambda u'$, $(Mu')'$ and f all belong to $L^1(0, 2\pi; X)$. Then $\frac{d}{dt}(c * u)$ must be in $L^1(0, 2\pi; X)$. Therefore $c * u \in W_{\rm per}^{1,1}(0, 2\pi; X)$ and $\widehat{\frac{d}{dt}(c * u)}(k) = ik\tilde{c}(ik)\hat{u}(k)$ by (2.4).

Taking Fourier series on both sides of (1.1) we obtain

$$[-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I]\hat{u}(k) = \hat{f}(k), \quad k \in \mathbb{Z}.$$

□

When (3.1) is $\mathcal{Y}$ well-posed, the map $\mathcal{S} : \mathcal{Y} \to \mathcal{Y}$, $f \mapsto u$ where u is the unique strong solution, is linear. We adopt the following definition of well-posedness.

Definition 3.4. We say that (3.1) is $\mathcal{Y}$-well-posed, if for each $f \in \mathcal{Y}$, there exists a unique strong $\mathcal{Y}$-solution u of (3.1) which depends continuously on f in the sense that the operator $\mathcal{S} : \mathcal{Y} \to \mathcal{Y}$ defined by $\mathcal{S}(f) = u$ where u is the unique strong $\mathcal{Y}$-solution of (3.1) is continuous.

Remark 3.5. We note that, according to Section 2, [7, 8, 17], all the spaces of vector-valued functions $\mathcal{Y}$ concerned in this paper are continuously embedded in $L^1(0, 2\pi, X)$. It follows that: If $f_n \to f$ in $\mathcal{Y}$, then $f_n \to f$ in $L^1(0, 2\pi, X)$ and consequently for each $k \in \mathbb{Z}$, $\lim_{n\to\infty} \hat{f}_n(k) = f(k)$ in X.

Our definition imposes an additional condition to that given in the previous works such as [13], [42] that allows us to establish the following characterization of well-posed of (3.1) in terms of Fourier multipliers. Actually, the above definition stems from the Hadamard concept of well-posedness in partial differential equations. We refer for example to [29] and [6] for the presentation of this fundamental concept.

Theorem 3.6. *Let X be a Banach space and A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that γ is a constant, $b, c \in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in* (3.3)*. Then the following assertions are equivalent.*

- (i) (3.1) *is $\mathcal{Y}$-well-posed.*
- (ii) *$i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A, B)$ and $(k^2 MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers, where*
$$N_k = [k^2 M + A + b_k B + ik\Lambda + ikc_k I + \gamma I]^{-1}$$

In this case the following maximal regularity property holds: The unique strong $\mathcal{Y}$-solution u is such that Au, $b * Bu$, Λu, $\Lambda u'$, $c * u$, $\frac{d}{dt}(c * u)$, Mu, Mu' and $(Mu')'$ all belong to $\mathcal{Y}$ and there exists a constant $C > 0$ independent of $f \in \mathcal{Y}$ such that
$$\begin{aligned} &\|u\|_{\mathcal{Y}} + \|Au\|_{\mathcal{Y}} + \|b * Bu\|_{\mathcal{Y}} + \|\Lambda u\|_{\mathcal{Y}} + \|\Lambda u'\|_{\mathcal{Y}} + \|c * u\|_{\mathcal{Y}} \\ &+ \|\frac{d}{dt}(c * u)\|_{\mathcal{Y}} + \|Mu\|_{\mathcal{Y}} + \|Mu'\|_{\mathcal{Y}} + \|(Mu')'\|_{\mathcal{Y}} \le C\|f\|_{\mathcal{Y}} \end{aligned}$$

Proof. (i) $\Rightarrow$ (ii). Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. Then $\hat{f}(k) = y$. By assumption, there exists a unique strong $\mathcal{Y}$-solution u of (3.1). By Lemma 3.3, we have that for all $k \in \mathbb{Z}$,
$$[-k^2 M - A - b_k B - ik\Lambda - ikc_k I - \gamma I]\hat{u}(k) = y$$
It follows that
$$[-k^2 M - A - b_k B - ik\Lambda - ikc_k I - \gamma I]$$
is surjective for each $k \in \mathbb{Z}$. Next we prove that for each $k \in \mathbb{Z}$,
$$[-k^2 M - A - b_k B - ik\Lambda - ikc_k I - \gamma I]$$
is injective. Let $x \in D(A) \cap D(B)$ such that
$$[-k^2 M - A - b_k B - ik\Lambda - ikc_k I - \gamma I]x = 0 \tag{3.4}$$
Define $u(t) = e^{ikt}x$ when $t \in [0, 2\pi]$. Then $\hat{u}(k) = x$ and $\hat{u}(n) = 0$ for all $n \in \mathbb{Z}$, $n \neq k$. By (3.4) we have
$$\widehat{(Mu')'}(n) - \widehat{\Lambda u'}(n) - \widehat{\frac{d}{dt}(c * u)}(n) = \gamma\hat{u}(n) + \widehat{Au}(n) + (\widehat{b * Bu})(n),$$
for all $n \in \mathbb{Z}$. From uniqueness theorem of Fourier coefficients, we conclude that u satisfies
$$(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}(c * u)(t) = \gamma u(t) + Aw(t) + (b * Bu)(t)$$
for almost all $t \in [0, 2\pi]$. Thus u is a strong $\mathcal{Y}$-solution of (3.1) with $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that
$$[-k^2 M - A - b_k B - ik\Lambda - ikc_k I - \gamma I]$$
is injective for each $k \in \mathbb{Z}$. Now we show that
$$N_k = [k^2 M + A + b_k B + ik\Lambda + ikc_k I + \gamma I]^{-1} \in \mathcal{L}(X)$$

Let $k \in \mathbb{Z}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to x$. For each $n \in \mathbb{N}$ we define $f_n(t) = e^{ikt}x_n$ and $f(t) = e^{ikt}x$. Then $f_n, f \in \mathcal{Y}$, for every $n \in \mathbb{N}$ and $f_n \to f$ in $\mathcal{Y}$. Since (3.1) is $\mathcal{Y}$-well-posed, for each $f_n, f \in \mathcal{Y}$ there exists a unique strong $\mathcal{Y}$-solution $\mathcal{S}(f_n) = u_n$, $\mathcal{S}(f) = u$. Since $f_n \to f$ in $\mathcal{Y}$, we have $u_n \to u$ in $\mathcal{Y}$ by continuity of $\mathcal{S}$. Therefore $\hat{u}_n(k) \to \hat{u}(k)$ by Remark 3.5. Since

$$-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I$$

is bijective, we obtain $\hat{u}_n(k) = -N_kx_n, \hat{u}(k) = -N_kx$ by Lemma 3.3; then $N_kx_n \to N_kx$. Thus by the Closed Graph Theorem, $N_k \in \mathcal{L}(X)$. Thus $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A,B)$.

We now set for each $k \in \mathbb{Z}$:

$$\begin{gathered} M_k = k^2MN_k \quad B_k = AN_k \\ S_k = BN_k \quad H_k = kN_k. \end{gathered}$$

Next we show that $(M_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(H_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers. Since $N_k \in \mathcal{L}(X)$, B, Λ, M are closed, M_k, B_k, H_k and S_k are bounded for all $k \in \mathbb{Z}$. Now let $f \in \mathcal{Y}$, then there exists a strong $\mathcal{Y}$-solution u of (3.1). Then $\hat{u}(k) = -N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 3.3. Therefore

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$. Since B is closed,

$$\widehat{Bu}(k) = B\hat{u}(k) = -BN_k\hat{f}(k) = -B_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ by [7, Lemma 3.1]. Since Λ, M are closed, $u \in \mathcal{Y}^{[1]}_{\rm per}$, $u' \in \mathcal{Y}(D(\Lambda)) \cap \mathcal{Y}(D(M))$, and $Mu' \in \mathcal{Y}^{[1]}_{\rm per}$, we have

$$\begin{gathered} \widehat{u}'(k) = ik\hat{u}(k) = -ikN_k\hat{f}(k) = -iH_k\hat{f}(k), \\ \widehat{\Lambda u'}(k) = \Lambda\widehat{u}'(k) = ik\Lambda\hat{u}(k) = -ik\Lambda N_k\hat{f}(k) = -iS_k\hat{f}(k), \\ \widehat{(Mu')'}(k) = ik\widehat{Mu'}(k) = ikM\widehat{u}'(k) = -k^2M\hat{u}(k) = k^2MN_k\hat{f}(k) = M_k\hat{f}(k) \end{gathered}$$

for all $k \in \mathbb{Z}$ by (2.4) and [7, Lemma 3.1]. It follows that $(M_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(H_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers. Therefore the implication (i) $\Rightarrow$ (ii) is true.

