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FIRST CURVE OF FUČIK SPECTRUM FOR THE p-FRACTIONAL LAPLACIAN OPERATOR WITH NONLOCAL NORMAL BOUNDARY CONDITIONS

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ABSTRACT. In this article, we study the Fučik spectrum of the p-fractional Laplace operator with nonlocal normal derivative conditions which is defined as the set of all $(a,b)\in\mathbb{R}^2$ such that

$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)_p^\alpha u+|u|^{p-2}u=\frac{\chi_{\Omega_\epsilon}}{\epsilon}(a(u^+)^{p-1}-b(u^-)^{p-1})\quad\text{in }\Omega,\\ \mathcal{N}_{\alpha,p}u=0\quad\text{in }\mathbb{R}^n\setminus\overline{\Omega},\end{gathered}$$

has a non-trivial solution u, where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $p\geq 2$, $n>p\alpha$, $\epsilon,\alpha\in(0,1)$ and $\Omega_\epsilon:=\{x\in\Omega: d(x,\partial\Omega)\leq\epsilon\}$. We show existence of the first non-trivial curve $\mathcal{C}$ of the Fučik spectrum which is used to obtain the variational characterization of a second eigenvalue of the problem defined above. We also discuss some properties of this curve $\mathcal{C}$, e.g. Lipschitz continuous, strictly decreasing and asymptotic behavior and non-resonance with respect to the Fučik spectrum.

1. Introduction

The Fučik spectrum of p-fractional Laplacian with nonlocal normal derivative is defined as the set Σ_p of all $(a,b)\in\mathbb{R}^2$ such that

$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)_p^\alpha u+|u|^{p-2}u=\frac{\chi_{\Omega_\epsilon}}{\epsilon}(a(u^+)^{p-1}-b(u^-)^{p-1})\quad\text{in }\Omega,\\ \mathcal{N}_{\alpha,p}u=0\quad\text{in }\mathbb{R}^n\setminus\overline{\Omega},\end{gathered}\tag{1.1}$$

has a non-trivial solution u, where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $p\geq 2$, $\alpha,\epsilon\in(0,1)$ and $\Omega_\epsilon:=\{x\in\Omega: d(x,\partial\Omega)\leq\epsilon\}$. The $(-\Delta)_p^\alpha$ is the p-fractional Laplacian operator defined as

$$(-\Delta)_p^\alpha u(x):=2\,\text{p.v.}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}dy\quad\text{for all }x\in\mathbb{R}^n,$$

and $\mathcal{N}_{\alpha,p}$ is the associated nonlocal derivative defined in [8] as

$$\mathcal{N}_{\alpha,p}u(x):=2\int_{\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}dy\quad\text{for all }x\in\mathbb{R}^n\setminus\overline{\Omega}.$$

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Bourgain, Brezis and Mironescu [3] proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, there exist a constant $\Lambda_{n,p}$ such that

$$\lim_{\alpha\to 1^-} \Lambda_{n,p}(1-\alpha)\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\, dxdy = \int_\Omega |\nabla u|^p\, dx.$$

The constant $\Lambda_{n,p}$ can be explicitly computed and is given by

$$\Lambda_{n,p} = \frac{p\Gamma(\frac{n+p}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(\frac{p+1}{2})}.$$

For $a = b = \lambda$, the Fučik spectrum in (1.1) becomes the usual spectrum that satisfies

$$\begin{gathered} \Lambda_{n,p}(1-\alpha)(-\Delta)_p^\alpha u + |u|^{p-2}u = \frac{\lambda}{\epsilon}\chi_{\Omega_\epsilon}|u|^{p-2}u \quad \text{in } \Omega, \\ \mathcal{N}_{\alpha,p}u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{gathered} \tag{1.2}$$

