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BLOW-UP OF SOLUTIONS TO THE ROTATION B-FAMILY SYSTEM MODELING EQUATORIAL WATER WAVES

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Abstract. We consider the blow-up mechanism to the periodic generalized rotation b-family system (R-b-family system). This model can be derived from the f-plane governing equations for the geographical water waves with a constant underlying current in the equatorial water waves with effect of the Coriolis force. When $b = 2$, it is a rotation two-component Camassa-Holm (R2CH) system. We consider the periodic R2CH system when linear dispersion is absent (which model is called r2CH system) and derive two finite-time blow-up results.

1. Introduction

It is known that for the geophysical water waves the forces with primary influence are the gravity and the Coriolis force induced by the Earth's rotation. When considering waves propagating in the equatorial ocean regions throughout the extent of the Pacific Ocean, it is found however that the Equatorial Undercurrent is one essential feature and the effect of the Coriolis force is small, because of the smallness of the variation in latitude of the EUC in the equatorial region. There have recently appeared several works involving steady periodic rotational Equatorial water waves in the f-plane on topics like existence, regularity of free surface and of the stream lines, symmetry and stability. This paper is to study the following periodic generalized rotation b-family system (R-b-family system).

$$
\begin{gathered}
u_t - u_{xxt} - Au_x + (b+1)uu_x \\
= \sigma(bu_xu_{xx} + uu_{xxx}) - \mu u_{xxx} - (1-2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, \\
\rho_t + (\rho u)_x = 0, \\
u(t,x) = u(t,x+1), \rho(t,x) = \rho(t,x+1) \\
u(0,x) = u_0(x), \rho(0,x) = \rho_0(x).
\end{gathered}
\tag{1.1}
$$

for $t > 0$ and $x \in \mathbb{S}$, where $u(x,t)$ is a horizontal velocity, $\rho(t,x)$ is related to the free surface elevation from equilibrium, the parameter A characterizes a linear underlying shear flow, the real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening

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and amplification due to stretching, μ is a non-dimensional parameter and Ω characterizes the constant rotational speed of the Earth. We can rewrite the periodic R-b-family system (1.1) into the system

$$
\begin{gathered}
u_t + (\sigma u - \mu)u_x = -\partial_x G * \Big((\mu - A)u + \frac{b+1-\sigma}{2}u^2 + \frac{(b-1)\sigma}{2}u_x^2 \\
+ \frac{1-2\Omega A}{2}\rho^2 - \Omega\rho^2 u\Big) + \Omega G * (\rho^2 u_x), \\
\rho_t + u\rho_x = -\rho u_x, \\
u(t,x) = u(t,x+1), \rho(t,x) = \rho(t,x+1), \\
u(0,x) = u_0(x), \rho(0,x) = \rho_0(x).
\end{gathered}
\tag{1.2}
$$

for $t > 0$ and $x \in \mathbb{S}$.

We denote $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, the fundamental solution of $1 - \partial_x^2$ on $\mathbb{S}$, that is, $(1-\partial_x^2)^{-1} f = G * f$, we have the relation

$$
G * f(t,x) = \int_0^1 \frac{\cosh\big((x-y) - [x-y] - \frac{1}{2}\big)}{2\sinh\big(\frac{1}{2}\big)} f(t,y)dy. \tag{1.3}
$$

The approach we adopt here to derive the R-b-family system is in the spirit of Ivanov's asymptotic perturbation analysis for the governing equations of two-dimensional rotational gravity water waves[23]. And there are two factors force us to do the asymptotic perturbation analysis. In the equatorial region there exist the shallow water waves, for which we mean that the shallow water parameter $\delta = h/\lambda < 0.07$[8], where h is the mean depth of water and λ is the wavelength. Actually, the equatorial region is characterized as a two-layer fluid with a shallow upper region of warmer and less dense water overlying a motionless deep region of cold water. The upper shallow water is less than 300 m deep [29] and usually the wavelength of the surface waves can be 100S km or more. The equatorial Rossby waves whose wavelength can be 500 km are evidence examples. Besides, the westward travelling waves with a wavelength of 1000 km near 3 in the central and eastern Pacific Ocean has been observed, and the wavelengths slightly in excess of 2000 km can arise from the instabilities of surface Equatorial Currents. On the other hand, to ensure the earth's rotation to have a significant impact on the fluid motion, one expects the Rossby number $R_0 = \frac{U}{\lambda\Omega} = O(1)$, where λ is the typical horizontal length scale for the flow, U is the typical horizontal length scale for the flow, U is the typical horizontal velocity scale for the fluid motion, and the symbol $O(1)$ means that the term is of the order of magnitude of one, or less. This in turn implies that the smaller the characteristic velocity U is, the smaller L can be and yet it still enables us to consider large-scale waves.

In fact, system (1.1) has significant relationship with several models describing the motion of waves at the free surface of shallow water under the influence of gravity.

When $b = 2$, it becomes the rotation Camassa-Holm system (R2CH system)

$$\begin{aligned} u_t - u_{xxt} - Au_x + 3uu_x &= \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} \\ &\quad - (1 - 2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, \\ \rho_t + (\rho u)_x &= 0, \\ u(t,x) = u(t,x+1), &\rho(t,x) = \rho(t,x+1), \\ u(0,x) = u_0(x), &\rho(0,x) = \rho_0(x). \end{aligned} \tag{1.4}$$

for $t > 0$ and $x \in \mathbb{S}$,

Moreover, we can rewrite the periodic R2CH system as the system

$$\begin{aligned} u_t + (\sigma u - \mu)u_x &= -\partial_x G * \Big((\mu - A)u + \frac{3-\sigma}{2}u^2 \\ &\quad + \frac{\sigma}{2}u_x^2 + \frac{1-2\Omega A}{2}\rho^2 - \Omega\rho^2 u\Big) + \Omega G * (\rho^2 u_x), \\ \rho_t + u\rho_x &= -\rho u_x, \\ u(t,x) = u(t,x+1), &\rho(t,x) = \rho(t,x+1), \\ u(0,x) = u_0(x), &\rho(0,x) = \rho_0(x). \end{aligned} \tag{1.5}$$

for $t > 0$ and $x \in \mathbb{S}$.

If $\Omega = 0$, without considering effect of the Earth's rotation, then the following functional is conserved

$$F(u,\rho) = \frac{1}{2}\int_{\mathbb{R}} (u^3 + \sigma u u_x^2 - Au^2 - \mu u_x^2 + 2(\rho - 1)u + u(\rho - 1)^2)dx.$$

When $b = 2, \Omega = 0$, system (1.1) is the generalized DGH system

$$\begin{aligned} u_t - u_{xxt} - Au_x + 3uu_x &= \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} - \rho\rho_x, \\ \rho_t + (\rho u)_x &= 0, \end{aligned} \tag{1.6}$$

in which $\sigma = 1$, $\mu = 0$. Then the equation recovers the standard two-component integrable Camassm-Holm system

$$\begin{aligned} u_t - u_{xxt} - Au_x + 3uu_x + \rho\rho_x &= 2u_x u_{xx} + uu_{xxx}, \\ \rho_t + (\rho u)_x &= 0, \end{aligned} \tag{1.7}$$

Moreover, in the case $\rho = 0$, (1.4) recovers the DGH equation and becomes the Camassa-Holm equation. The CH equation is completely integrable for a large class of initial data, for which it can be solved by the inverse scattering method [11, 18]. In contrast to the KdV equation, the CH equation has three remarkable distinctive properties. First, although CH is completely integrable, it can describe wave breaking phenomenon: the solution remains bounded while its slope becomes infinite in finite time. The second is the existence of peakons, which are nonanalytic solitary waves that are global weak solutions and interact cleanly like solitons. Indeed, the CH equation has the single peakon [6] and the multi-peakon solutions [22]. It is significant that the peakons are orbitally stable: the shape is stable under small perturbations [19, 25]. These peakons capture a feature of the waves of greatest height for the free-boundary incompressible Euler equations [12, 16, 32]. The last one is the variety of interesting geometric formulations of the CH equation [7, 17, 24, 27].

Well-posedness and wave breaking of the CH equation were studied in a number of papers. It has been shown [14, 26, 31, 28] that the Cauchy problem is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Moreover, if the initial momentum density

$$m_0(x) = m(0,x) = (1-\partial_x^2)u_0 = u_0(x) - u_0''(x)$$

does not change sign, the Cauchy problem admits global solution for certain initial values [10, 14, 15], whereas solutions may blow up if their initial momentum density m_0 changes sign [10, 13, 14, 15]. After blow-up, the solutions to CH can be continued uniquely as global weak solutions [4, 5]. Moreover, the existence of global weak solution was investigated in [33, 34].