(ii) $\Rightarrow$ (i). Since

$$k^2MN_k + AN_k + b_kBN_k + ik\Lambda N_k + ikc_kN_k + \gamma N_k = I,$$

we have

$$AN_k = I - \left(k^2MN_k + AN_k + b_kBN_k + ikc_kN_k + \gamma N_k\right)$$

for each $k \in \mathbb{Z}$. Therefore, $(AN_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier by Remarks 2.5, 2.7, and 3.1. Since $(k^2MN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers, it follows that $(N_k)_{k\in\mathbb{Z}}$, $(ikc_kN_k)_{k\in\mathbb{Z}}$, $(c_kN_k)_{k\in\mathbb{Z}}$, $(ikN_k)_{k\in\mathbb{Z}}$, $(ik\Lambda N_k)_{k\in\mathbb{Z}}$, $(\Lambda N_k)_{k\in\mathbb{Z}}$, $(-k^2MN_k)_{k\in\mathbb{Z}}$ $(ikMN_k)_{k\in\mathbb{Z}}$, and $(MN_k)_{k\in\mathbb{Z}}$ are also $\mathcal{Y}$-Fourier multipliers again by Remarks 2.5, 2.7, and 3.1. From the fact that $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(\Lambda N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, and $(c_kN_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers, then for all $f \in \mathcal{Y}$, we conclude that exist u, v_1, v_2, v_3, v_4, and $v_5 \in \mathcal{Y}$ such that

$$\hat{u}(k) = N_k\hat{f}(k), \tag{3.5}$$

and

$$\begin{aligned}
\hat{v}_1(k) &= AN_k\hat{f}(k) = A\hat{u}(k) = \widehat{Au}(k),\\
\hat{v}_2(k) &= BN_k\hat{f}(k) = B\hat{u}(k) = \widehat{Bu}(k),\\
\hat{v}_3(k) &= \Lambda N_k\hat{f}(k) = \Lambda\hat{u}(k) = \widehat{\Lambda u}(k),\\
\hat{v}_4(k) &= MN_k\hat{f}(k) = M\hat{u}(k) = \widehat{Mu}(k),\\
\hat{v}_5(k) &= c_kN_k\hat{f}(k) = c_k\hat{u}(k) = \widehat{c*u}(k),
\end{aligned} \tag{3.6}$$

for all $k \in \mathbb{Z}$ by the closedness of A, B, Λ, M, and (2.2). Since $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A,B)$, it follows that

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$ by (3.5). Since A, B, Λ, and M are closed,

$$u(t) \in D(A) \cap D(B)$$

and $Au(t) = v_1(t)$, $Bu(t) = v_2(t)$, $\Lambda u(t) = v_3(t)$, $Mu(t) = v_4(t)$ and $(c*u)(t) = v_5(t)$ a.e. $t \in [0, 2\pi]$ by (3.6) and [7, Lemma 3.1] (here we also use the fact that $\mathcal{Y} \subset L^p(0, 2\pi, X)$). Therefore

$$u \in \mathcal{Y}(D(A)) \cap \mathcal{Y}(D(B)),$$

and $c*u$, Λu, $Mu \in \mathcal{Y}$. Since $(ikN_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier, there exists $v_6 \in \mathcal{Y}$ such that

$$\hat{v}_6(k) = ikN_k\hat{f}(k) = ik\hat{u}(k) \in D(\Lambda) \cap D(M). \tag{3.7}$$

for all $k \in \mathbb{Z}$. Therefore by (2.4) and (3.7), $u \in \mathcal{Y}^{[1]}_{\rm per}$, $\widehat{u}'(k) = ik\hat{u}(k)$ and

$$\widehat{u}'(k) \in D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$. Since $(ik\Lambda N_k)_{k\in\mathbb{Z}}$ and $(ikMN_k)_{k\in\mathbb{Z}}$ are $\mathcal{Y}$-Fourier multipliers, there exist v_7, $v_9 \in \mathcal{Y}$ such that

$$\begin{aligned}
\hat{v}_7(k) &= ik\Lambda N_k\hat{f}(k) = \Lambda(ik\hat{u}(k)) = \Lambda\widehat{u}'(k) = \widehat{\Lambda u'}(k),\\
\hat{v}_8(k) &= ikMN_k\hat{f}(k) = M(ik\hat{u}(k)) = M\widehat{u}'(k) = \widehat{Mu'}(k),
\end{aligned} \tag{3.8}$$

for al $k \in \mathbb{Z}$. Since Λ and M are closed,

$$u'(t) \in D(\Lambda) \cap D(M)$$

and $\Lambda u'(t) = v_7(t)$, $Mu'(t) = v_8(t)$ a.e. $t \in [0, 2\pi]$ by (3.8) and [7, Lemma 3.1] (here again, we also use the fact that $\mathcal{Y} \subset L^p(0, 2\pi, X)$). Therefore

$$u' \in \mathcal{Y}(D(\Lambda)) \cap \mathcal{Y}(D(M)).$$

Since $(-k^2MN_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier, there exists $v_9 \in \mathcal{Y}$ such that

$$\hat{v}_9(k) = -k^2kMN_k\hat{f}(k) = ik(ikM\hat{u}(k)) = ikM\widehat{u}'(k) = ik\widehat{Mu}'(k), \tag{3.9}$$

for al $k \in \mathbb{Z}$ by (3.8). Then $Mu' \in \mathcal{Y}^{[1]}_{\rm per}$. Since $(ikc_kN_k)_{k\in\mathbb{Z}}$ is a $\mathcal{Y}$-Fourier multiplier, there exists $v_{10} \in \mathcal{Y}$ such that

$$\hat{v}_{10}(k) = ikc_kN_k\hat{f}(k) = ikc_k\hat{u}(k) = ik\widehat{(c*u)}(k), \tag{3.10}$$

for al $k \in \mathbb{Z}$ by (3.6). Then $c*u \in \mathcal{Y}^{[1]}_{\rm per}$ by (2.4). Since $\hat{u}(k) = N_k\hat{f}(k)$, we have

$$[-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I](-\hat{u}(k)) = \hat{f}(k),$$

this means that

$$(\widehat{Mw'})'(k) - \widehat{\Lambda w'}(k) - \widehat{\frac{d}{dt}(c*w)}(k) = \gamma\hat{w}(k) + \widehat{Aw}(k) + (\widehat{b*Bw})(k) + \hat{f}(k),$$

for all $k \in \mathbb{Z}$ where $w = -u$. From the uniqueness theorem of Fourier coefficients, we conclude that w satisfies

$$(Mw')'(t) - \Lambda w'(t) - \frac{d}{dt}(c*w)(t) = \gamma w(t) + Aw(t) + (b*Bw)(t) + f(t)$$

for almost all $t \in [0, 2\pi]$. Thus w is a strong $\mathcal{Y}$-solution of (3.1). To prove uniqueness, let u be a strong $\mathcal{Y}$-solution of (3.1) with $f = 0$. Then

$$[-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I]\hat{u}(k) = 0$$

for all $k \in \mathbb{Z}$ by Lemma 3.3. Since $ik \in \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A, B)$ for all $k \in \mathbb{Z}$, it follows that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. From the uniqueness theorem of Fourier coefficients we have that $u = 0$. Now we show the continuous dependence of u on f. Let $f \in \mathcal{Y}$, then the unique strong $\mathcal{Y}$-solution of (3.1), u, is such that $\hat{u}(k) = -N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 3.3 and $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A, B)$. Since N_k is a $\mathcal{Y}$-Fourier multiplier, there exists a bounded linear operator $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ such that $\widehat{Tf}(k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$ by Remark 2.7. Then $Tf = u$, so u depends continuously on f.

The last assertion of the theorem is a direct consequence of the fact that Au, $b*Bu$, Λu, $\Lambda u'$, $c*u$, $\frac{d}{dt}(c*u)$, Mu, Mu' and $(Mu')' \in \mathcal{Y}$ are defined through the following operator valued Fourier multipliers $(-AN_k)_{k\in\mathbb{Z}}$, $(-b_kBN_k)_{k\in\mathbb{Z}}$, $(-\Lambda N_k)_{k\in\mathbb{Z}}$, $(-k\Lambda N_k)_{k\in\mathbb{Z}}$, $(-c_kN_k)_{k\in\mathbb{Z}}$, $(-kc_kN_k)_{k\in\mathbb{Z}}$, $(-MN_k)_{k\in\mathbb{Z}}$, $(kMN_k)_{k\in\mathbb{Z}}$, $(k^2MN_k)_{k\in\mathbb{Z}}$ (here we use the Remarks 2.5, 2.7, and 3.1). □

The last assertion of the previous theorem is known as the *maximal regularity* property for (3.1).

Remark 3.7. We can construct the solution $u(\cdot)$ given by the above theorems using Proposition 2.6 and the fact that $\mathcal{Y}$ is continuously embedded in $L^p(0, 2\pi; X)$. More precisely,

$$u(\cdot) = -\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k(\cdot)N_k\hat{f}(k), \tag{3.11}$$

with convergence in $L^p(0, 2\pi; X)$.

Remark 3.8. If at most one operator of those that appear in (1.1) is unbounded, then the additional condition in our definition of well-posedness is obtained automatically. In that case the operators

$$-k^2M - A - b_kB - ik\Lambda - ikc_kI - \gamma I$$

are closed for all $k \in \mathbb{Z}$ and once we show that they are bijective, continuity follows from the Closed Graph Theorem.