In [7], authors proved that there exists a sequence of eigenvalues $\lambda_{k,\epsilon}(\Omega_\epsilon)$ of (1.2) such that $\lambda_{k,\epsilon}(\Omega_\epsilon) \to \infty$ as $k \to \infty$. Moreover, $0 < \lambda_{1,\epsilon}(\Omega_\epsilon) < \lambda_{2,\epsilon}(\Omega_\epsilon) \le \dots \le \lambda_{k,\epsilon}(\Omega_\epsilon) \le \dots$, and the first eigenvalue $\lambda_{1,\epsilon}(\Omega_\epsilon)$ of (1.2) is simple, isolated and can be characterized as follows

$$\begin{aligned} &\lambda_{1,\epsilon}(\Omega_\epsilon) \\ &= \inf_{u\in\mathcal{W}^{\alpha,p}} \Big\{\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\, dx\, dy + \int_\Omega |u|^p\, dx : \int_{\Omega_\epsilon} |u|^p\, dx = \epsilon\Big\}. \end{aligned}$$

The Fučik spectrum was introduced by Fučik (1976) who studied the problem in one dimension with periodic boundary conditions. In higher dimensions, the non-trivial first curve in the Fučik spectrum of Laplacian with Dirichlet boundary for bounded domain has been studied in [10]. Later in [6] Cuesta, de Figueiredo and Gossez studied this problem for p-Laplacian operator with Dirichlet boundary condition.

The Fučik spectrum in the case of Laplacian, p-Laplacian operator with Dirichlet, Neumann and Robin boundary condition has been studied by many authors, for instance [2, 5, 18, 20, 7, 22]. Goyal and Sreenadh [14] extended the results of [6] to nonlocal linear operators which include fractional Laplacian. The existence of Fučik eigenvalues for p-fractional Laplacian operator with Dirichlet boundary conditions has been studied by many authors, for instance refer [23, 24]. Also, in [15], Goyal discussed the Fučik spectrum of of p-fractional Hardy Sobolev-Operator with weight function. A non-resonance problem with respect to Fučik spectrum is also discussed in many papers [6, 21, 16]. We also refer to the related papers [9, 12, 13, 17].

The inspiring point of our work is [14, 15], where the existence of a nontrivial curve is studied only for $p = 2$ but the nature of the curve is left open for $p \neq 2$. In the present work, we extend the results obtained in [14] to the nonlinear case of p-fractional operator for any $p \geq 2$ and also show that this curve is the first curve. We also showed the variational characterization of the second eigenvalue of the operator associated with (1.1). There is a substantial difference while handling the nonlinear nature of the operator. This difference is reflected while constructing the paths below a mountain-pass level (see the proof of Theorem 1.1). To the best of our knowledge, no work has been done on the Fučik spectrum for nonlocal operators with nonlocal normal derivative. We would like to remark that the main result obtained in this paper is new even for the following p-fractional Laplacian equation with Dirichlet boundary condition:

$$(-\Delta)_p^\alpha u + |u|^{p-2}u = a(u^+)^{p-1} - b(u^-)^{p-1} \quad \text{in } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n\backslash\overline{\Omega}.$$

With this introduction, we state our main result.

Theorem 1.1. *Let $s \geq 0$ then the point $(s + c(s), c(s))$ is the first nontrivial point of Σ_p in the intersection between Σ_p and the line $(s, 0) + t(1, 1)$ of (1.1).*

This article is organized as follows: In section 2 we give some preliminaries. In section 3 we construct a first nontrivial curve in Σ_p, described as $(s + c(s), c(s))$. In section 4 we prove that the lines $\lambda_{1,\epsilon}(\Omega_\epsilon) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_{1,\epsilon}(\Omega_\epsilon)$ are isolated in Σ_p, the curve that we obtained in section 3 is the first nontrivial curve and give the variational characterization of second eigenvalue of (1.1). In section 5 we prove some properties of the first curve and non resonance problem.