It is observed that system (1.1) we derived is a generalization of system with the rotation of Earth-these effects feature significantly for such large scale phenomena as currents. It is found that the consideration of the Coriolis force has introduced a higher order nonlinear term into the generalized two-component b-family system, which has interesting implications for the fluid motion, particular in the relation to the wave breaking phenomena and the permanent waves.

In our case, appearance of the Earth's rotation, however, introduces a cubic-order nonlinear term $\rho(\rho u)_x$ to R2CH system, which is difficult to estimate as usually by using the conservation laws. To deal with this higher nonlinearity, we reformulate the first equation in (1.4) into (1.5) with a nonlocal translate $\Omega\partial_x G*\rho^2$, and establish the Riccati differential inequality for $K_x = u_x + \Omega\partial_x G * \rho^2$. Then by solving the inequality and use the fact that the term $\Omega G * \rho^2$ is bounded, u_x blows up if and only if K_x blows up. But the advantage of considering K is that in the equation for K and K_x, the cubic terms can be bounded by the conservation laws, which enables one to carry out a standard procedure to reach a Riccati type inequality for K_x

$$\frac{d}{dt}K_x \le -K_x^2 + C,$$

and thus by choosing K_x sufficiently negative initially, the corresponding solution blows up in finite time. A crucial ingredient in this argument is the use of the "global" information of solutions (like the conservation laws) in deriving various estimates. However the "local" structure of solutions is under appreciated. On the other hand, the non-diffusive nature of the system indicates that the local structure of data may strongly affect the evolution of the solutions, in particular, the blow-ups. This has recently been evidenced in a class of CH-type equations in a series of works of Brandolese and Cortez [1, 2, 3]and Zhu [36, 37, 38], and later extended to some other quasilinear model equations with higher order nonlinearities. One of the main ideas lies in understanding of the interplay between the solution and its gradient. For this amounts to tracking the dynamics of $K \pm K_x$ along the characteristics. Due to the nonlocal character involved in K, the conservation law is still needed to establish the convolution estimate. However it is now much apparent to see how rotation affects the wave-breaking. In particular, when the Coriolis effect is turned off our wave-breaking criteria recovers the one for the classical CH equation.

Our goal in the present paper to investigate the conditions for R2CH system to ensure the occurrence of the wave-breaking phenomena or permanent waves. In Section 2, we derived the R-b-family system as a model in the equatorial water waves. In Section 3, we recall some basic results concerning the formation of singularities in the R-b-family system and R2CH system. In Section 4, we obtain the r2CH system when linear dispersion is absent in R2CH system. Then we give two kinds of the wave-breaking criterion on which addresses the local structure of the solutions and also indicates explicitly how rotation is involved. Moreover, we further provide an upper bound of u_x along each characteristics emanating from a vanishing point of ρ_0.

2. Derivation of the model

2.1. **Governing equations.** Assume that the Earth to be a perfect sphere of radius 6371 km and with a constant rotational speed $\Omega = 73 \times 10^{-6}$ rad/s round the polar axis towards east. We choose a reference frame with the origin located at a point on the earth's surface and which is rotating with earth, setting x-axis horizontally due east, the y-axis horizontally due north and the z-axis upwards. We employ the f-plane approximation from the full geophysical governing equations [9]:

$$\begin{gathered} u_t + uu_x + vu_y + \omega u_z + 2\Omega\omega = -p_x \\ v_t + uv_x + vv_y + \omega u_z = -p_y, \\ \omega_t + u\omega_x + v\omega_y + \omega\omega_z - 2\Omega u = -p_z - g, \end{gathered} \tag{2.1}$$

here (u, v, ω) is the fluid velocity, p is the pressure, t represents time.

At the wave surface, the pressure of the fluid matches the atmospheric pressure p_{atm} and assume the fluid bed is impermeable, we impose the no-flux condition, then follow the idea in to derive the rotation-two-component CH system with effect of the Coriolis force. We use the undisturbed depth of the water h, as the vertical scale, a typical wavelength λ, as the horizontal scale, and a typical amplitude of the surface a, and we denote the dimensionless parameters $\varepsilon = a/h$ and $\delta = h/\lambda$. Let $z = h + \eta(t, x, y)$ be the surface of the ocean, and set $z = h$ to be the mean surface level for the flow, with $z = 0$ we denote the lower boundary of the water. Under the assumption that the constant density of the water is one, the governing equations in the region $0 \le z \le h+\eta(t, x, y)$ in the f-plane approximation comprise the Euler equation we summarize the above equations then we have the form

$$\begin{gathered} u_t + uu_x + \omega u_x + 2\Omega\omega = -p_x, \quad 0 < z < h + \eta(t, x) \\ \omega_t + u\omega_x + \omega\omega_z - 2\Omega u = -p_z - g, \quad 0 < z < h + \eta(t, x) \\ u_x + \omega_z = 0, \quad 0 < z < h + \eta(t, x) \\ u_z - \omega_x = \gamma, \quad 0 < z < h + \eta(t, x) \\ p = p_{atm}, \quad \text{on } z = h + \eta(t, x), \\ \omega = \eta_t + u\eta_x, \quad \text{on } z = h + \eta(t, x), \\ \omega = 0, \quad \text{on } z = 0, \end{gathered} \tag{2.2}$$

2.2. **Derivation of the model.** In this section, we follow the ideas in to derived the rotation-two-component CH system with effect of the Coriolis force. We first introduce a non-dimensional of the variables. For this purpose we use the undisturbed depth of the water h, as the vertical scale, a typical wavelength λ as the horizontal scale, and a typical amplitude of the surface a, and we denote the dimensionless parameters $\varepsilon = a/h$ and $\delta = h/\lambda$. Then we make the following change of variables

$$x \to \lambda x, \quad z \to hz, \quad \eta \to a\eta, \quad t \to \frac{\lambda}{\sqrt{gh}} t, \quad u \to \sqrt{gh} u,$$

$$\Omega \to \frac{\sqrt{gh}}{h}\Omega, \quad p \to p_{atm} + g(h-z) + ghp, \quad \gamma \to \frac{\sqrt{gh}}{h}\gamma$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u and ω, on the right-hand side. Therefore, the geographic water-wave problem transforms into

$$\begin{gathered} u_t + uu_x + \omega u_z + 2\omega\omega = -p_x, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ \delta^2(\omega_t + u\omega_x + \omega\omega_z) - 2\Omega u = -p_z, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ u_x + \omega_z = 0, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ u_z - \delta^2\omega_x = \gamma, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ p = \varepsilon\eta, \quad \text{on } z = 1 + \varepsilon\eta(t,x), \\ \omega = \varepsilon(\eta_t + u\eta_x), \quad \text{on } z = 1 + \varepsilon\eta(t,x), \\ \omega = 0, \quad \text{on } z = 0, \end{gathered} \tag{2.3}$$

We now consider the constant vorticity $\gamma = A$. Using the following scaling around a laminar flow (a simplest nontrivial case):

$$u \to U + \varepsilon u, \quad \omega \to W + \varepsilon w, \quad p \to P + \varepsilon p,$$

where (U, W, P) is the solution to system (2.2), characterized by a flat surface $\eta = 0$ and for which every particle moves horizontally, with a speed that depends linearly on the depth, that is,

$$U = Az, W = 0, P = \Omega A z^2 - \Omega A,$$

the geophysical water-wave problem writes in the new scaling as

$$\begin{gathered} u_t + Azu_x + (A + 2\Omega)\omega + \varepsilon(uu_x + \omega u_x) = -p_x, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ \delta^2(\omega_t + Az\omega_x + \varepsilon(u\omega_x + \omega\omega_z)) - 2\Omega u = -p_z, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ u_x + \omega_z = 0, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ u_z - \delta^2\omega_x = 0, \quad 0 < z < 1 + \varepsilon\eta(t,x) \\ p - (1 - 2\Omega A)\eta + \varepsilon\Omega A\eta^2, \quad \text{on } z = 1 + \varepsilon\eta(t,x), \\ \omega - (\eta_t + \varepsilon u\eta_x + \varepsilon A\eta\eta_x + A\eta_x) = 0, \quad \text{on } z = 1 + \varepsilon\eta(t,x), \\ \omega = 0, \quad \text{on } z = 0, \end{gathered} \tag{2.4}$$

It then follow from (2.4) that

$$u = u_0 - \delta\frac{z^2}{2}u_{0xx} + o(\varepsilon^2, \delta^4, \varepsilon\delta^2), \tag{2.5}$$

$$\omega = -zu_{0,x} + \delta^2\frac{z^3}{6}u_{0xxx} + o(\varepsilon^2, \delta^4, \varepsilon\delta^2), \tag{2.6}$$

where $u_0(x,t)$ is the leading order approximation for u. Note that u_0 does not depend on z in view of the above equation, since $u_z = 0$ when $\delta \to 0$.