4. Concrete characterization on periodic Lebesgue, Besov and Triebel-Lizorkin spaces

In this section, we give concrete conditions that allow us to apply Theorem 3.6. Specifically we obtain conditions under which the sequences $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are Fourier multipliers in the scale of spaces under consideration by use of the operator valued multiplier theorems established in [5, 7, 8, 17]. Versions of the multiplier theorems on the real line can be found in [3, 33, 34] (the reference [34] contains concrete criteria for R-boundedness of operator families), [56, 57]. The L^p-case is much different from the other scales of spaces in that it involves the notion of R-boundedness and one has to restrict consideration to UMD Banach spaces. Fortunately, many Banach spaces, for example $L^p(\Omega,\mu)$, $1<p<\infty$ are UMD spaces. In addition, the R-boundedness condition holds for resolvents of many classical operators in the analysis of partial differential equations of evolution type (see for example Kunstmann-Weis [39] and Girardi-Weis [34]).

Let $\{a_k : k\in\mathbb{Z}\}\subset\mathbb{C}$ be a scalar sequence, we denote by $\Delta a_k = a_{k+1}-a_k$. It is obvious that Δ is linear: $\Delta(a_k+b_k)=\Delta a_k+\Delta b_k$; $\Delta(\lambda a_k)=\lambda\Delta a_k$. Another property used frequently is $\Delta(a_kb_k)=a_k\Delta b_k+(\Delta a_k)b_k$. Define $\Delta^{n+1}\alpha_k=\Delta\Delta^n a_k$ for all $n\in\mathbb{N}$, $k\in\mathbb{Z}$. Δ^n is the n^{th} order difference operator:

$$\Delta^n a_k=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}a_{k+j}.$$

We will use the following hypotheses:

(H0) $\{a_k : k\in\mathbb{Z}\}$ is bounded.
(H1) $\{a_k : k\in\mathbb{Z}\}$, $\{k\Delta a_k : k\in\mathbb{Z}\}$ are bounded.
(H2) $\{a_k : k\in\mathbb{Z}\}$, $\{k\Delta a_k : k\in\mathbb{Z}\}$, $\{k^2\Delta^2 a_k : k\in\mathbb{Z}\}$ are bounded.
(H3) $\{a_k : k\in\mathbb{Z}\}$, $\{k\Delta a_k : k\in\mathbb{Z}\}$, $\{k^2\Delta^2 a_k : k\in\mathbb{Z}\}$, $\{k^3\Delta^3 a_k : k\in\mathbb{Z}\}$ are bounded.

Clearly (H0) is weaker than (H1) which in turn is weaker than (H2), and the latter is weaker than (H3). In our cases (H0) is obtained automatically from the Riemann-Lebesgue Lemma. The condition (H1) will be used for L^p well-posedness, while (H2) and (H3) are needed for Besov spaces and Triebel-Lizorkin spaces respectively. Some variations to this rule will occur when the Banach space X satisfies a special geometric property such as being UMD or having nontrivial Fourier type.

Examples of functions $a(t)$ such that $a_k=\tilde{a}(ik)$ satisfies (H3) are $a(t)=Ce^{-\omega t}t^{\nu}$ where $\omega>0$, $\nu>-1$ and C is a constant. We give a class of functions which discriminate between the above conditions in the following example.

Example 4.1. Let $\beta>0$, $\omega>0$, $c\in\mathbb{R}$ and consider the family of functions

$$b(t)=\begin{cases}0 & \text{if } 0<t\le\beta,\\ Ce^{-\omega t}(t-\beta)^{\nu} & \text{if } t>\beta\end{cases}$$

$b_k=\tilde{b}(ik)$. Then

(a) For $-1<\nu<0$ and $\beta\notin 2\pi\mathbb{Z}$, b_k satisfies (H0) but not (H1).
(b) For $0\le\nu<1$ and $\beta\notin 2\pi\mathbb{Z}$, b_k satisfies (H1) but not (H2).
(c) For $1\le\nu<2$ and $\beta\notin 2\pi\mathbb{Z}$, b_k satisfies (H2) but not (H3).
(d) For $\nu\ge 2$ or $\beta\in 2\pi\mathbb{Z}$, b_k satisfies (H3).

In the following theorem, we characterize well-posedness in the vector-valued L^p spaces.

Theorem 4.2. *Let X be a UMD Banach space, $1<p<\infty$ and A, B, Λ, M be closed linear operators in X such that $D(A)\cap D(B)\subset D(\Lambda)\cap D(M)$. Suppose that γ is a constant, $b,c\in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in (3.3) such that $\{b_k : k\in\mathbb{Z}\}$ and $\{c_k : k\in\mathbb{Z}\}$ satisfy (H1). Then the following assertions are equivalent.*

(i) (3.1) *is L^p-well-posed.*

(ii) *$i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A,B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where*
$$N_k = [k^2M + A + b_kB + ik\Lambda + ikc_kI + \gamma I]^{-1}$$

Proof. (i) ⇒ (ii) Assume that (3.1) is L^p-well-posed. Then by Theorem 3.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. The R-boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $\{kN_k : k \in \mathbb{Z}\}$ now follows from [7, Proposition 1.11].

(ii) ⇒ (i) In view of Theorem 3.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

For each $k \in \mathbb{Z}$ we define $M_k = k^2MN_k$, $B_k = BN_k$, $H_k = kN_k$ and $S_k = k\Lambda N_k$. These operators are bounded because $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A,B)$. Since $\{kN_k : k \in \mathbb{Z}\}$ is R-bounded, $\{N_k : k \in \mathbb{Z}\}$ is R-bounded by Remark 2.9. We observe that
$$\begin{aligned}
N_{k+1}^{-1}N_k &= \left[(k+1)^2M + A + b_{k+1}B + i(k+1)\Lambda + i(k+1)c_{k+1}I + \gamma I\right] N_k \\
&= [N_k^{-1} + (2k+1)M + \Delta b_kB + ik\Delta c_kI + ic_{k+1}I + i\Lambda]N_k \\
&= I + (2k+1)MN_k + \Delta b_kBN_k + ik\Delta c_kN_k + ic_{k+1}N_k + i\Lambda N_k \\
&= I + \frac{2k+1}{k^2}M_k + \Delta b_kB_k + i\Delta c_kH_k + \frac{ic_{k+1}}{k}H_k + \frac{i}{k}S_k
\end{aligned}$$
for all $k \in \mathbb{Z}$, $k \neq 0$. If we define
$$T_k = \frac{2k+1}{k^2}M_k + \Delta b_kB_k + i\Delta c_kH_k + i\frac{c_{k+1}}{k}H_k + \frac{i}{k}S_k, \tag{4.1}$$
then $N_{k+1}^{-1}N_k = I + T_k$ for all $k \in \mathbb{Z}$, $k \neq 0$. Define
$$\begin{aligned}
Q_k &= -kT_k \\
&= -[\frac{2k+1}{k}M_k + k\Delta b_kB_k + ik\Delta c_kH_k + ic_{k+1}H_k + iS_k].
\end{aligned}$$
for all $k \in \mathbb{Z}$, $k \neq 0$. Since $\{b_k : k \in \mathbb{Z}\}$ and $\{c_k : k \in \mathbb{Z}\}$ satisfy (H1), $\{Q_k : k \in \mathbb{Z}\}$ is R-bounded by Remark 2.9 and 3.1. We observe that
$$\begin{aligned}
k\Delta N_k &= k(N_{k+1} - N_k) = kN_{k+1}(I - N_{k+1}^{-1}N_k) \\
&= kN_{k+1}[I - (I + T_k)] = kN_{k+1}[-T_k] = N_{k+1}Q_k
\end{aligned}$$
Thus, we have
$$\begin{aligned}
k\Delta B_k &= k\Delta(BN_k) = B(k\Delta N_k) = BN_{k+1}Q_k = B_{k+1}Q_k, \\
k\Delta H_k &= k[(k+1)N_{k+1} - kN_k] \\
&= k[(k+1)N_{k+1} - (k+1)N_k + (k+1)N_k - k\Lambda N_k] \\
&= k[(k+1)\Delta N_k + N_k] = (k+1)(k\Delta N_k) + kN_k \\
&= (k+1)N_{k+1}Q_k + kN_k = H_{k+1}Q_k + H_k, \\
k\Delta S_k &= \Lambda(k[(k+1)N_{k+1} - kN_k]) \\
&= \Lambda[H_{k+1}Q_k + H_k] = S_{k+1}Q_k + S_k,
\end{aligned}$$

$$
\begin{aligned}
k\Delta M_k &= k((k+1)^2MN_{k+1} - k^2MN_k) \\
&= k((k+1)^2MN_{k+1} - (k+1)^2MN_k + (k+1)^2MN_k - k^2MN_k) \\
&= k[(k+1)^2M\Delta N_k + (2k+1)MN_k \\
&= (k+1)^2M[k\Delta N_k] + k(2k+1)MN_k \\
&= (k+1)^2MN_{k+1}Q_k + k(2k+1)MN_k \\
&= M_{k+1}Q_k + \frac{2k+1}{k}M_k
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0$. Then $\{k\Delta B_k : k \in \mathbb{Z}\}$, $\{k\Delta H_k : k \in \mathbb{Z}\}$, $\{k\Delta S_k : k \in \mathbb{Z}\}$, and $\{k\Delta M_k : k \in \mathbb{Z}\}$ are R-bounded by Remark 2.9. Therefore by [7, Theorem 1.3] we obtain that $(B_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(M_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. □

From the proof of Theorem 4.2, we deduce the following result for B^s_{pq}-solutions in case X has nontrivial Fourier type.