2. Preliminaries

In this section we assemble some requisite material. By [8] we know the nonlocal analogue of divergence theorem which states that for any bounded functions u and $v \in C^2$, it holds that

$$\int_\Omega (-\Delta)_p^\alpha u(x)\,dx = -\int_{\Omega^c} \mathcal{N}_{\alpha,p} u(x)\,dx.$$

More generally, we have following integration by parts formula

$$\mathcal{H}_{\alpha,p}(u, v) = \int_\Omega v(x)(-\Delta)_p^\alpha u(x)\,dx + \int_{\Omega^c} v(x)\mathcal{N}_{\alpha,p} u(x)\,dx,$$

where $\mathcal{H}_{\alpha,p}(u, v)$ is defined as

$$\mathcal{H}_{\alpha,p}(u, v) := \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+p\alpha}} dy, \quad Q := \mathbb{R}^{2n}\backslash(\Omega^c)^2.$$

Now, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we set

$$\|u\|_{\alpha,p} := (\|u\|^p_{L^p(\Omega)} + [u]^p_{\alpha,p})^{1/p}, \quad \text{where } [u]_{\alpha,p} := (\mathcal{H}_{\alpha,p}(u, u))^{1/p}. \tag{2.1}$$

Then $\|\cdot\|_{\alpha,p}$ defines a norm on the space

$$\mathcal{W}^{\alpha,p} := \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable } : \|u\|_{\alpha,p} < \infty\}.$$

Clearly $\mathcal{W}^{\alpha,p} \subset W^{\alpha,p}(\Omega)$, where $W^{\alpha,p}(\Omega)$ denotes the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{\alpha,p}} = \|u\|_{L^p} + \Big(\int_{\Omega\times\Omega} \frac{(u(x) - u(y))^p}{|x - y|^{n+p\alpha}}\,dx\,dy\Big)^{1/p}.$$

To study the fractional Sobolev space in detail see [19].

Definition 2.1. A function $u \in \mathcal{W}^{\alpha,p}$ is a weak solution of (1.1), if for every $v \in \mathcal{W}^{\alpha,p}$, u satisfies

$$\Lambda_{n,p}(1 - \alpha)\mathcal{H}_{\alpha,p}(u, v) + \int_\Omega |u|^{p-2}uv - \frac{a}{\epsilon}\int_{\Omega_\epsilon} (u^+)^{p-1}v + \frac{b}{\epsilon}\int_{\Omega_\epsilon} (u^-)^{p-1}v = 0.$$

Now, we define the functional J associated to problem (1.1) as $J : \mathcal{W}^{\alpha,p} \to \mathbb{R}$ such that

$$\begin{aligned} J(u) = \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u|^p\,dx \\ - \frac{a}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx + \frac{b}{\epsilon}\int_{\Omega_\epsilon}(u^-)^p\,dx. \end{aligned}$$

Then J is Fréchet differentiable in $\mathcal{W}^{\alpha,p}$ and for all $v \in \mathcal{W}^{\alpha,p}$.

$$\langle J'(u), v\rangle = \Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u,v) + \int_\Omega |u|^{p-2}uv - \frac{a}{\epsilon}\int_{\Omega_\epsilon}(u^+)^{p-1}v + \frac{b}{\epsilon}\int_{\Omega_\epsilon}(u^-)^{p-1}v.$$

For the sake of completeness, we describe the Steklov problem

$$\begin{gathered} (-\Delta)_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where Ω is a bounded domain and $p > 1$. By [7], (1.1) is related to (2.2) in the sense that if Ω be a bounded smooth domain in $\mathbb{R}^n$ with Lipschitz boundary and $p \in (1,\infty)$. For a fixed $u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$, we have

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{\Omega_\epsilon}|u|^p dx = \int_{\partial\Omega}|u|^p dS \quad \text{and} \quad \lim_{\alpha\to 1^-}\Lambda_{n,p}(1-\alpha)[Eu]^p_{\alpha,p} = \|\nabla u\|^p_{L^p(\Omega)},$$

where E is a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W_0^{1,p}(B_R)$ such that $Eu = u$ in Ω and Ω is relatively compact in B_R, the ball of radius R in $\mathbb{R}^n$. This leads to the following Lemma in [20].