Combining (2.4) with (2.5) and (2.6), we obtain

$$\eta_t + A\eta_x + ((1 + \varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2)_x - \delta^2\frac{1}{6}u_{0xxx} = 0, \tag{2.7}$$

ignoring terms of order $o(\varepsilon^2, \delta^4, \varepsilon\delta^2)$. With the same method, we ignore the terms of order $o(\varepsilon^2, \delta^4, \varepsilon\delta^2)$.

$$p = (1 - 2\Omega A)\eta - \varepsilon\Omega A\eta^2 - \delta^2(\frac{1 - z^2}{2}u_{0xt} + A\frac{1 - z^3}{3}u_{0xx})$$

$$
- 2\Omega u_0(1+\varepsilon\eta - z) + \delta^2\Omega\frac{1-z^3}{3}u_{0xx}.
$$

It is then inferred from (2.4) that

$$
\begin{aligned}
&(u_0 - \frac{\delta^2}{2}u_{0xx})_t + \eta_x + \varepsilon u_0 u_{0x} - \delta^2\frac{A}{3}u_{0xxx} + 2\Omega(\eta_t + a\eta_x - \frac{\delta^2}{6}u_{0xxx})\\
&- 2\Omega A\eta_x + \delta^2\frac{\Omega}{3}u_{0xxx} = 0
\end{aligned}
$$

as a result of (2.7). Consequently, we deduce the following two equations

$$
\begin{gathered}
\eta_t + A\eta_x + ((1+\varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2)_x - \delta^2\frac{1}{6}u_{0xxx} = 0,\\
(u_0 - \frac{\delta^2}{2}u_{0xx})_t + \eta_x + \varepsilon u_0u_{0x} - \delta^2\frac{A}{3}u_{0xxx} + 2\Omega\eta_t = 0.
\end{gathered} \tag{2.8}
$$

Letting both the parameters ε and δ tend to zero, we obtain from the system of linear equations

$$
\begin{gathered}
\eta_t + A\eta_x + u_{0x} = 0,\\
u_{0t} + 2\Omega\eta_t + \eta_x = 0.
\end{gathered} \tag{2.9}
$$

The equivalence of the above systems then gives

$$
\begin{gathered}
\eta_t + A\eta_x + u_{0x} = 0,\\
u_{0t} + (1-2\Omega A)\eta_x - 2\Omega u_{0x} = 0.
\end{gathered} \tag{2.10}
$$

which is useful in our later calculation. In view of (2.9), we obtain

$$
\eta_{tt} + (A - 2\Omega)\eta_{xt} - \eta_{xx} = 0 \tag{2.11}
$$

The linear equation has a travelling wave solution $\eta = \eta(x-ct)$ with a velocity c satisfying

$$
c^2 - (A-2\Omega)c - 1 = 0.
$$

There is one positive and one negative solution, representing left and right running waves. We assume that we have only one of these waves. Then

$$
\eta = \frac{1}{c-A}u_0 = o(\varepsilon, \delta^2) = \frac{2\Omega + c}{1-2\Omega A}u_0 + o(\varepsilon,\delta^2).
$$

Here we choose $A \neq c$ as if $A = c$ then we have from (2.10) that u_0 is a constant, and this is not the case we consider.

Let us introduce a new variable

$$
\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\nu u_{0xx},
$$

for some constants α, β and ν. These constants will be determined in our further considerations. The variable ρ will be used instead of η has a tool for mathematical simplification of our equations. The expansion of ρ^2 in the same order of ε and δ^2 is

$$
\rho^2 = 1 + 2\varepsilon\alpha\eta + \varepsilon^2(\alpha^2 + 2\beta)\eta^2 + 2\varepsilon\delta^2\nu u_{0xx}.
$$

With this definition it is found

$$
\rho_t + A\rho_x = \varepsilon\alpha(\eta_t + A\eta_x) - 2\varepsilon^2\beta\eta u_{0x} + \varepsilon\delta^2\nu(A-c)u_{0xxx}.
$$

which implies

$$
\eta_t + A\eta_x = \frac{\rho_t + A\rho_x}{\varepsilon\alpha} + \varepsilon\frac{\beta}{\alpha}(\eta u_0)_x - \delta^2\frac{\nu}{\alpha}(A-c)u_{0xxx}.
$$

Then we rewrite in the form

$$\frac{\rho_t + A\rho_x}{\alpha\varepsilon} + \delta^2(\frac{\nu}{\alpha}(c-A) - \frac{1}{6})u_{0xxx} + ((1+\varepsilon(1+\frac{A}{2}\frac{2\Omega + c}{1-2\Omega A} + \frac{\beta}{\alpha})\eta)u_0)_x = 0. \quad (2.12)$$

One can eliminate the u_{0xxx} term by choosing

$$\frac{\nu}{\alpha} = \frac{1}{6(c-A)},$$

and with the choice

$$\alpha = 1 + \frac{A}{2}\frac{2\Omega + c}{1-2\Omega A} + \frac{\beta}{\alpha},$$

we can write (2.12) in the form

$$\rho_t + A\rho_x + \alpha\varepsilon(\rho u_0)_x = 0,$$

which contains only the variables ρ and u_0 but not η. Expressing η in terms of ρ,

$$\begin{aligned}\frac{(\rho^2)_x}{2\varepsilon\alpha} &= \eta_x + \varepsilon\frac{\alpha^2 + 2\beta}{\alpha}\eta\eta_x + \delta^2\frac{\nu}{\alpha}u_{0xxx},\\ \frac{(\rho^2)_t}{2\varepsilon\alpha} &= \eta_t + \varepsilon\frac{\alpha^2 + 2\beta}{\alpha}\eta\eta_t + \delta^2\frac{\nu}{\alpha}u_{0xxt},\end{aligned}$$

we have

$$\begin{aligned}\eta_x + 2\Omega\eta_t = &\frac{1-2\Omega A}{\varepsilon\alpha}\rho\rho_x - 2\Omega\rho(\rho u_0)_x - \varepsilon\frac{\alpha^2 + 2\beta}{\alpha}\eta c u_{0x}\\ &- \delta^2\frac{\nu}{\alpha}u_{0xxx} - 2\Omega\delta^2\frac{\nu}{\alpha}u_{0xxt}.\end{aligned} \quad (2.13)$$

Replacing $\eta_x + 2\Omega\eta_t$ in (2.8) by the above equation and a simple computation shows that thus

$$\begin{aligned}&(u_0 - \delta^2 u_{0,xx})_t + A(u_0 - \delta^2 u_{0,xx})_x - Au_{0,x} + \delta^2(\frac{2A}{3} - \frac{\nu}{\alpha} - \frac{c}{2} + 2\Omega c\frac{\nu}{\alpha})u_{0xxx}\\ &+ \varepsilon(1 - \frac{\alpha^2 + 2\beta}{\alpha}\cdot\frac{c(2\Omega + c)}{1-2\Omega A})u_0u_{0,x} + (\frac{1-2\Omega A}{\varepsilon\alpha})\rho\rho_x - 2\Omega\rho(\rho u_0)_x = 0.\end{aligned} \quad (2.14)$$

Then breaking $u_0u_{0,x}$ as

$$u_0u_{0,x} = s(bmu_{0,x} + u_0m_x) + (1-(b+1)s)u_0u_{0,x} + O(\delta^2). \quad (2.15)$$

Here $m = u_0 - \delta^2 u_{0,xx}$, then we obtain from (2.14) and (2.15) that

$$\begin{aligned}&(u_0 - \delta^2 u_{0,xx})_t + A(u_0 - \delta^2 u_{0,xx})_x - Au_{0,x} + \delta^2(\frac{2A}{3} - \frac{\nu}{\alpha} - \frac{c}{2} + 2\Omega c\frac{\nu}{\alpha})u_{0xxx}\\ &+ \varepsilon s(1 - \frac{\alpha^2 + 2\beta}{\alpha}\cdot\frac{c(2\Omega + c)}{1-2\Omega A})(bmu_{0x} + u_0m_x) + \frac{1-2\Omega A}{\epsilon\alpha}\rho\rho_x - 2\Omega\rho(\rho u_0)_x\\ &+ \varepsilon(1-(b+1)s)(1 - \frac{\alpha^2 + 2\beta^2}{\alpha}\cdot\frac{c(2\Omega + c)}{1-2\Omega A})u_0u_{0x} = 0\end{aligned} \quad (2.16)$$