Theorem 4.3. *Let X be a Banach space with nontrivial Fourier type and A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that γ is constant, $b, c \in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in* (3.3) *such that $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$ satisfy (H1). Then for $s > 0$ and $1 \leq p, q \leq \infty$, the following are equivalent.*

(i) (3.1) *is $B^s_{p,q}$-well-posed.*

(ii) *$i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are bounded, where*

$$N_k = [k^2M + A + b_kB + ik\Lambda + ikc_kI + \gamma I]^{-1}$$

Proof. (i) $\Rightarrow$ (ii). Assume that (3.1) is B^s_{pq}-well-posed. Then by Theorem 3.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A, B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. The boundedness of $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ now follows from Remark 2.7.

(ii) $\Rightarrow$ (i). In view of Theorem 3.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. By [8, Theorem 4.5] the proof follows the same lines as that of the preceding theorem. □

We now consider the problem of well-posedness in Besov spaces $B^s_{pq}(0, 2\pi, X)$ for arbitrary Banach spaces X. For this, assumption (H0) and (H1) are no longer sufficient. It is proved in [8, Theorem 4.2] that for any sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$, the so-called variational Marcinkiewicz condition; that is,

$$\sup_{k\in\mathbb{Z}} \|M_k\| + \sup_{j\geq 0} \Big(\sum_{2^j \leq |k| < 2^{j+1}} \|\Delta M_k\| \Big) < \infty \tag{4.2}$$

implies that $(M_k)_{k\in\mathbb{Z}}$ is a B^s_{pq}-Fourier multiplier if and only if $1 < p < \infty$ and X is a UMD space.

For Banach spaces with nontrivial Fourier type, a condition which implies that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier for the scale $B^s_{p,q}$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ is the Marcinkiewicz condition of order one:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\|) < \infty, \tag{4.3}$$

see [8, Theorem 4.5], which is used in the proof of Theorem 4.3.

For arbitrary Banach spaces, a Marcinkiewicz condition of order two is needed, namely,

$$\sup_{k\in\mathbb{Z}}(\|M_k\|+\|k\Delta M_k\|+k^2\|\Delta^2 M_k\|)<\infty, \tag{4.4}$$

see [8, Theorem 4.5]. Our next result uses this condition to obtain maximal regularity of (3.1) when X does not necessarily have nontrivial Fourier type.

Theorem 4.4. *Let X be a Banach space and A, B, Λ, M be closed linear operators in X such that $D(A)\cap D(B)\subset D(\Lambda)\cap D(M)$. Suppose that γ is a constant, $b,c\in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in (3.3) such that $(b_k)_{k\in\mathbb{Z}}$, and $(c_k)_{k\in\mathbb{Z}}$ satisfy (H2). Then for $s>0$ and $1\le p,q\le\infty$, the following statements are equivalent.*

(i) *(3.1) is B^s_{pq}-well-posed.*
(ii) *$i\mathbb{Z}\subset\rho_{\Lambda,M,\tilde b,\tilde c}(A,B)$ and $\{k^2MN_k:k\in\mathbb{Z}\}$, $\{BN_k:k\in\mathbb{Z}\}$, $\{k\Lambda N_k:k\in\mathbb{Z}\}$, and $\{kN_k:k\in\mathbb{Z}\}$ are bounded, where*
$$N_k=[k^2M+A+b_kB+ik\Lambda+ikc_kI+\gamma I]^{-1}$$

Proof. (i) $\Rightarrow$ (ii). Assume that (3.1) is B^s_{pq}-well-posed. Then by Theorem 3.6, $i\mathbb{Z}\subset\rho_{\Lambda,M,\tilde b,\tilde c}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k:k\in\mathbb{Z}\}$, $\{BN_k:k\in\mathbb{Z}\}$, $\{kN_k:k\in\mathbb{Z}\}$, and $\{kN_k:k\in\mathbb{Z}\}$ now follows from Remark 2.7.

(ii) $\Rightarrow$ (i). By Theorem 3.6, it suffices to show that the families $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. Let $M_k=k^2MN_k$, $B_k=BN_k$, $H_k=kN_k$, and $S_k=k\Lambda N_k$. Since (H2) implies (H1), the verification of the Marcinkiewicz condition of order one is similar to what was done in the proof of Theorem 4.2. It remains to prove that $\sup_{k\in\mathbb{Z}}\|k^2\Delta^2M_k\|<\infty$, $\sup_{k\in\mathbb{Z}}\|k^2\Delta^2B_k\|<\infty$, $\sup_{k\in\mathbb{Z}}\|k^2\Delta^2S_k\|<\infty$, and $\sup_{k\in\mathbb{Z}}\|k^2\Delta^2H_k\|<\infty$.

We recall from the proof of Theorem 4.2 that the family $(T_k)_{k\in\mathbb{Z}}$ defined through
$$T_k=\frac{2k+1}{k^2}M_k+\Delta b_kB_k+i\Delta c_kH_k+i\frac{c_{k+1}}{k}H_k+i\frac{1}{k}S_k,\ k\neq0$$
is such that $N_{k+1}^{-1}N_k=I+T_k$, $Q_k=-kT_k$, $k\Delta N_k=N_{k+1}Q_k$ for all $k\in\mathbb{Z}$, $k\neq0$, and $\{kT_k:k\in\mathbb{Z}\}$ is bounded.

We observe that
$$\begin{aligned}\Delta T_k=\Delta(\frac{2k+1}{k^2}M_k)+\Delta(\Delta b_k)B_k+i\Delta(\Delta(c_k)H_k)\\ +i\Delta(\frac{c_{k+1}}{k}H_k)+i\Delta(\frac{1}{k}S_k))\end{aligned}$$

However,
$$\begin{aligned}\Delta(\frac{2k+1}{k^2}M_k)&=\frac{2k+3}{(k+1)^2}M_{k+1}-\frac{2k+1}{k^2}M_k\\ &=\frac{2k+3}{(k+1)^2}M_{k+1}-\frac{2k+3}{(k+1)^2}M_k+\frac{2k+3}{(k+1)^2}M_k-\frac{2k+1}{k^2}M_k\\ &=\frac{2k+3}{(k+1)^2}\Delta M_k-\frac{2k^2+4k+1}{k^2(k+1)^2}M_k\\ &=\frac{2k+3}{k(k+1)^2}(k\Delta M_k)-\frac{2k^2+4k+1}{k^2(k+1)^2}M_k,\end{aligned}$$

$$
\begin{aligned}
\Delta(\frac{1}{k}S_k) &= \frac{1}{k+1}S_{k+1} - \frac{1}{k}S_k \\
&= \frac{1}{k+1}S_{k+1} - \frac{1}{k+1}S_k + \frac{1}{k+1}S_k - \frac{1}{k}S_k \\
&= \frac{1}{k+1}\Delta S_k - \frac{1}{k(k+1)}S_k \\
&= \frac{1}{k(k+1)}(k\Delta S_k) - \frac{1}{k(k+1)}S_k,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(\frac{c_{k+1}}{k}H_k) &= \frac{c_{k+2}}{k+1}H_{k+1} - \frac{c_{k+1}}{k}H_k \\
&= \frac{c_{k+2}}{k+1}H_{k+1} - \frac{c_{k+2}}{k+1}H_k + \frac{c_{k+2}}{k+1}H_k - \frac{c_{k+2}}{k}H_k + \frac{c_{k+2}}{k}H_k - \frac{c_{k+1}}{k}H_k \\
&= \frac{c_{k+2}}{k+1}\Delta H_k + \frac{\Delta c_{k+1}}{k}H_k - \frac{c_{k+2}}{k(k+1)}H_k \\
&= \frac{c_{k+2}}{k(k+1)}(k\Delta H_k) + \frac{(k+1)\Delta c_{k+1}}{k(k+1)}H_k - \frac{c_{k+2}}{k(k+1)}H_k,
\end{aligned}
$$

$$
\begin{aligned}
\Delta[k(\Delta b_k)B_k] &= (\Delta b_{k+1})B_{k+1} - (\Delta b_k)B_k \\
&= (\Delta b_{k+1})B_{k+1} - (\Delta b_{k+1})B_k + (\Delta b_{k+1})B_k - (\Delta b_k)B_k \\
&= (\Delta b_{k+1})\Delta B_k + (\Delta^2 b_k)B_k \\
&= \frac{1}{k(k+1)}((k+1)\Delta b_{k+1})(k\Delta B_k) + \frac{1}{k^2}(k^2\Delta^2 b_k)B_k,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta((\Delta c_k)H_k)) &= (\Delta c_{k+1})H_{k+1} - (\Delta c_k)H_k \\
&= (\Delta c_{k+1})H_{k+1} - (\Delta c_{k+1})H_k + (\Delta c_{k+1})H_k - (\Delta c_k)H_k \\
&= (\Delta c_{k+1})\Delta H_k + (\Delta^2 c_k)H_k \\
&= \frac{1}{k(k+1)}((k+1)\Delta c_{k+1})(k\Delta H_k) + \frac{1}{k^2}(k^2\Delta^2 c_k)H_k
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1$. Since $\{b_k : k \in \mathbb{Z}\}$ and $\{c_k : k \in \mathbb{Z}\}$ satisfy (H2), we have $(M_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$ satisfy the Marcinkiewicz condition of order one, and $\{c_k : k \in \mathbb{Z}\}$ is bounded by Remark 3.1. It follows that $\sup_{k\in\mathbb{Z}}\{k^2\|\Delta T_k\|\} < \infty$.