Lemma 2.2. *Let Ω be a smooth domain in $\mathbb{R}^n$ with Lipschitz boundary and $p \in (1,\infty)$. For a fixed $u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$, it holds*

$$\lim_{\alpha\to 1^-}\frac{\Lambda_{n,p}(1-\alpha)[Eu]^p_{\alpha,p} + \|Eu\|^p_{L^p(\Omega)}}{\frac{1}{1-\alpha}\|Eu\|^p_{L^p(\Omega_{1-\alpha})}} = \frac{\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}.$$

Taking $\epsilon = 1-\alpha$, by Lemma 2.2 the eigenvalue $\lambda_{1,1-\alpha}(\Omega_{1-\alpha}) \to \lambda_1$ as $\alpha \to 1^-$, where λ_1 is the first eigenvalue of the operator associated with (2.2). Similarly, we obtain that as $\alpha \to 1^-$ the Fučik Spectrum of the operator associated with (1.1) tends to Fučik Spectrum of the Steklov problem.

We shall throughout use the function space $\mathcal{W}^{\alpha,p}$ with the norm $\|\cdot\|$ and we use the standard $L^p(\Omega)$ space whose norms are denoted by $\|u\|_{L^p(\Omega)}$. Also, we denote $\lambda_{n,\epsilon}(\Omega_\epsilon)$ by $\lambda_{n,\epsilon}$. Here $\phi_{1,\epsilon}$ is the eigenfunction corresponding to $\lambda_{1,\epsilon}$.

3. The Fučik spectrum Σ_p

In this section, we study existence of the first nontrivial curve in the Fučik spectrum Σ_p of (1.1). We find that the points in Σ_p are associated with the critical value of some restricted functional. For this, for fixed $s \in \mathbb{R}$ and $s \geq 0$, we consider the functional $J_s : \mathcal{W}^{\alpha,p} \to \mathbb{R}$ defined by

$$J_s(u) = \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u|^p\,dx - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx.$$

Then $J_s \in C^1(\mathcal{W}^{\alpha,p},\mathbb{R})$ and for any $\phi \in \mathcal{W}^{\alpha,p}$

$$\langle J_s'(u), \phi\rangle = p\ \Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u,\phi) + p\int_\Omega |u|^{p-2}u\phi\,dx - \frac{ps}{\epsilon}\int_{\Omega_\epsilon}(u^+)^{p-1}\phi\,dx.$$

Also $\tilde{J}_s := J_s|_{\mathcal{S}}$ is $C^1(\mathcal{W}^{\alpha,p},\mathbb{R})$, where $\mathcal{S}$ is defined as

$$\mathcal{S} := \{u \in \mathcal{W}^{\alpha,p} : I(u) := \frac{1}{\epsilon}\int_{\Omega_\epsilon} |u|^p = 1\}.$$

We first note that $u \in \mathcal{S}$ is a critical point of $\tilde{J}_s$ if and only if there exists $t \in \mathbb{R}$ such that

$$\Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u,v) - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^{p-1}v\,dx = \frac{t}{\epsilon}\int_{\Omega_\epsilon}|u|^{p-2}uv\,dx, \tag{3.1}$$

for all $v \in \mathcal{W}^{\alpha,p}$. Hence $u \in \mathcal{S}$ is a nontrivial weak solution of the problem

$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)_p^\alpha + |u|^{p-2}u = \frac{\chi_{\Omega_\epsilon}}{\epsilon}\left((s+t)(u^+)^{p-1} - t(u^-)^{p-1}\right) \quad \text{in } \Omega,\\ \mathcal{N}_{\alpha,p}u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega},\end{gathered}$$

which exactly means $(s+t,t) \in \Sigma_p$. Substituting $v = u$ in (3.1), we obtain $t = \tilde{J}_s(u)$. Thus we obtain the following Lemma which links the critical point of $\tilde{J}_s$ and the spectrum Σ_p.

Lemma 3.1. *For $s \ge 0$, $(s+t,t) \in \mathbb{R}^2$ belongs to the spectrum Σ_p if and only if there exists a critical point $u \in \mathcal{S}$ of $\tilde{J}_s$ such that $t = \tilde{J}_s(u)$, a critical value.*

Proposition 3.2. *The first eigenfunction $\phi_{1,\epsilon}$ is a global minimum for $\tilde{J}_s$ with $\tilde{J}_s(\phi_{1,\epsilon}) = \lambda_{1,\epsilon} - s$. The corresponding point in Σ_p is $(\lambda_{1,\epsilon}, \lambda_{1,\epsilon} - s)$ which lies on the vertical line through $(\lambda_{1,\epsilon}, \lambda_{1,\epsilon})$.*