By the scaling $u_0 \to \frac{1}{\alpha\varepsilon}u_0, x \to \delta x, t \to \delta t$, we obtain

$$
\begin{aligned}
&m_t + Am_x - Au_{0x} + (\frac{2A}{3} - \frac{\nu}{\alpha} - \frac{c}{2} + 2\Omega c\frac{\nu}{\alpha})u_{0xxx} \\
&+ \frac{s}{\alpha}(1 - \frac{\alpha^2+2\beta}{\alpha}\cdot\frac{c(2\Omega+c)}{1-2\Omega A})(bmu_{0x} + u_0m_x) \\
&+ (1-2\Omega A)\rho\rho_x - 2\Omega\rho(\rho u_0)_x \\
&+ \frac{1-(b+1)s}{\alpha}(1 - \frac{\alpha^2+2\beta}{\alpha}\frac{c(2\Omega+c)}{1-2\Omega A})u_0u_{0,x} = 0, \\
&m = u_0 - u_{0xx}, \\
&\rho_t + A\rho_x + (\rho u_0)_x = 0,
\end{aligned}
\tag{2.17}
$$

If we choose

$$
1 - \frac{\alpha^2+2\beta}{\alpha}\frac{c(2\Omega+c)}{1-2\Omega A} = (b+1)\alpha
$$

and denote

$$
\mu = \frac{2A}{3} - \frac{\nu}{\alpha} - \frac{c}{2} + 2\Omega c\frac{\nu}{\alpha}, \quad \sigma = (b+1)s.
$$

then we arrive at

$$
\begin{aligned}
&m_t + Am_x - Au_{0x} + \mu u_{0xxx} + \sigma(bmu_{0x} + u_0m_x) \\
&+ (b+1)(1-\sigma)uu_{0x} + (1-2\Omega A)\rho\rho_x - 2\Omega\rho(\rho u_0)_x = 0, \\
&m = u_0 - u_{0xx}, \\
&\rho_t + A\rho_x + (\rho u_0)_x = 0,
\end{aligned}
\tag{2.18}
$$

with the constant α, β, ν, μ and c satisfying

$$
\begin{gathered}
c^2 - (A-2\Omega)c - 1 = 0, \\
\alpha = \frac{(1-2\Omega A) + 2c(2\Omega+c)(1+\frac{A}{2}\frac{2\Omega+c}{1-2\Omega A})}{3(1-2\Omega A + c(2\Omega+c))}, \\
\beta = \alpha^2 - \alpha(1 + \frac{A}{2}\frac{2\Omega+c}{1-2\Omega A}), \\
\nu = \frac{\alpha}{6(c-A)}, \\
\mu = \frac{2A}{3} - \frac{\nu}{\alpha} - \frac{c}{2} + 2\Omega c\frac{\nu}{\alpha}.
\end{gathered}
$$

With a further Galilean transformation $x \to x - At, t \to t$, we drop the terms Am_x and $A\rho_x$ in (2.18) and hence get

$$
\begin{gathered}
u_t - u_{xxt} - Au_x + (b+1)uu_x \\
= \sigma(bu_xu_{xx} + uu_{xxx}) - \mu u_{xxx} - (1-2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, \\
\rho_t + (\rho u)_x = 0,
\end{gathered}
\tag{2.19}
$$

Recalling the change of variables $\Omega \to \frac{\sqrt{gh}}{h}\Omega, A \to \frac{\sqrt{gh}}{h}A$, and noticing that $\Omega = 7.3\times10^{-5}$rad/s, A is of order 10^{-2} and h is less than 300 m in the physical variables. One then can assume that $1 - 2\Omega A > 0$.

3. Preliminaries

In this section, we recall some basic results concerning the formation of singularities in the R-b-family system (1.1) and the R2CH system (1.4).

Let $\sigma = 1, \mu = 0$, then the periodic of the R2CH system is been written as the system

$$
\begin{aligned}
u_t + uu_x = -\partial_x G * \Big(- Au + u^2 + \frac{1}{2}u_x^2 + \frac{1-2\Omega A}{2}\rho^2 - \Omega\rho^2 u\Big) \\
+ \Omega G * (\rho^2 u_x), \\
\rho_t + u\rho_x = -\rho u_x, \\
u(0,x) = u(1,x), \\
\rho(0,x) = \rho(1,x).
\end{aligned}
\tag{3.1}
$$

for $t > 0$ and $x \in \mathbb{S}$, which is called r2CH system. Similarity [21], we can get the following the theorem about the periodic R-b-family system (1.1).

Theorem 3.1. *Given $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, there exist a maximal $T = T(z_0) > 0$ and a unique solution $z = (u, \rho)$ to system (1.1) such that*

$$
z = z(\cdot; z_0) \in C([0,T); H^s(\mathbb{S}) \times H(\mathbb{S})^{s-1}) \cap C^1([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})). \tag{3.2}
$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$
z_0 \to z(\cdot; z_0) : H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to C([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0,T); H^{s-1} \times H^{s-2})
$$

is continuous.

Similar to the proof of [30, Theorem 6.2]. We have the following blow-up criterion for the periodic R-b-family system (1.1).

Lemma 3.2 (Wave-breaking criteria)**.** *Assume that $1 - 2\Omega A > 0$. Let $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > 3/2$, and $T > 0$ be the maximal time of existence of the solution (u, ρ) to system (1.1) with initial data (u_0, ρ_0). Then the corresponding solution (u, ρ) blows up in finite time $T < \infty$ if and only if*

$$
\lim_{t \to T^-} \sup_{x \in \mathbb{S}} |u_x(t,x)| = +\infty.
$$

Lemma 3.3 ([35])**.** *For every $f \in H^1(\mathbb{S})$, we have*

$$
\max_{x \in [0,1]} f^2(x) \le C \int_{\mathbb{S}} (f^2 + \alpha^2 f_x^2) dx,
$$

where

$$
C = \frac{\cosh(\frac{1}{2\alpha})}{2\alpha \sinh(\frac{1}{2\alpha})}.
$$

Moreover C is the minimum value. so in this sense, C is the optimal constant which is obtained by the associated Green function.

$$
G = \frac{\cosh(\frac{x}{\alpha} - \frac{[x]}{\alpha} - \frac{1}{2\alpha})}{2\alpha \sinh(\frac{1}{2\alpha})}.
$$

Note that when $\alpha = 1$, the constant $C_1 = \frac{e+1}{2(e-1)}$ is sharp.

To study the wave-breaking problem, we now briefly give the needed results without proof to pursue our goal. We consider the following associated Lagrangian scales of the system (3.1),

$$\begin{aligned} \frac{\partial q}{\partial t} &= u(t,q), \quad 0<t<T,\\ q(0,x) &= x, \quad x\in\mathbb{S}, \end{aligned} \tag{3.3}$$

where $u\in C^1([0,T),H^{s-1}(\mathbb{S}))$ is the first component of the solution (u,ρ) to (3.1).

Lemma 3.4 ([36])**.** *Let (u,ρ) be the solution of system (3.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. Then system (3.1) has a unique solution $q\in C^1([0,T)\times\mathbb{S},\mathbb{S})$. This equation satisfies $q(t,x+1)=q(t,x)+1$. Moreover, the map $q(t,\cdot)$ is increasing diffeomorphisms of $\mathbb{S}$ with*

$$q_x(t,x)=\exp\Big(\int_0^t u_x(\tau,q(\tau,x))d\tau\Big)>0, \quad (t,x)\in[0,T)\times\mathbb{S},$$

The above lemmas indicate that $q(t,\cdot):\mathbb{S}\to\mathbb{S}$ is diffeomorphisms of the line for each $t\in[0,T)$. Hence, the L^∞ norm of any function $u(t,\cdot)\in L^\infty(\mathbb{S})$ is preserved under the family of diffeomorphisms $q(t,\cdot)$ with $t\in[0,T)$, that is

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{S})}=\|u(t,q(t,\cdot))\|_{L^\infty(\mathbb{S})}, \quad t\in[0,T).$$

Similarly, we have

$$\begin{aligned} \inf_{x\in\mathbb{S}} u(t,x) &= \inf_{x\in\mathbb{S}} u(t,q(t,x)), \quad t\in[0,T),\\ \sup_{x\in\mathbb{S}} u(t,x) &= \sup_{x\in\mathbb{S}} u(t,q(t,x)), \quad t\in[0,T). \end{aligned}$$

Lemma 3.5. *[[20]] For all $u\in H^1(\mathbb{S})$, the following inequality holds*

$$G*(u^2+\frac{1}{2}u_x^2)\geq \kappa u^2(x),$$

with

$$\kappa=\frac{1}{2}+\frac{\arctan(\sinh(1/2))}{2\sinh(1/2)+2\arctan(\sinh(1/2))\sinh^2(1/2)}\approx 0.869.$$

Moreover, κ is the optimal constant obtained by the function

$$f_0=\frac{1+\arctan(\sinh(x-[x]-1/2))\sinh(x-[x]-1/2)}{1+\arctan(\sinh(1/2))\sinh(1/2)}.$$

We then prove several useful conservation laws of strong solutions to r2CH system (3.1).