We observe that from (4.1) we have

$$
\begin{aligned}
k^2\Delta^2 N_k &= k^2[\Delta N_{k+1} - \Delta N_k] \\
&= k^2[-N_{k+2}T_{k+1} + N_{k+1}T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - N_{k+2}^{-1}N_{k+1}T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - (I + T_{k+1})T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - T_k - T_{k+1}T_k] \\
&= -k^2 N_{k+2}[\Delta T_k - T_{k+1}T_k]
\end{aligned}
$$

$$= -N_{k+2}[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k] = N_{k+2}R_k$$

where we have set $R_k = -[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k]$ for all $k \in \mathbb{Z}$, $k \neq 0, -1$. Since $\{Q_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta T_k : k \in \mathbb{Z}\}$ are bounded, $\{R_k : k \in \mathbb{Z}\}$ is bounded.

Now, we have

$$k^2\Delta^2 B_k = k^2\Delta^2(BN_k) = B(k^2\Delta^2 N_k) = BN_{k+2}R_k = B_{k+2}R_k,$$

$$\begin{aligned}
k^2\Delta^2 H_k &= k^2\Delta^2(kN_k) \\
&= k^2[(k+2)N_{k+2} - 2(k+1)N_{k+1} + kN_k] \\
&= k^2[kN_{k+2} - 2kN_{k+1} + kN_k] + 2k^2N_{k+2} - 2k^2N_{k+1} \\
&= k^3\Delta^2 N_k + 2k^2\Delta N_{k+1} \\
&= k(k^2\Delta^2 N_k) + \frac{2k^2}{k+1}[(k+1)\Delta N_{k+1}] \\
&= kN_{k+2}R_k + \frac{2k^2}{k+1}N_{k+2}Q_{k+1} \\
&= \frac{k}{k+2}H_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}H_{k+2}Q_{k+1},
\end{aligned}$$

$$\begin{aligned}
k^2\Delta^2 S_k &= k^2\Delta^2(k\Lambda N_k) = k^2\Lambda\Delta^2(kN_k) = \Lambda(k^2\Delta^2 H_k) \\
&= \Lambda\Big(\frac{k}{k+2}H_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}H_{k+2}Q_{k+1}\Big) \\
&= \frac{k}{k+2}S_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}S_{k+2}Q_{k+1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
k^2\Delta^2 M_k &= k^2\Delta^2(k^2MN_k) \\
&= k^2[(k+2)^2MN_{k+2} - 2(k+1)^2MN_{k+1} + k^2MN_k] \\
&= k^2[k^2MN_{k+2} - 2k^2MN_{k+1} + k^2MN_k] + k^2(4k+4)MN_{k+2} \\
&\quad - 2k^2(2k+1)MN_{k+1} \\
&= k^2M(k^2\Delta^2 N_k) + \frac{2k^2(2k+1)}{k+1}M[(k+1)\Delta N_{k+1}] + 2k^2MN_{k+2} \\
&= k^2MN_{k+2}R_k + \frac{2k^2(2k+1)}{k+1}MN_{k+2}Q_{k+1} + 2k^2MN_{k+2} \\
&= \frac{k^2}{(k+2)^2}M_{k+2}R_k + \frac{2k^2(2k+1)}{(k+1)(k+2)^2}M_{k+2}Q_{k+1} + \frac{2k^2}{(k+2)^2}M_{k+2}
\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$. Since $\{B_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, $\{H_k : k \in \mathbb{Z}\}$, $\{M_k : k \in \mathbb{Z}\}$, $\{Q_k : k \in \mathbb{Z}\}$, and $\{R_k : k \in \mathbb{Z}\}$ are bounded, $\{k^2\Delta^2 B_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 H_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 S_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta^2 M_k : k \in \mathbb{Z}\}$ are bounded. This completes the proof. $\square$

From the proof of Theorem 4.4 and using [17, Theorem 3.2], we deduce the following result for F^s_{pq}-solutions in the case that $1 < p < \infty$, $1 < q \leq \infty$ and $s > 0$.

Theorem 4.5. *Let X be a Banach space and A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that γ is a constant, $b, c \in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in* (3.3) *such that $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$ satisfy* (H2). *Then for $s > 0$ and $1 < p < \infty$, $1 < q \le \infty$, the following are equivalent.*

(i) (3.1) *is $F^s_{p,q}$-well-posed.*
(ii) *$i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde b,\tilde c}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are bounded, where*
$$N_k = [k^2M + A + b_kB + ik\Lambda + ikc_kI + \gamma I]^{-1}$$

Proof. (i) $\Rightarrow$ (ii). Follows from Theorem 3.6 and Remark 2.7.

(ii) $\Rightarrow$ (i). Follows from [17, Theorem 3.2] using the same lines as the proof of the preceding theorem. $\square$

We now consider the problem of well-posedness in the vector-valued Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi, X)$ with parameters $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$. For this, assumption (H2) is no longer sufficient.

A condition which implies that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier for the scale F^s_{pq}, $s \in \mathbb{R}$, $1 < p < \infty$, $1 < q \le \infty$ is the Marcinkiewicz condition of order two which is used in the proof of Theorem 4.5.

For $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$, a Marcinkiewicz condition of order three is needed, namely,
$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\| + |k|^3\|\Delta^3 M_k\|) < \infty. \tag{4.5}$$
Our next result uses this condition to obtain characterization of F^s_{pq}-well-posedness of the Problem (3.1).

Theorem 4.6. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that γ is a constant, $b, c \in L^1(\mathbb{R}_+)$, and consider b_k, c_k as in* (3.3) *such that $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$ satisfy* (H3). *Then for $s > 0$ and $1 \le p < \infty$, $1 \le q \le \infty$, the following assertion are equivalent.*

(i) (3.1) *is $F^s_{p,q}$-well-posed.*
(i) *$i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde b,\tilde c}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are bounded, where*
$$N_k = [k^2M + A + b_kB + ik\Lambda + ikc_kI + \gamma I]^{-1}$$

Proof. (i) $\Rightarrow$ (ii). Assume that (3.1) is F^s_{pq}-well-posed. Then by Theorem 3.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde b,\tilde c}(A, B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $\{kN_k : k \in \mathbb{Z}\}$ follows of Remark 2.7.

(ii) $\Rightarrow$ (i). In view of Theorem 3.6, it suffices to show that the families $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. Let $M_k = k^2MN_k$, $B_k = BN_k$, $H_k = kN_k$ and $S_k = k\Lambda N_k$. Since (H3) implies (H2) and (H2) implies (H1), the verification of the Marcinkiewicz condition of order two and one is equal to what was done in the proof of Theorem 4.4.

It remains to prove the following inequalities:
$$\sup_{k\in\mathbb{Z}}\|k^3\Delta^3 M_k\| < \infty, \quad \sup_{k\in\mathbb{Z}}\|k^3\Delta^3 B_k\| < \infty,$$

$$\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 S_k\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k^3\Delta^3 H_k\| < \infty.$$

But we obtain this using the same technique as used in the proof of the previous theorems. □

The following remark concerns the independence on the parameters regarding the results of Section 4.

Remark 4.7. • In Theorem 4.2, if the problem is well-posed for some $p \in (1, \infty)$, then it well-posed for all $p \in (1, \infty)$.

• Likewise, in Theorems 4.3, 4.4, 4.5, and 4.6, if the problem under consideration is well-posed for one set of parameters in the range afforded by the corresponding theorem then it is well-posed for any set of parameters in that range.

This is a direct consequence of statement (ii) in each of the mentioned theorems.

5. Examples and applications

A large number of partial differential equations arising in physics and in applied sciences can be written in the form of equation (1.1); among them there are some famous examples, such as the pseudo-parabolic equations and the Sobolev type equations. Sobolev type equations have the form

$$\Lambda u' = Au + f, \tag{5.1}$$

generally denoting equations or systems in which spatial derivatives are mixed with the time derivative of highest order. Showalter [54, 55] studied Sobolev type equations of the first and second order in time. Specifically, Equation 5.1 is called *strongly regular* if $\Lambda^{-1}A$ is continuous, *weakly regular* if Λ is invertible but does not dominate A and *degenerate* if Λ is not invertible. Strongly regular Sobolev type equations are also widely known as pseudoparabolic. The Sobolev type equations are of interest not only for the sake of generalizations but also because they arise naturally in a variety of applications (e.g. in acoustics, electromagnetics, viscoelasticity, heat conduction etc., see e.g. [40]). A general theory in the context of generalized semigroups is developed in the monograph [44].

For the periodic case initially, Arendt and Bu [7] deal with the problem $u'(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$. This problem corresponds to (3.1) with $M = B = 0$, $\Lambda = -I$, $c = 0$, and $\gamma = 0$. The additional condition of our definition of well-posedness is obtained automatically by Remark 3.8. In this case their result are equivalent to our result by Remarks 2.5 and 2.7.