Proof. We have

$$\begin{aligned}\tilde{J}_s(u) =&\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u|^p dx - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx\\ \ge&\frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon}|u|^p\,dx - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx \ge \lambda_{1,\epsilon} - s.\end{aligned}$$

Thus $\tilde{J}_s$ is bounded below by $\lambda_{1,\epsilon} - s$. Moreover,

$$\tilde{J}_s(\phi_{1,\epsilon}) = \lambda_{1,\epsilon} - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(\phi_{1,\epsilon}^+)^p\,dx = \lambda_{1,\epsilon} - s.$$

Thus $\phi_{1,\epsilon}$ is a global minimum of $\tilde{J}_s$ with $\tilde{J}_s(\phi_{1,\epsilon}) = \lambda_{1,\epsilon} - s$. $\square$

Proposition 3.3. *The negative eigenfunction $-\phi_{1,\epsilon}$ is a strict local minimum for $\tilde{J}_s$ with $\tilde{J}_s(-\phi_{1,\epsilon}) = \lambda_{1,\epsilon}$. The corresponding point in Σ_p is $(\lambda_{1,\epsilon} + s, \lambda_{1,\epsilon})$, which lies on the horizontal line through $(\lambda_{1,\epsilon}, \lambda_{1,\epsilon})$.*

Proof. Suppose by contradiction that there exists a sequence $u_k \in \mathcal{S}$, $u_k \ne -\phi_{1,\epsilon}$ with $\tilde{J}_s(u_k) \le \lambda_{1,\epsilon}$, $u_k \to -\phi_{1,\epsilon}$ in $\mathcal{W}^{\alpha,p}$. We claim that u_k changes sign for sufficiently large k. Since $u_k \to -\phi_{1,\epsilon}$, u_k must be < 0 for sufficiently large k. If $u_k \le 0$ for a.e $x \in \Omega$, then

$$\tilde{J}_s(u_k) = \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u_k|^p\,dx > \lambda_{1,\epsilon},$$

since $u_k \not\equiv \pm\phi_{1,\epsilon}$ and we obtain contradiction as $\tilde{J}_s(u_k) \le \lambda_{1,\epsilon}$. Therefore the claim is proved.

Now, define $w_k := \frac{\epsilon^{1/p} u_k^+}{\|u_k^+\|_{L^p(\Omega_\epsilon)}}$ and

$$r_k := \Lambda_{n,p}(1-\alpha) \int_Q \frac{|w_k(x) - w_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |w_k|^p\,dx.$$

We claim that $r_k \to \infty$ as $k \to \infty$. Assume by contradiction that r_k is bounded. Then there exists a subsequence (still denoted by $\{w_k\}$) of $\{w_k\}$ and $w \in \mathcal{W}^{\alpha,p}$ such that $w_k \rightharpoonup w$ weakly in $\mathcal{W}^{\alpha,p}$ and $w_k \to w$ strongly in $L^p(\Omega)$. It implies $w_k \to w$ strongly in $L^p(\Omega_\epsilon)$. Therefore $\frac{1}{\epsilon}\int_{\Omega_\epsilon} w^p\,dx = 1$, $w \geq 0$ a.e. in Ω_ϵ and so for some $\eta > 0$, $\delta = |\{x \in \Omega_\epsilon : w(x) \geq \eta\}| > 0$. Since, $u_k \to -\phi_{1,\epsilon}$ in $\mathcal{W}^{\alpha,p}$ and hence in $L^p(\Omega)$. Therefore, for each $\eta > 0$, $|\{x \in \Omega_\epsilon : u_k(x) \geq \eta\}| \to 0$ as $k \to \infty$ and $|\{x \in \Omega_\epsilon : w_k(x) \geq \eta\}| \to 0$ as $k \to \infty$, which is a contradiction to $\eta > 0$. Hence, $r_k \to \infty$. Clearly, one can have