Lemma 3.6. *Let $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s>3/2$, and T be the maximal existence time of the solution (u,ρ) in the periodic of the r2CH system (3.1). Then for all $t\in[0,T)$, we have*

$$\int_{\mathbb{S}} u^2+u_x^2+(1-2\Omega A)(\rho-1)^2dx=\int_{\mathbb{S}} u_0^2+u_{0x}^2+(1-2\Omega A)(\rho_0-1)^2dx \tag{3.4}$$

which means that

$$E(u,\rho)=\frac{1}{2}\int_{\mathbb{S}}\big(u^2+u_x^2+(1-2\Omega A)(\rho-1)^2\big)\,dx=E_0(u_0,\rho_0).$$

Proof. Multiplying the first equation of r2CH (3.1) by u and integrating by parts, in view of the periodicity of u and ρ, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}(u^2+u_x^2)dx = -(1-2\Omega A)\int_{\mathbb{S}} u\rho\rho_x\,dx,$$
$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}(1-2\Omega A)(\rho-1)^2dx = (1-2\Omega A)\int_{\mathbb{S}} u\rho\rho_x\,dx$$

Adding the above two equations, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u^2+u_x^2+(1-2\Omega A)(\rho-1)^2dx = 0\,. \tag{3.5}$$

From the above equation, we obtain the statement of the lemma. □

Lemma 3.7. *Let $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s>3/2$, and T be the maximal existence time of the solution (u,ρ) in the periodic of the r2CH system (3.1). Then for all $t\in[0,T)$, we have*

$$\int_{\mathbb{S}} u+\Omega(\rho-1)^2dx = \int_{\mathbb{S}} u_0+\Omega(\rho_0-1)^2dx, \tag{3.6}$$
$$\int_{\mathbb{S}}(\rho-1)dx = \int_{\mathbb{S}}(\rho_0-1)dx. \tag{3.7}$$

which means that

$$I_1(u,\rho) = \int_{\mathbb{S}} u+\Omega(\rho-1)^2dx = \int_{\mathbb{S}} u_0+\Omega(\rho_0-1)^2dx,$$
$$I_2(u,\rho) = \int_{\mathbb{S}}(\rho-1)dx = \int_{\mathbb{S}}(\rho_0-1)dx.$$

Proof. Integrating the first equation of (3.1) by parts, in view of the periodicity of u and G, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}} u(t,x)dx\\ &= \int_{\mathbb{S}} -\partial_x G*(-Au+u^2+\frac{1}{2}u_x^2+\frac{1-2\Omega A}{2}\rho^2-\Omega\rho^2u)+\Omega G*(\rho^2u_x)dx\end{aligned} \tag{3.8}$$

Multiplying the second equation by $\Omega(\rho-1)$ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}}\Omega(\rho-1)^2dx = \int_{\mathbb{S}}\Omega u(\rho^2)_x\,dx. \tag{3.9}$$

Adding the above two equations and using the identity $G*f = f+\partial_x^2G*f$, we obtain $\frac{d}{dt}\int_{\mathbb{S}} u+\Omega(\rho-1)^2dx = 0$. On the other hand, integrating the second equation, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}(\rho-1)dx = 0. \tag{3.10}$$

This completes the proof. □

4. Blow-up criteria

In this subsection we address the problem of the blow-up criteria of the periodic r2CH system (3.1). The following blow-up criterion can be proved easily by Lemma 3.2, so that we omit its proof.

Lemma 4.1. *Assume that* $1-2\Omega A>0$. *Let* $(u_0,\rho_0-1)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, *with* $s>3/2$, *and* $T>0$ *be the maximal time of existence of the solution* (u,ρ) *to r2CH system* (3.1) *with initial data* (u_0,ρ_0). *Then the corresponding solution* (u,ρ) *blows up in finite time* $T<\infty$ *if and only if*

$$\lim_{t\to T^-}\big\{\inf_{x\in\mathbb{S}} u_x(t,x)\big\}=-\infty.$$

For wave-breaking, one would like to choose some initial data such that u_x approaches $-\infty$ in finite time. The difficulty in the analysis of the dynamics of u_x sources from the last term $G_x*(\rho^2u_x)$, which fails to be controlled by the conservation laws. Our idea is to absorb this term by considering the dynamics of the quantity $K:=u+\Omega G*\rho^2$. The following lemma is about K, which is important to Theorem 4.3.

Lemma 4.2. *Let* $K:=u+\Omega G*\rho^2$. *Then, R2CH system* (1.4) *can be written with the following equation,*

$$K_t+uK_x=\Omega(A-\mu)\partial_xG*\rho^2+\Omega\sigma u\partial_xG*\rho^2-\partial_xG*\big((\mu-A)u+\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\big)$$

and

$$\begin{aligned}
&K_{xt}+(\sigma u-\mu)K_{xx}\\
&=-\frac{\sigma}{2}(K_x-\Omega\partial_xG*\rho^2)^2+\frac{1+2\Omega(\mu-A)-2\Omega\sigma u}{2}\rho^2+\Omega\sigma uG*\rho^2\\
&\quad+(A-\mu)\partial_x^2G*u+\frac{3-\sigma}{2}u^2-G*\big(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1-2\Omega(A-\mu)}{2}\rho^2\big).
\end{aligned}$$

Proof. Recall the first equation in (1.4) in the form

$$\begin{aligned}
&u_t-u_{xxt}-Au_x+3uu_x\\
&=\sigma(2u_xu_{xx}+uu_{xxx})-\mu u_{xxx}-(1-2\Omega A)\rho\rho_x+2\Omega\rho(\rho u)_x.
\end{aligned}\tag{4.1}$$

To deal with the high-order nonlinear term $\rho(\rho u)_x$, it is found that

$$\begin{aligned}
u_t-u_{xxt}+2\Omega\rho\rho_t&=A(u-u_{xx}+\Omega\rho^2)_x+(A-\mu)u_{xxx}-3uu_x\\
&\quad+\sigma(2u_xu_{xxx}+uu_{xxx})-\mu u_{xxx}-\rho\rho_x.
\end{aligned}\tag{4.2}$$

Applying the operator $(1-\partial_x^2)^{-1}$ to both sides of (4.2), we have

$$\begin{aligned}
&(u+\Omega(1-\partial_x^2)^{-1}\rho^2)_t-A(u+\Omega(1-\partial_x^2)^{-1}\rho^2)_x\\
&=(\mu-A)[u+\Omega(1-\partial_x^2)^{-1}\rho^2]_x-(\mu-A)\Omega\partial_x(1-\partial_x^2)^{-1}\rho^2\\
&\quad-\sigma u[u+\Omega(1-\partial_x^2)^{-1}\rho^2]_x+\Omega\sigma u\partial_x(1-\partial_x^2)^{-1}\rho^2\\
&\quad-\partial_x(1-\partial_x^2)^{-1}[(\mu-A)u_x+\frac{\sigma}{2}u_x^2+\frac{3-\sigma}{2}u^2+\frac{1}{2}\rho^2].
\end{aligned}\tag{4.3}$$

Taking $K = u + \Omega p * \rho^2$, we deduce that

$$
\begin{aligned}
&K_t - AK_x \\
&= (\mu - A)K_x - \sigma u K_x - \Omega(\mu - A)\partial_x(1-\partial_x^2)^{-1}\rho^2 + \Omega\sigma u\partial_x(1-\partial_x^2)^{-1}\rho^2 \\
&\quad - \partial_x(1-\partial_x^2)^{-1}\big((\mu - A)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2\big).
\end{aligned}
\tag{4.4}
$$

And using that $(1-\partial_x^2)^{-1}f = G * f$, we obtain the statement of Lemma.