Arendt and Bu [7] (see also the review paper [4]) consider the problem $u''(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. This problem corresponds to (3.1) with $M = I$, $\Lambda = B = 0$ $c = 0$, and $\gamma = 0$. Here again the additional condition of our definition of well-posedness is obtained automatically by Remark 3.8. In this case their result are equivalent to our result by Remarks 2.7.

Keyantuo and Lizama [35, 36] considered well-posedness of (3.1) when $B = M = 0$ and Λ is a scalar operator. As noted earlier, this problem is relevant for viscoelasticity and was previously studied in the framework of periodic solutions by Da Prato-Lunardi [25] among other references, and on the real line by [23, 26]. Second order equations are considered in this context in [37, 48]

The additional condition of our definition of well-posedness is obtained automatically by Remark 3.8. Their results can be deduced from ours. Some additional papers on the subject are Bu [12, 13, 15]. Delay equations are considered in [16, 49] with the method of operator-valued Fourier multipliers.

Bu [13] considered the well-posedness of (3.1) when $B = \Lambda = 0$, $c = 0$, and γ. His results follow from ours. With our definition of well-posedness we do not need the a priori the estimate [13, (2.2)]. Thus, in the reference [13], the author considers the problem

$$\begin{aligned} &(Mu')'(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \\ &u(0) = u(2\pi), \quad (Mu')(0) = (Mu')(2\pi). \end{aligned}$$

It follows from Theorem 4.2 that this problem is L^p-well-posed if and only if $i\mathbb{Z} \subset \rho_{0,M,\tilde{0},\tilde{0}}(A,0) = \rho_M(A)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (k^2M + A)^{-1}$. In a similar way, we deduce the results in $B^s_{p,q}$ and $F^s_{p,q}$ using Theorem 4.4 and Theorem 4.6 respectively.

We introduce some facts on uniformly elliptic operators on domains of $\mathbb{R}^n$ to discuss the examples that follow. Let $\Omega \subset \mathbb{R}^n$ be open, $n \ge 1$. We consider measurable functions α_{ik}, β_k, γ_k, and α_0 $(1 \le j, k \le n)$ on Ω. We assume that the following uniform ellipticity condition holds: The functions α_{kj}, β_k, γ_k, α_0 are bounded on Ω, i.e., α_{kj}, β_k, γ_k, $\alpha_0 \in L^\infty(\Omega, \mathbb{C})$ for $1 \le j, k \le n$ and the principal part is elliptic; i.e., there exists a constant $\eta > 0$ such that

$$\operatorname{Re}\Big(\sum_{j,k=1}^{n} \alpha_{kj}(x)\xi_j\overline{\xi_k}\Big) \ge \eta|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n, \text{ a.e. } x \in \Omega. \tag{5.2}$$

The largest possible η in (5.2) is called the ellipticity constant of the matrix $(\alpha_{jk})_{1\le j,k\le n}$. Then we consider the elliptic operator $L : W^{1,2}_{\rm loc}(\Omega) \to \mathcal{D}(\Omega)'$ given by

$$Lu = -\sum_{k,j=1}^{n} D_j(\alpha_{kj}D_k u) + \sum_{k=1}^{n}(\beta_k D_k u - D_k(\gamma_k u)) + \alpha_0 u.$$

With the help of bilinear forms we will define various realizations of $L \in L^2(\Omega)$ corresponding to diverse boundary conditions. Let V be a closed subspace of $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$. We define the form $\alpha_V : V \times V \to \mathbb{C}$ by

$$\alpha_V(u,v) = \int_\Omega \Big[\sum_{k,j=1}^{n} \alpha_{kj} D_k u\overline{(D_j v)} + \sum_{k=1}^{n}(\beta_k \overline{v} D_k u + \gamma_k u\overline{D_k v}) + \alpha_0 u\overline{v}\Big]dx.$$

Then α_V is densely defined, accretive, and closed sesquilinear form on $L^2(\Omega)$ (see [46, Chapter 4 p. 100-101]). Denote by A_V the operator on $L^2(\Omega)$ associated with α_V. Then $-A_V$ generates a C_0-semigroup T_V on $L^2(\Omega)$ (see [46, Proposition 1.51]). It follows from the definition of the associated operator that $A_V u = Lu$ for all $u \in D(A_V)$. We will say that we have:

- Dirichlet boundary conditions if $V = W^{1,2}_0(\Omega)$;
- Neumann boundary conditions if $V = W^{1,2}(\Omega)$;

We consider Dirichlet boundary conditions with Ω bounded and we assume the following additional conditions: α_{kj} is real-valued with $\alpha_{kj} = \alpha_{jk}$, $\beta_k = \gamma_k = 0$, $\alpha_0 \ge 0$. Then, in this case the semigroup T_V is positive, $\|T_V(t)\|_{\mathcal{L}(L^2(\Omega))} \le 1$ for all $t \ge 0$, and T_V is given by an integral kernel $p_V(t,x,y)$ such that there exist constants $C > 0$, $b > 0$, and $\delta > 0$ such that

$$|p_V(t,x,y)| \le Ct^{-n/2}e^{-\delta t}e^{-\frac{|x-y|^2}{4bt}} \tag{5.3}$$

for every $t > 0$ and a.e. $x, y \in \Omega$, see [46, Theorem 4.2, Corollary 6.14 and Theorem 4.28] and [27]. For every $r \in (1, \infty)$, the C_0-semigroup T_V extends to a bounded C_0-semigroup T_r on $L^r(\Omega)$ with $\|T_r(t)\|_{\mathcal{L}(L^r(\Omega))} \leq 1$ for all $t \geq 0$, by [46, Theorem 4.28]. By (5.3) there exist $M_r > 0$, and $\delta_r > 0$ depending only on r such that $\|T_r(t)\|_{\mathcal{L}(L^r(\Omega))} \leq M_r e^{-\delta_r t}$ for all $t > 0$ and $r \in (1, \infty)$. Denote now by $-A_r$ the corresponding infinitesimal generator on $L^r(\Omega)$. If $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > -\delta$, then $\lambda \in \rho(-A_r)$ and

$$R(\lambda, -A_r)u = \int_0^\infty e^{-\lambda t} T_r(t) u dt \text{ for all } u \in L^r(\Omega), \tag{5.4}$$

by [6, Theorem 3.1.7].

Let $r \in (1, \infty)$. The C_0-semigroup T_r extends to a bounded holomorphic semigroup on the sector $\Sigma_{\pi/2}$, where Σ_θ is the sector in the complex right half plane of angle $\theta \in (0, \pi]$. By [6, Theorem 3.7.11] we have that $\Sigma_\pi \subset \rho(-A_r)$ and $\sup_{\lambda \in \Sigma_{\pi-\varepsilon}} \|\lambda R(\lambda, -A_r)\| < \infty$ for all $\varepsilon > 0$. Denote by $\sigma(A_r)$ the spectrum of the operator A_r on $L^r(\Omega)$. By [46, Theorem 7.10], we have that $\sigma(A_r) = \sigma(A_2) \subset (0, \infty)$ for all $r \in (1, \infty)$. By [4, Section 7.2.6] we have that $\lambda R(\lambda, -A_r)$ is R-bounded for all $\lambda \in \Sigma_{\pi/2+\theta_r}$ with $0 < \theta_r \leq \pi/2$. Since $\lambda \to R(\lambda, -A_r)$ is analytic on $\Sigma_\pi \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\delta_r\}$, it follows that $R(\lambda, -A_r)$ is R-bounded on every compact subset of $\Sigma_\pi \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\delta_r\}$ by [28, Proposition 3.10]. By Remark 2.9, we have that $R(\lambda, -A_r)$ and $\lambda R(\lambda, -A_r)$ are R-bounded on $\Sigma_{\pi/2+\theta_r} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\delta_r/2\}$. Using Kahane's principle, we obtain that $R(\lambda, A_r)$ and $\lambda R(\lambda, A_r)$ are R-bounded on $\mathbb{C} \setminus \Sigma_{\theta_r} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq \delta_r/2\}$.

We conclude, with some examples using uniformly elliptic operators in $L^r(\Omega)$ just discussed. General references on uniformly elliptic operators in L^p-spaces and the associated heat kernel estimates are [27] and [46].

Example 5.1. Let us consider the boundary value problem (in which L is a uniformly elliptic operator as defined above)

$$\begin{gathered}
\frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t}) + L\frac{\partial u(t,x)}{\partial t} \\
= -Lu(t,x) - \int_{-\infty}^t b(t-s)Lu(s,x)ds + f(t,x), \quad (t,x) \in [0,2\pi] \times \Omega, \\
u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0, \quad (t,x) \in [0,2\pi] \times \partial\Omega, \\
u(0,x) = u(2\pi,x), \quad m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x \in \Omega,
\end{gathered} \tag{5.5}$$

where $f \in L^p(0, 2\pi; L^r(\Omega))$ for $1 < p, r < \infty$, m is a real-valued measurable function on Ω such that $m \in L^\infty(\Omega)$. This is the degenerate wave equation with fading memory. The non-degenerate equation is studied in [1], and the reference list of this paper contains additional works on that topic. Maximal regularity for the damped wave equation in the absence of memory effects has been studied in [20] and [37]. The problem (5.5) can also be considered as a modified version of a problem which is considered in Favini-Yagi [32, Example 6.24 p. 197]. They do not incorporate the delay aspect of the equation. They restrict their study to the Hölder spaces. The authors are considered with the evolutionary problem as well. For periodic boundary conditions, we obtain complete characterization of well-posedness in the three scales of spaces: L^p, B^s_{pq}, and F^s_{pq}.