$$\begin{aligned}
&|u_k(x) - u_k(y)|^p \\
&= (|u_k(x) - u_k(y)|^2)^{p/2} = [((u_k^+(x) - u_k^+(y)) - (u_k^-(x) - u_k^-(y)))^2]^{p/2} \\
&= [(u_k^+(x) - u_k^+(y))^2 + (u_k^-(x) - u_k^-(y))^2 - 2(u_k^+(x) - u_k^+(y))(u_k^-(x) - u_k^-(y))]^{p/2} \\
&= [(u_k^+(x) - u_k^+(y))^2 + (u_k^-(x) - u_k^-(y))^2 + 2u_k^+(x)u_k^-(y) + 2u_k^-(x)u_k^+(y)]^{p/2} \\
&\geq |u_k^+(x) - u_k^+(y)|^p + |u_k^-(x) - u_k^-(y)|^p.
\end{aligned}$$

Using the above inequality, we have

$$\begin{aligned}
\tilde{J}_s(u_k) &= \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u_k|^p - \frac{s}{\epsilon}\int_{\Omega_\epsilon} (u_k^+)^p\,dx \\
&\geq \Big[\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k^+(x) - u_k^+(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u_k^+|^p\Big] \\
&\quad + \Big[\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k^-(x) - u_k^-(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u_k^-|^p \\
&\quad - \frac{s}{\epsilon}\int_{\Omega_\epsilon} (u_k^+)^p\,dx\Big] \\
&\geq \frac{(r_k - s)}{\epsilon}\int_{\Omega_\epsilon} (u_k^+)^p\,dx + \frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon} (u_k^-)^p\,dx.
\end{aligned} \tag{3.2}$$

On the other hand, since $u_k \in \mathcal{S}$, we obtain

$$\tilde{J}_s(u_k) \leq \lambda_{1,\epsilon} = \frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega\epsilon} (u_k^+)^p\,dx + \frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega\epsilon} (u_k^-)^p\,dx. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\frac{(r_k - s - \lambda_{1,\epsilon})}{\epsilon}\int_{\Omega_\epsilon} (u_k^+)^p\,dx \leq 0,$$

and this implies $r_k - s \leq \lambda_{1,\epsilon}$, which contradicts that $r_k \to +\infty$. Therefore, $-\phi_{1,\epsilon}$ is the strict local minimum. $\square$

Proposition 3.4 ([1]). *Let Y be a Banach space, $g, f \in C^1(Y, \mathbb{R})$, $M = \{u \in Y : g(u) = 1\}$ and u_0, $u_1 \in M$. Let $\epsilon > 0$ such that $\|u_1 - u_0\| > \epsilon$ and*

$$\inf\{f(u) : u \in M \text{ and } \|u - u_0\|_Y = \epsilon\} > \max\{f(u_0), f(u_1)\}.$$

Assume that f satisfies the (PS) condition on M and that

$$\Gamma = \{\gamma \in C([-1,1], M) : \gamma(-1) = u_0 \text{ and } \gamma(1) = u_1\}$$

is non empty. Then $c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} f(u)$ is a critical value of $f|_M$.

We now find the third critical point via mountain pass Theorem as stated above. A norm of derivative of the restriction $\tilde{J}_s$ of J_s at $u \in \mathcal{S}$ is defined as

$$\|\tilde{J}_s'(u)\|_* = \min\{\|\tilde{J}_s'(u) - tI'(u)\| : t \in \mathbb{R}\}.$$

Lemma 3.5. *J_s satisfies the (PS) condition on $\mathcal{S}$.*

Proof. Let $J_s(u_k)$ and $t_k \in \mathbb{R}$ be a sequences such that for some $K > 0$,

$$|J_s(u_k)| \le K, \tag{3.4}$$

$$\begin{aligned} &\Big|\Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u_k, v) + \int_\Omega |u_k|^{p-2}u_k v - \frac{s}{\epsilon}\int_{\Omega_\epsilon} (u_k^+)^p v\,dx \\ &- \frac{t_k}{\epsilon}\int_{\Omega_\epsilon} |u_k|^{p-2}u_k v\,dx\Big| \\ &\le \eta_k\|v\| \end{aligned} \tag{3.5}$$