Now we take derivative to equation (4.4) with respect to x, and use $-\partial_x^2 G * f = f - G * f$ to obtain

$$
\begin{aligned}
&K_{xt} + (\sigma u - \mu)K_{xx} \\
&= -\sigma u_x^2 + \Omega(\mu - A)(\rho^2 - G * \rho^2) + \Omega\sigma u(G * \rho^2 - \rho^2) - (\mu - A)\partial_x^2 G * u \\
&\quad + \frac{b+1-\sigma}{2}(u^2 - G * u^2) + \frac{\sigma}{2}u_x^2 - \frac{\sigma}{2}G * u_x^2 + \frac{\rho^2}{2} - G * (\frac{\rho^2}{2})
\end{aligned}
$$

A rearrangement of the equation leads to the lemma. This completes the proof. $\square$

Theorem 4.3. *Suppose that $1 - 2\Omega A > 0$. Let (u, ρ) be the solution of r2CH system* (3.1) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > \frac{3}{2}$ and T be the maximal time of existence. Assume there exists a x_0 such that*

$$
\rho_0(x_0) = 0, \tag{4.5}
$$

and

$$
u_{0,x}(x_0) < -|u_0(x_0) - \frac{A}{2}| - 4\Omega C_1 - \sqrt{\frac{2A^2}{e-1} + 16\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)}}. \tag{4.6}
$$

where the constant C_1 is the following

$$
C_1 = \frac{e}{2(e-1)}\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)} + \frac{e}{e-1}\Big(\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)}\Big)^{1/2} + \frac{e}{2(e-1)}.
$$

Then the corresponding solution (u, ρ) to system (3.1) *will blow up in finite time in the following sense, there is a T_1 with*

$$
0 < T_1 \le \frac{8}{\sqrt{K_{0,x}^2(x_0) - (K_0(x_0) - \frac{A}{2})^2}}. \tag{4.7}
$$

with

$$
K_0(x) = u_0(x) + \Omega(1-\partial_x^2)^{-1}(\rho^2)(0,x) \tag{4.8}
$$

such that

$$
\lim_{t\to T_1}\inf_{x\in\mathbb{S}} u_x(t,x) = -\infty,
$$

Remark 4.4. Note that in the case when $\Omega = 0$, the condition on the velocity u reduces to the same one as for the classical Camassa-Holm equation with linear dispersion. Here the appearance of the Coriolis effect brings up delicate interaction between the surface and the velocity. To control the additional terms in the blow-up analysis we are forced to use the conservation law of $E(u, \rho)$, as can be seen from the following proof.

Proof. We have just need to consider $s \geq 3$. We follow the characteristics of the r2CH system to generate finite-time blow-up. Hence we define the characteristics $q(t,x)$ as

$$\begin{gathered} q_t(t,x) = u(t, q(t,x)), \quad x \in \mathbb{S},\ t \in [0,T] \\ \rho_t + u\rho_x = -\rho u_x, \end{gathered} \tag{4.9}$$

Then we can easily check that $q \in C^1([0,T] \times \mathbb{S}, \mathbb{S})$ with $q_x(t,x) > 0$ for all $([0,T] \times \mathbb{S}, \mathbb{S})$. First we take $\sigma = 1, \mu = 0$ then

$$\begin{aligned} K_{xt} + uK_{xx} &= -\frac{1}{2}(K_x - \Omega\partial_x G * \rho^2)^2 + \frac{1 - 2\Omega A - 2\Omega u}{2}\rho^2 + \Omega u G * \rho^2 \\ &\quad + A\partial_x^2 G * u + u^2 - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2). \end{aligned} \tag{4.10}$$

From now on, we abuse of notation by denoting

$$\begin{gathered} u(t) = u(t, q(t, x_0), \quad u_x(t) = u_x(t, q(t, x_0)), \\ K(t) = K(t, q(t, x_0)), \quad K_x(t) = K_x(t, q(t, x_0)). \end{gathered}$$

We further denote by $'$ the material derivative $\partial_t + u\partial_x$ along the characteristics $q(t, x_0)$. Let

$$2G_- = G + G_x = \frac{e^x}{2\sinh(\frac{1}{2})}, \quad 2G_+ = G - G_x = -\frac{e^{-x}}{2\sinh(\frac{1}{2})}$$

for $-\frac{1}{2} < x < \frac{1}{2}$. Then from Lemma 4.2 and (4.10) we see that

$$(K + K_x)' = -2G_- * (u^2 - Au + \frac{1}{2}u_x^2) - \frac{1}{2}u_x^2 + u^2 - Au - (1 - 2\Omega A - 2\Omega u)G_- * \rho^2,$$

$$(K - K_x)' = 2G_+ * (u^2 - Au + \frac{1}{2}u_x^2) + \frac{1}{2}u_x^2 - u^2 + Au + (1 - 2\Omega A - 2\Omega u)G_+ * \rho^2,$$

Applying Lemma 3.3, we have the following convolution estimates

$$\begin{aligned} \frac{e^x}{2\sinh(\frac{1}{2})} * (u^2 - Au + \frac{1}{2}u_x^2) &= \frac{e^x}{2\sinh(\frac{1}{2})} * \big((u - \frac{A}{2})^2 + \frac{1}{2}u_x^2 - \frac{A^2}{4}\big) \\ &\geq \frac{1}{2}(u - \frac{A}{2})^2 - \frac{A^2}{4(e-1)} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \frac{e^{-x}}{2\sinh(\frac{1}{2})} * (u^2 - Au + \frac{1}{2}u_x^2) &= \frac{e^{-x}}{2\sinh(\frac{1}{2})} * \big((u - \frac{A}{2})^2 + \frac{1}{2}u_x^2 - \frac{A^2}{4}\big) \\ &\leq \frac{1}{2}(u - \frac{A}{2})^2 + \frac{A^2}{4(e-1)} \end{aligned}$$

Then, these equations provide the bounds for $(K \pm K_x)'$ as

$$(K + K_x)' \leq -\frac{1}{2}[u_x^2 - (u - \frac{A}{2})^2] + \frac{A^2}{4(e-1)} - (1 - 2\Omega A - 2\Omega u)G_+ * \rho^2. \tag{4.12}$$

Using the same method, we have the inequality

$$(K - K_x)' \geq \frac{1}{2}[u_x^2 - (u - \frac{A}{2})^2] - \frac{A^2}{4(e-1)} + (1 - 2\Omega A - 2\Omega u)G_- * \rho^2. \tag{4.13}$$

Using the fact that

$$(K \pm K_x)' = [(K - \frac{A}{2}) \pm K_x]', \quad (1 - 2\Omega A)p_\pm * \rho^2 \geq 0, \tag{4.14}$$

we can further deduce that

$$
\begin{aligned}
{}' &\le -\frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]+\frac{A^2}{4(e-1)}+2\Omega uG_+*\rho^2,\\
[(K-\frac{A}{2})-K_x]' &\ge \frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]-\frac{A^2}{4(e-1)}-2\Omega uG_-*\rho^2.
\end{aligned}
\tag{4.15}
$$

Using convolution Lemma 3.5 and Lemma 3.6, the above estimates can be bounded by

$$
\begin{aligned}
0 &\le G_\pm*\rho^2 = G_\pm*(\rho-1)^2+2G_\pm*(\rho-1)+G_\pm*1\\
&\le \|G_\pm\|_{L^\infty}\|\rho-1\|_{L^2}^2+2\|G_\pm\|_{L^2}\|\rho-1\|_{L^2}+\frac{e^{1/2}}{4\sinh\frac{1}{2}}\\
&\le \frac{e^{1/2}}{4\sinh(\frac{1}{2})}\|\rho-1\|_{L^2}^2+\frac{2e^{1/2}}{4\sinh(\frac{1}{2})}\|\rho-1\|_{L^2}+\frac{e^{1/2}}{4\sinh(\frac{1}{2})}\\
&\le \frac{e^{1/2}}{4\sinh(\frac{1}{2})}\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)}+\frac{e^{1/2}}{2\sinh(\frac{1}{2})}(\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)})^{1/2}+\frac{e^{1/2}}{4\sinh(\frac{1}{2})}\\
&= \frac{e}{2(e-1)}\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)}+\frac{e}{e-1}(\frac{E_0(u_0,\rho_0)}{2(1-2\Omega A)})^{1/2}+\frac{e}{2(e-1)}\\
&\equiv C_1.
\end{aligned}
\tag{4.16}
$$

where we have used Lemma 3.6 and the fact that

$$
\|G_\pm\|_{L^\infty}=\frac{e^{1/2}}{4\sinh(\frac{1}{2})}=\frac{e}{2(e-1)},\quad \|\rho-1\|_{L^2}^2=\frac{E_0(u_0,\rho_0)}{1-2\Omega A}.
$$

and

$$
|uG_\pm*\rho^2|\le\|u\|_{L^\infty}\|G_\pm*\rho^2|_{L^\infty}\le\Big(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\Big)^{1/2}C_1. \tag{4.17}
$$

Putting these equations together into (4.15), we can further conclude that

$$
\begin{aligned}
&[(K-\frac{A}{2})+K_x]'\\
&\le -\frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]+\frac{A^2}{4(e-1)}+2\Omega C_1\left(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\right)^{1/2},\\
&[(K-\frac{A}{2})-K_x]'\\
&\ge \frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]-\frac{A^2}{4(e-1)}-2\Omega C_1\left(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\right)^{1/2}.
\end{aligned}
\tag{4.18}
$$

Then, using $(K-\frac{A}{2})+K_x=u+\Omega G*\rho^2-\frac{A}{2}+u_x=(u-\frac{A}{2})+u_x+\Omega G*\rho^2$, we can obtain the inequalities

$$
\begin{aligned}
(u-\frac{A}{2})+u_x\le(K-\frac{A}{2})+K_x\le(u-\frac{A}{2})+u_x+2\Omega C_1,\\
(u-\frac{A}{2})-u_x\le(K-\frac{A}{2})-K_x\le(u-\frac{A}{2})-u_x+2\Omega C_1.
\end{aligned}
\tag{4.19}
$$