We can rewrite problem (5.5) in as follows (where A_r was defined above):

$$\begin{gathered}\frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t})+A_r\frac{\partial u(t,x)}{\partial t}\\ =-A_r u(t,x)-\int_{-\infty}^{t} b(t-s)A_r u(s,x)ds+f(t,x),\quad (t,x)\in[0,2\pi]\times\Omega, \\ u(0,x)=u(2\pi,x),\quad m(x)\frac{\partial u(0,x)}{\partial t}=m(x)\frac{\partial u(2\pi,x)}{\partial t},\quad x\in\Omega.\end{gathered}\tag{5.6}$$

If we suppose that b_k defined by (3.3) satisfies (H1) and the additional condition $|\operatorname{Im} b_k|<1$ for all $k\in\mathbb{Z}$, it follows that $\frac{k^2 m(x)}{1+b_k+ik}\notin(0,\infty)$ for all $x\in\Omega$ and all $k\in\mathbb{Z}$. Therefore $i\mathbb{Z}\subset\rho_{-A_r,M,\tilde{b},\tilde{0}}(-A_r,-A_r)$, where M is the multiplication operator by m. By Remark 3.1, we have that there exists $N\in\mathbb{N}$ such that $\frac{k^2 m(x)}{ik+1+b_k}\in\mathbb{C}\setminus\Sigma_{\theta_r}\cup\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\le\delta_r/2\}$ for all $x\in\Omega$ and all $k\in\mathbb{Z}$ whit $|k|\ge N$. Then $\{(\frac{k^2}{ik+1+b_k}M-A_r)^{-1}:k\in\mathbb{Z},|k|\ge N\}$ and $\{\frac{k^2}{ik+1+b_k}M(\frac{k^2}{ik+1+b_k}M-A_r)^{-1}:k\in\mathbb{Z},|k|\ge N\}$ are R-bounded. Since

$$\begin{aligned}&\frac{k}{ik+1+b_k}A_r(\frac{k^2}{ik+1+b_k}M-A_r)^{-1}\\ &=-\frac{k}{ik+1+b_k}I+\frac{k}{ik+1+b_k}\frac{k^2}{ik+1+b_k}M(\frac{k^2}{ik+1+b_k}M-A_r)^{-1},\end{aligned}$$

it follows that $\{\frac{k}{ik+1+b_k}A_r(\frac{k^2}{ik+1+b_k}M-A_r)^{-1}:k\in\mathbb{Z},|k|\ge N\}$ is R-bounded as well by Remark 2.9. Since $N_k=\frac{1}{ik+1+b_k}(\frac{k^2}{ik+1+b_k}M-A_r)^{-1}$, we have shown that $\{k^2MN_k:k\in\mathbb{Z},|k|\ge N\}$, $\{kA_rN_k:k\in\mathbb{Z},|k|\ge N\}$ and $\{kN_k:k\in\mathbb{Z},|k|\ge N\}$ are R-bounded. Also by Remark 2.9, we have that $\{kN_k:k\in\mathbb{Z}\}$, $\{kA_rN_k:k\in\mathbb{Z}\}$ and $\{k^2MN_k:k\in\mathbb{Z}\}$ are R-bounded. Therefore, by Theorem 4.2, it follows that (5.6) is $L^p(0,2\pi;L^r(\Omega))$-well-posed for all $1<p<\infty$. Since R-boundedness implies uniformly boundedness, if we suppose that $f\in B^s_{pq}(0,2\pi;L^r(\Omega))$ and b_k satisfies (H2) with $|\operatorname{Im} b_k|<1$ for all $k\in\mathbb{Z}$, then we have that (5.6) is $B^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p,q\le\infty$ by Theorem 4.4. Observe that here we include the scale of vector-valued Hölder spaces C^s, $0<s<1$. In the F^s_{pq} case if $f\in F^s_{pq}(0,2\pi;L^r(\Omega))$ and b_k satisfies (H3) with $|\operatorname{Im} b_k|<1$ for all $k\in\mathbb{Z}$, then we have that (5.6) is $F^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p<\infty$, $1\le q\le\infty$, by Theorem 4.6. Observe that if $s>$, $1<p<\infty$ and $1<q\le\infty$ we only need the (H2) condition for this scale. As a particular example we have that $b(t)=e^{-\varepsilon t}\frac{t^{\nu-1}}{\Gamma(\nu)}$ with $\varepsilon>0$ and $\nu>0$ satisfies the required conditions for b_k for all the cases.

Example 5.2. Let us consider the boundary value problem

$$\begin{gathered}\frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t})+L\frac{\partial u(t,x)}{\partial t}\\ =Lu(t,x)+\int_{-\infty}^{t} b(t-s)Lu(s,x)ds+f(t,x),\ (t,x)\in[0,2\pi]\times\Omega,\\ u(t,x)=\frac{\partial u(t,x)}{\partial t}=0,\quad (t,x)\in[0,2\pi]\times\partial\Omega,\\ u(0,x)=u(2\pi,x),\quad m(x)\frac{\partial u(0,x)}{\partial t}=m(x)\frac{\partial u(2\pi,x)}{\partial t},\quad x\in\Omega,\end{gathered}\tag{5.7}$$

where $f \in L^p(0,2\pi;L^r(\Omega))$ for $1<p,r<\infty$, m is a complex-valued measurable function on Ω such that $m \in L^\infty(\Omega)$, $m(x) \in \sum_{\theta_r} \cup \{0\}$ for all $x \in \Omega$. Here, as in the previous example (and similarly in Example 5.3 below), L is a uniformly elliptic operator.

Following Example 5.1, we can rewrite the problem (5.7) in the form

$$\begin{aligned} &\frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t}) + A_r\frac{\partial u(t,x)}{\partial t} \\ &= A_r u(t,x) + \int_{-\infty}^t b(t-s)A_r u(s,x)ds + f(t,x), \quad (t,x)\in[0,2\pi]\times\Omega, \qquad (5.8)\\ &\quad u(0,x)=u(2\pi,x), \quad m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x\in\Omega. \end{aligned}$$

If we suppose that b_k defined by (3.3) satisfies (H1) and the additional condition $\operatorname{Re} b_k > -1$ for all $k \in \mathbb{Z}$, then $\frac{k^2 m(x)}{1+b_k-ik} \in \Sigma_{\pi/2+\theta_r} \cup \{0\}$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{-A_r,M,\tilde{b},\tilde{0}}(A_r,A_r)$ and $\{(\frac{k^2}{1+b_k-ik}M + A_r)^{-1} : k \in \mathbb{Z}\}$, $\{\frac{k^2}{1+b_k-ik}M(\frac{k^2}{1+b_k-ik}M+A_r)^{-1} : k\in\mathbb{Z}\}$ are R-bounded, here M is the multiplication operator by m. By Remarks 2.9 and 3.1, we have that $\{\frac{k}{1+b_k-ik}(\frac{k^2}{1+b_k-ik}M + A_r)^{-1} : k\in\mathbb{Z}\}$ is also R-bounded. Since

$$\begin{aligned} &\frac{k}{1+b_k-ik}A_r(\frac{k^2}{1+b_k-ik}M+A_r)^{-1}\\ &= \frac{k}{1+b_k-ik}I - \frac{k}{1+b_k-ik}\frac{k^2}{1+b_k-ik}M(\frac{k^2}{1+b_k-ik}M+A_r)^{-1}, \end{aligned}$$

it follows that $\{\frac{k}{1+b_k-ik}A_r(\frac{k^2}{1+b_k-ik}M + A_r)^{-1} : k \in \mathbb{Z}\}$ is R-bounded as well by Remark (2.9). Since $N_k = \frac{1}{1+b_k-ik}(\frac{k^2}{1+b_k-ik}M + A_r)^{-1}$, we have shown that $\{k^2MN_k : k\in\mathbb{Z}\}$, $\{kA_rN_k : k\in\mathbb{Z}\}$ and $\{kN_k : k\in\mathbb{Z}\}$ are R-bounded. Therefore, by Theorem 4.2, we have that (5.8) is $L^p(0,2\pi;L^r(\Omega))$-well-posed for all $1<p<\infty$. Since R-boundedness implies uniformly boundedness, if we suppose that $f\in B^s_{pq}(0,2\pi;L^r(\Omega))$ and b_k satisfies (H2) with $\operatorname{Re} b_k > -1$ for all $k\in\mathbb{Z}$, then we have that (5.8) is $B^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p,q\le\infty$ by Theorem 4.4. Observe that here we include the scale of vector-valued Hölder spaces C^s, $0<s<1$. In the F^s_{pq} case if $f\in F^s_{pq}(0,2\pi;L^r(\Omega))$ and b_k satisfies (H3) with $\operatorname{Re} b_k > -1$ for all $k\in\mathbb{Z}$, then we have that (5.8) is $F^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p<\infty$, $1\le q\le\infty$, by Theorem 4.6. Observe that if $s>0$, $1<p<\infty$ and $1<q\le\infty$ we only need the (H2) condition for this scale. As in the Example 5.1, a particular example of $b(t)$ we have $b(t) = e^{-\varepsilon}\frac{t^{\nu-1}}{\Gamma(\nu)}$ with $\varepsilon>0$ and $\nu>0$ that fulfills the required conditions for b_k in all the cases.