Now if the assumption holds, we have

$$
\frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]-\frac{A^2}{4(e-1)}-2\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)}>0, \tag{4.20}
$$

which implies

$$[(K-\frac{A}{2})+K_x]'(0)<0,\quad [(K-\frac{A}{2})-K_x]'(0)>0. \tag{4.21}$$

Hence at least for a short time t, $K(t)+K_x(t)$ is non-increasing and $K(t)-K_x(t)$ is non-decreasing, then we have

$$\begin{aligned}
&(K(0)-\frac{A}{2})+K_x(0)\\
&<-\Big(\frac{2A^2}{e-1}+16\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)}\Big)^{1/2}-2\Omega C_1,\\
&(K(0)-\frac{A}{2})-K_x(0)\\
&>\Big(\frac{2A^2}{e-1}+16\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)}\Big)^{1/2}+2\Omega C_1.
\end{aligned} \tag{4.22}$$

The short time monotonicity indicates that the above bounds continue to hold, at least for a short time. Therefore, we have

$$\begin{aligned}
(u(t)-\frac{A}{2})+u_x(t)&<-\Big(\frac{2A^2}{e-1}+16\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E(0)}\Big)^{1/2}-2\Omega C_1,\\
(u(t)-\frac{A}{2})-u_x(t)&>\Big(\frac{2A^2}{e-1}+16\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E(0)}\Big)^{1/2}+2\Omega C_1.
\end{aligned} \tag{4.23}$$

Then, plugging these to (4.19), shows that the monotonicity of $(K-\frac{A}{2})+K_x$ persists and thus the bounds of the form in (4.22) continue to hold true, pushing the monotonicity even further in time. Hence, we always have $K(t)-\frac{A}{2}-K_x(t)<0$ is non-increasing, and $K(t)-\frac{A}{2}-K_x(t)>0$ is non-decreasing, which allows us to define the function

$$h(t)=\sqrt{K_x^2(t)-[K(t)-\frac{A}{2}]^2}>0. \tag{4.24}$$

Computing the derivative of h leads to

$$\begin{aligned}
h'(t)&=\frac{-(K-\frac{A}{2}+K_x)'(K-\frac{A}{2}-K_x)-(K-\frac{A}{2}+K_x)(K-\frac{A}{2}-K_x)'}{2\sqrt{K_x^2(t)-[K(t)-\frac{A}{2}]^2}}\\
&\geq\Big(\frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]-\frac{A^2}{4(e-1)}-2\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_(u_0,\rho_0)}\Big)\\
&\quad\times\frac{(K-\frac{A}{2}-K_x)-(K-\frac{A}{2}+K_x)}{2\sqrt{K_x^2(t)-[K(t)-\frac{A}{2}]^2}}\\
&\geq\frac{1}{2}[u_x^2-(u-\frac{A}{2})^2]-\frac{A^2}{4(e-1)}-2\Omega C_1\sqrt{\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)}>0,
\end{aligned} \tag{4.25}$$

where we have used that

$$\frac{(K-\frac{A}{2}-K_x)-(K-\frac{A}{2}+K_x)}{2}\geq h. \tag{4.26}$$

From (4.19) and (4.23), it follows that

$$\begin{aligned} &0 < (K - \frac{A}{2}) - K_x \le 2[(u - \frac{A}{2}) - u_x], \\ &0 < -(K - \frac{A}{2}) - K_x \le -(u - \frac{A}{2}) - u_x. \end{aligned} \tag{4.27}$$

Therefore,

$$K_x^2 - (K - \frac{A}{2})^2 \le 2[u_x^2 - (u - \frac{A}{2})^2],$$

and hence

$$h' \ge \frac{1}{4}h^2 - \frac{A^2}{4(e-1)} - 2\Omega C_1 \sqrt{\frac{e+1}{2(e-1)} E_0(u_0, \rho_0)}. \tag{4.28}$$

Evaluating (4.19) at initial time we have

$$\begin{aligned} (K(0) - \frac{A}{2}) + K_x(0) &\le (u_0(x_0) - \frac{A}{2}) + u_{0,x}(x_0) + 2\Omega C_1 \\ &< -\Big(\frac{A^2}{2(e-1)} + 4\Omega C_1 \sqrt{\frac{e+1}{2(e-1)} E_0(u_0, \rho_0)}\Big)^{1/2}, \\ (K(0) - \frac{A}{2}) - K_x(0) &\ge (u_0(x_0) - \frac{A}{2}) - u_{0,x}(x_0) \\ &> \Big(\frac{A^2}{2(e-1)} + 4\Omega C_1 \sqrt{\frac{e+1}{2(e-1)} E_0(u_0, \rho_0)}\Big)^{1/2}. \end{aligned} \tag{4.29}$$

Therefore

$$h^2(0) > \frac{2A^2}{e-1} + 16\Omega C_1 \sqrt{\frac{e+1}{2(e-1)} E_0(u_0, \rho_0)}.$$

We see that h is increasing and in fact we have

$$h' \ge \frac{1}{8}h^2.$$

This is enough to show that h blows up in finite time. Indeed, we can solve the above equation to get

$$h(t) \ge \frac{8h(0)}{8 - th(0)}. \tag{4.30}$$

Therefore

$$h(t) \to +\infty, \quad t \to \frac{8}{h(0)}.$$

On the other hand, since

$$h(t) \le -K_x = u_x - \Omega p_x * \rho^2$$

and from (4.16) we know that

$$h(t) \le -u_x + 2C_1. \tag{4.31}$$

Therefore $-u_x$ must blow up at time T^* with

$$T^* \le \frac{8}{h(0)} \tag{4.32}$$

This completes the proof. □

Now we give the other blow-up condition about the r2CH system. The following method is an important method to obtain the blow-up time about the shallow wave system.

Theorem 4.5. *Suppose that $1-2\Omega A>0$. Let(u,ρ) be the solution of r2CH*(3.1) *with initial data $(u_0,\rho_0-1)\in H^s\times H^{s-1}$ with $s>\frac{3}{2}$ and T be the maximal time of existence. Assume there exists a x_0 such that*

$$\rho_0(x_0)=0, \tag{4.33}$$

$$u_{0,x}(x_0)<-C_3-2\Omega C_2. \tag{4.34}$$

Then the corresponding solution (u,ρ) to system (3.1) *blows up in finite time in the following sense, there is a T_2 with*

$$\begin{aligned} 0<T_2&<\frac{1}{C_3}\ln\frac{M_1(0)+\Omega C_2-C_3}{M_1(0)+\Omega C_2+C_3}\\ &=\frac{1}{C_3}\ln\frac{u_{0,x}(x_0)+\Omega\partial_{0,x}G*\rho^2(x_0)+\Omega C_2-C_3}{u_{0,x}(x_0)+\Omega\partial_{0,x}G*\rho^2(x_0)+\Omega C_2+C_3}. \end{aligned} \tag{4.35}$$

such that $\lim_{t\to T_1}\inf_{x\in\mathbb{S}}u_x(t,x)=-\infty$, *with*

$$\begin{aligned} C_2&=\frac{1}{2}\frac{1}{1-2\Omega A}E(u_0,\rho_0-1)+I_2(u_0,\rho_0)+\frac{1}{2},\\ \frac{1}{2}C_3^2&=\Omega\left(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\right)^{1/2}C_1+\frac{|A|}{2}E_0^{1/2}(u_0,\rho_0)\\ &\quad+\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)+\frac{e}{4(e-1)}E_0(u_0,\rho_0). \end{aligned}$$

Proof. We have just need to consider $s\ge 3$. Given $x\in\mathbb{S}$, let

$$M_1(t)=K_x(t,q(t,x)),\quad \gamma(t)=\rho(t,q(t,x)),\quad t\in[0,T), \tag{4.36}$$

where $q(t,x)$ is defined by (3.3). Along with the trajectory of $q(t,x)$, we have

$$\gamma'(t)=-\gamma u_x,\quad t\in[0,T). \tag{4.37}$$

Taking $x=x_0$, the assumption $\gamma(0)=\rho_0(x_0)=0$ and Lemma 3.4 imply

$$\gamma(t)\equiv 0,\quad t\in[0,T). \tag{4.38}$$

Then (4.10) has the form

$$M_1'(t)=-\frac{1}{2}(M_1-\Omega\partial_xG*\rho^2)^2+f(t,q(t,x_0)) \tag{4.39}$$

at $(t,q(t,x_0))$, where "$'$" is the derivative with respect to t and

$$f(t,q(t,x_0))=\Omega uG*\rho^2+A\partial_x^2G*u+u^2-G*(u^2+\frac{1}{2}u_x^2+\frac{1-2\Omega A}{2}\rho^2). \tag{4.40}$$

Combining the estimates

$$\begin{aligned}|\partial_x G*\rho^2| &= |\int_0^x \frac{\sinh(x-y-\frac{1}{2})}{2\sinh(\frac{1}{2})}\rho^2 dy + \int_x^1 \frac{\sinh(x-y+\frac{1}{2})}{2\sinh(\frac{1}{2})}\rho^2 dy| \\ &\le \frac{\sinh(\frac{1}{2})}{2\sinh(\frac{1}{2})}\int_0^x \rho^2 dy + \frac{\sinh(\frac{1}{2})}{2\sinh(\frac{1}{2})}\int_x^1 \rho^2 dy \\ &= \frac{1}{2}\int_0^1 \rho^2 dy \\ &= \frac{1}{2}(\int_0^1 [(\rho-1)^2 + 2(\rho-1)+1]dx) \\ &\le \frac{1}{2}\frac{1}{1-2\Omega A}E(u_0,\rho_0-1) + I_2(u_0,\rho_0) + \frac{1}{2} \equiv C_2\end{aligned} \tag{4.41}$$

and

$$|uG*\rho^2| \le \|u\|_{L^\infty}\|G*\rho^2\|_{L^\infty} \le \left(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\right)^{1/2} C_1, \tag{4.42}$$

$$u^2 \le \int_{\mathbb{S}} (u^2+u_x^2)dx \le \frac{e+1}{2(e-1)}E(u_0,\rho_0), \tag{4.43}$$

$$|A\partial_x^2 G*u| \le |A|\|\partial_x G\|_{L^2}\|u_x\|_{L^2} \le \frac{|A|}{2}\|u_x\|_{L^2} \le \frac{|A|}{2}E^{1/2}(u_0,\rho_0), \tag{4.44}$$

$$|G*(u^2+\frac{1}{2}u_x^2)| \le \|G\|_{L^\infty}\|u^2+\frac{1}{2}u_x^2\|_{L^2} \le \frac{e}{4(e-1)}E_0(u_0,\rho_0) \tag{4.45}$$

Then, from (4.41)-(4.45), it follows that

$$\begin{aligned} f &\le \Omega\left(\frac{e+1}{2(e-1)}E_0(u_0,\rho_0)\right)^{1/2} C_1 + \frac{|A|}{2}E^{1/2}(u_0,\rho_0) \\ &\quad + \frac{e+1}{2(e-1)}E_0(u_0,\rho_0) + \frac{e}{4(e-1)}E_0(u_0,\rho_0) \\ &= \frac{1}{2}C_3^2.\end{aligned} \tag{4.46}$$

By (4.46), we deduce the inequality

$$M_1'(t) \le -\frac{1}{2}(M_1 - \Omega\partial_x G*\rho^2)^2 + \frac{1}{2}C_3^2, \quad t\in[0,T). \tag{4.47}$$

If the assumption (4.34) holds, then

$$M_1'(0) = u_{0,x}(x_0) + \Omega\partial_{0,x}G*\rho^2(x_0) < u_{0,x}(x_0) + \Omega C_2 < -C_3 - \Omega C_2. \tag{4.48}$$

We now claim that

$$M_1(t) < -C_3 - \Omega C_2, \quad \forall t\in[0,T). \tag{4.49}$$

In fact, as $M_1(0) < -C_3 - \Omega C_2$ and $M_1(t)$ is continuous, failure of (4.47) would ensure the existence of some $t_0 \in (0,T_0)$ such that $M_1 < -C_3 - \Omega C_2$ on $[0,t_0)$, while $M_1(t_0) = -C_3 - \Omega C_2$. But then we would have

$$\frac{dM_1(t)}{dt} < 0, \quad \text{a.e. } t\in[0,t_0). \tag{4.50}$$

Being locally Lipschitz, the function $M_1(t)$ is absolutely continuous on $[0,t_0]$, and therefore an integration of the previous inequality would lead us to

$$M_1(t_0) \le M_1(0) < -C_3 - \Omega C_2, \tag{4.51}$$

which contradicts our assumption $M_1(t_0) = -C_2 - \Omega C_1$. Hence (4.49) holds, implying that $M'(t)$ is strictly decrease on $[0,T)$. Then

$$M_1'(t) \le -\frac{1}{2}(M_1 + \Omega C_2)^2 + \frac{1}{2}C_3^2, \quad t \in [0,T). \tag{4.52}$$

Solving the inequality gives

$$\frac{M_1(0) + \Omega C_2 + C_3}{M_1(0) + \Omega C_2 - C_3} e^{C_2 t} - 1 \le \frac{2C_3}{M_1(t) + \Omega C_2 - C_3} \le 0. \tag{4.53}$$

In view of $0 < \frac{M_1(0)+\Omega C_1 + C_2}{M_1(0)+\Omega C_1 - C_2} < 1$, we deduce that there exists T_1 satisfying

$$\begin{aligned} 0 < T_2 &< \frac{1}{C_2} \ln \frac{M_1(0) + \Omega C_2 - C_3}{M_1(0) + \Omega C_2 + C_3} \\ &= \frac{1}{C_2} \ln \frac{u_{0,x}(x_0) + \Omega \partial_{0,x} G * \rho^2(x_0) + \Omega C_2 - C_3}{u_{0,x}(x_0) + \Omega \partial_{0,x} G * \rho^2(x_0) + \Omega C_2 + C_3}. \end{aligned}$$

such that $\lim_{t\to T_1} M_1(t) = -\infty$, $i.e. \lim_{t\to T_1} u_x(t) = -\infty$, as a result of the boundness of $\partial_x G * \rho^2$. This completes the proof. □

An interesting question is weather u_x has an upper bound. The investigation on this issue gives the following result.

Proposition 4.6. *Assume that $1 - 2\Omega A > 0$. Let $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s > 3/2$, and $T > 0$ be the maximal time of existence of the solution (u,ρ) to system r2CH with initial data (u_0, ρ_0). Then for $x \in \{\Lambda := x \in \mathbb{S} : \rho_0(x) = 0\}$, we have that $u_x(t, q(t,x))$ is bounded from above for $t \in [0,T)$.*

Proof. We need only to prove this proposition for $s > 3$. Given $x \in \mathbb{S}$. From Theorem 4.5, we have

$$f \le C_{E(0)}^2, \tag{4.54}$$

where the $C_{E(0)}$ denotes a constant that depends only on $E(0)$. Given any $x \in \mathbb{S}$, let us define

$$P(t) = M_1(t) - \|u_{0,x}\|_{L^\infty} - 2\Omega C_1 - 2C_{E(0)},$$

where C_1 is defined by (4.16). Observing $P(t)$ is a C^1-differentiable function in [0,t) and satisfies

$$\begin{aligned} P(0) &= M_1(0) - \|u_{0,x}\|_{L^\infty} - 2\Omega C_1 - 2C_{E(0)} \\ &\le u_{0,x}(x) + \Omega G_x * \rho^2(0,x) - \|u_{0,x}\|_L^\infty - 2\Omega C_1 \le 0 \end{aligned}$$

where we used the estimate (4.42). We now claim that

$$P(t) \le 0, \quad \forall t \in [0,T). \tag{4.55}$$

On the contrary assume that there is $t_0 \in [0,T)$ such that $P(t_0) > 0$. Let $t_1 = \{\max t < t_0; P(t) = 0\}$. Then $P(t_1) = 0$ and $P'(t_1) \ge 0$, or equivalently,

$$M_1(t_1) = \|u_{0,x}\|_{L^\infty} + 2\Omega C_1 + 2C_{E(0)}, \tag{4.56}$$

$$M_1'(t_1) \ge 0. \tag{4.57}$$

By (4.54) and (4.56), it follows that

$$\begin{aligned} M_1'(t_1) &= -\frac{1}{2}(M_1(t_1) - \Omega \partial_x G * \rho^2)^2 + f \\ &\le -\frac{1}{2}(\|u_{0,x}\|_L^\infty + 2C_{E(0)})^2 + C_{E(0)}^2 < 0 \end{aligned} \tag{4.58}$$

which is a contradiction to (4.57). This verifies the estimate in (4.55). Therefore, for any such that $\rho(x)=0$,

$$\sup_{t\in[0,T)} u_x(t,q(t,x))+\Omega\partial_x G*\rho^2(t,q(t,x))\le \|u_{0,x}\|_L^\infty+2C_{E(0)}+\|u_{0,x}\|_{L^\infty}.$$

which implies

$$\sup_{t\in[0,T)} u_x(t,q(t,x))\le \|u_{0,x}\|_L^\infty+4\Omega C_1+2C_{E(0)}.$$

This completes the proof. □

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