Example 5.3. Consider another initial-boundary value problem.

$$\begin{gathered}
\frac{\partial}{\partial t}(m_2(x)\frac{\partial u(t,x)}{\partial t}) - m_1(x)\frac{\partial u(t,x)}{\partial t} \\
= Lu(t,x) + \int_{-\infty}^{t} b(t-s)Lu(s,x)ds + f(t,x), \quad (t,x) \in [0,2\pi]\times\Omega, \\
u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0, \quad (t,x) \in [0,2\pi]\times\partial\Omega, \\
u(0,x) = u(2\pi,x), \quad m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x \in \Omega,
\end{gathered} \tag{5.9}$$

where m_1 and m_2 are real-valued measurable functions on Ω such that $m \in L^\infty(\Omega)$, $m_2(x) \geq 0$, $\tau < |m_1(x)| \leq \mu$ for some $\tau, \mu > 0$, and $f \in L^p(0,2\pi; L^r(\Omega))$ for $1 < p, r < \infty$.

Following the Example 5.1, we can rewrite the problem (5.9) in the form

$$\begin{gathered}
\frac{\partial}{\partial t}(m_2(x)\frac{\partial u(t,x)}{\partial t}) - m_1\frac{\partial u(t,x)}{\partial t} \\
= A_r u(t,x) + \int_{-\infty}^{t} b(t-s)A_r u(s,x)ds + f(t,x), \quad (t,x) \in [0,2\pi]\times\Omega, \\
u(0,x) = u(2\pi,x), \quad m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x \in \Omega.
\end{gathered} \tag{5.10}$$

If we assume that $\operatorname{Re} b_k > -1$ for all $k \in \mathbb{Z}$, then $\frac{k^2 m_2(x) + ikm_1(x)}{b_k+1} \notin (-\infty, 0)$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{0}}(A_r, A_r)$, where Λ and M are the multiplication operators by m_1 and m_2 respectively. By Remark 3.1, we have that there exists $N \in \mathbb{N}$ such that $\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1} \in \Sigma_{\pi/2+\theta_r}$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$ whit $|k| \geq N$. Then $\{\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded. Since $\{\frac{1}{km_2(x)+im_1(x)} : k \in \mathbb{Z}, x \in \Omega\}$ is bonded, $\{\frac{k}{b_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded by Remark 2.9. Since m_1 is bounded, by Remark 2.9 we have that $\{\frac{km_1(x)}{b_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded. Therefore, $\{\frac{k^2 m_2(x)}{b_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{b_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded. Since $N_k = \frac{1}{b_k+1}(\frac{k^2}{b_k+1}M + i\frac{k}{b_k+1}\Lambda + A_r)^{-1}$, we have show that $\{kN_k : k \in \mathbb{Z}, |k| \geq N\}$, $\{k\Lambda N_k : k \in \mathbb{Z}, |k| \geq N\}$ and $\{k^2 MN_k : k \in \mathbb{Z}, |k| \geq N\}$ are R-bounded. By Remark 2.9, we have that $\{kN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$ and $\{k^2 MN_k : k \in \mathbb{Z}\}$ are R-bounded. Under the same conditions over b_k in the Example 5.2 and $f \in \mathcal{Y}$ we can apply Theorems 4.2, 4.4 and 4.6 to obtain that the (5.10) is $\mathcal{Y}$-well-posed.

Example 5.4.

Let us now consider the boundary-value problem

$$
\begin{gathered}
\frac{\partial}{\partial t}(m_2(x)\frac{\partial u(t,x)}{\partial t}) - m_1(x)\frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)u(s,x)ds \\
= Lu(t,x) + \int_{-\infty}^{t} b(t-s)Lu(s,x)ds \\
+ \int_{-\infty}^{t} b(t-s)m_0(x)u(s,x)ds + f(t,x), \quad (t,x)\in[0,2\pi]\times\Omega, \\
u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0, \quad (t,x)\in[0,2\pi]\times\partial\Omega, \\
u(0,x) = u(2\pi,x), \quad m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x\in\Omega,
\end{gathered} \tag{5.11}
$$

where m_0, m_1, and m_2 are real-valued measurable functions on Ω such that $0 \le m_0(x) \le \mu$, $\tau < |m_1(x)| \le \mu$, $0 \le m_2(x)$, for some $\mu, \tau > 0$, all $x \in \Omega$, and $f \in L^p(0,2\pi; L^r(\Omega))$ for $1 < p, r < \infty$.

Following the Example 5.1, we can rewrite the problem (5.11) in the form

$$
\begin{gathered}
\frac{\partial}{\partial t}(m_2(x)\frac{\partial u(t,x)}{\partial t}) - m_1(x)\frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)u(s,x)ds \\
= A_r u(t,x) + \int_{-\infty}^{t} b(t-s)m_0(x)u(s,x)ds + f(t,x), \quad (t,x)\in[0,2\pi]\times\Omega, \\
u(0,x) = u(2\pi,x), \quad m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t}, \quad x\in\Omega.
\end{gathered} \tag{5.12}
$$

If we suppose that $\operatorname{Re} b_k \ge 0$ and $k \operatorname{Im} c_k \le 0$ for all $k \in \mathbb{Z}$, we then have that $k^2 m_2(x) + ikm_1(x) + b_k m_0(x) + ikc_k \notin (-\infty, 0)$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{b},\tilde{c}}(A_r, B)$, where B, Λ, and M are the multiplication operators by m_0, m_1, and m_2 respectively. In the similar way that in the Example 5.3 we can show that $\{kN_k : k\in\mathbb{Z}\}$, $\{kBN_k : k\in\mathbb{Z}\}$, $\{k\Lambda N_k : k\in\mathbb{Z}\}$ and $\{k^2 MN_k : k\in\mathbb{Z}\}$ are R-bounded where $N_k = (k^2 M + ik\Lambda + b_k B + ikc_k I + A_r)^{-1}$. Typical cases of functions b and c is the function $e^{-\varepsilon t}$, $\varepsilon > 0$. With $f \in \mathcal{Y}$ and the appropriate b, and c we can obtain that the (5.12) is $\mathcal{Y}$-well-posed.

In the case of Neumann boundary conditions, the operator A_r is not invertible. To apply the results to this case, we can add in the right side of each of the above equations the term $\eta u(t,x)$ for some $\eta > 0$. Then the above conclusions hold in this case as well.

Example 5.5. The following equation is a modification of the one studied by Chill and Srivastava [20]. Here we have include memory term.

$$
\begin{gathered}
u''(t) + \alpha A^{\frac{1}{2}} u'(t) = -Au(t) + \int_{-\infty}^{t} b(t-s)Au(s,x)ds + f(t), \\
t\in[0,2\pi], \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\end{gathered} \tag{5.13}
$$

where A is a invertible sectorial operator in a Banach space X which admits a bounded H^∞ functional calculus of angle β (see for example [20], [28]) with $\beta \in (0, \pi - 2\tan^{-1}\frac{\sqrt{4-\alpha^2}}{\alpha})$ if $0 < \alpha < 2$ or $\beta \in (0,\pi)$ if $\alpha \ge 2$, $f \in B^s_{pq}(0,2\pi;X)$, $(1 \le p, q \le \infty,\ s > 0)$, and $b \in L^1(\mathbb{R}_+)$ is such that $b_k = \tilde{b}(ik)$ satisfies $|CQb_k| < \frac{1}{2}$

where Q is a constant provided by [20, Lemma 4.1] and C is a constant provided by the H^∞ functional calculus.

In the same way as in the proof of theorem [20, Theorem 4.1] we have that for $k \in \mathbb{Z}$, $\|k^2(k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1}\| \le CP$, $\|kA^{\frac{1}{2}}(k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1}\| \le CP$, and $\|A(k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1}\| \le CP$. In this case for $k \in \mathbb{Z}$, we have that $N_k = (k^2 - \alpha kiA^{\frac{1}{2}} - A + b_kA)^{-1}$. Since $\|b_kA(k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1}\| \le \frac{1}{2}$, we have

$$N_k = (k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1} \sum_{n=0}^{\infty} (-1)^n \Big(b_kA(k^2 - \alpha kiA^{\frac{1}{2}} - A)^{-1} \Big)^n,$$

which implies that $\|k^2N_k\| \le CP$ and $\|\alpha kA^{\frac{1}{2}}N_k\| \le CP$ for $k \in \mathbb{Z}$. Now if b_k satisfy (H2), then we have that the problem 5.5 is B^s_{pq}-well-posed. This gives in particular well-posedness in the Hölder spaces $C^s(0, 2\pi; X)$, $0 < s < 1$.

In a similar way, one can handle the case of the vector-valued Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$, $1 \le p, q < \infty$, $s > 0$.

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