for all $v \in \mathcal{W}^{\alpha,p}$, $\eta_k \to 0$. From (3.4), using fractional Sobolev embedding, we obtain $\{u_k\}$ is bounded in $\mathcal{W}^{\alpha,p}$ which implies there is a subsequence denoted by u_k and $u_0 \in \mathcal{W}^{\alpha,p}$ such that $u_k \rightharpoonup u_0$ weakly in $\mathcal{W}^{\alpha,p}$, and $u_k \to u_0$ strongly in $L^p(\Omega)$ for all $1 \le p < p_\alpha^*$. Substituting $v = u_k$ in (3.5), we obtain

$$\begin{aligned} |t_k| &\le \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u_k|^p + \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u_k^+)^p\,dx + \eta_k\|u_k\| \\ &\le C. \end{aligned}$$

Hence, t_k is a bounded sequence so has a convergent subsequence say t_k that converges to t. Next, we claim that $u_k \to u_0$ strongly in $\mathcal{W}^{\alpha,p}$. Since $u_k \rightharpoonup u_0$ weakly in $\mathcal{W}^{\alpha,p}$, we obtain

$$\begin{aligned} &\int_Q \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))(u_k(x)-u_k(y))}{|x-y|^{n+p\alpha}}\,dx\,dy \\ &\to \int_Q \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy \quad \text{as } k \to \infty. \end{aligned} \tag{3.6}$$

Also $\langle \tilde{J}_s'(u_k), (u_k - u_0)\rangle = o(\eta_k)$. This implies

$$\begin{aligned} &\Big|\Lambda_{n,p}(1-\alpha)\int_Q \frac{1}{|x-y|^{n+p\alpha}}\Big(|u_k(x)-u_k(y)|^{p-2} \\ &\times (u_k(x)-u_k(y))((u_k-u_0)(x) - (u_k-u_0)(y))\Big)\,dx\,dy\Big| \\ &\le o(\eta_k) + \|u_k\|_{L^p(\Omega)}^{p-1}\|u_k - u_0\|_{L^p(\Omega)} + s\|u_k^+\|_{L^p(\Omega_\epsilon)}^{p-1}\|u_k-u_0\|_{L^p(\Omega_\epsilon)} \\ &\quad + |t_k|\|u_k\|_{L^p(\Omega_\epsilon)}^{p-1}\|u_k-u_0\|_{L^p(\Omega_\epsilon)} \to 0 \end{aligned}$$

as $k \to \infty$. Thus,

$$\begin{aligned} &\int_Q \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy \\ &- \int_Q \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(u_0(x)-u_0(y))}{|x-y|^{n+p\alpha}}\,dx\,dy \to 0, \end{aligned} \tag{3.7}$$

as $k \to \infty$. As we know that $|a-b|^p \le 2^p(|a|^{p-2}a - |b|^{p-2}b)(a-b)$ for all $a, b \in \mathbb{R}$. Therefore, from (3.6) and (3.7) we obtain

$$\int_Q \frac{|(u_k-u_0)(x)-(u_k-u_0)(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy \to 0 \quad \text{as } k \to \infty$$

Hence, u_k converges strongly to u_0 in $\mathcal{W}^{\alpha,p}$. □

Lemma 3.6. *Let $\eta_0 > 0$ be such that*

$$\tilde{J}_s(u) > \tilde{J}_s(-\phi_{1,\epsilon}) \tag{3.8}$$

for all $u \in B(-\phi_{1,\epsilon}, \eta_0) \cap \mathcal{S}$ with $u \not\equiv -\phi_{1,\epsilon}$, where the ball is taken in $\mathcal{W}^{\alpha,p}$. Then for any $0 < \eta < \eta_0$,

$$\inf\{\tilde{J}_s(u) : u \in \mathcal{S} \quad \textit{and} \quad \|u-(-\phi_{1,\epsilon})\| = \eta\} > \tilde{J}_s(-\phi_{1,\epsilon}). \tag{3.9}$$

Proof. If possible, let infimum in (3.9) is equal to $\tilde{J}_s(-\phi_{1,\epsilon}) = \lambda_{1,\epsilon}$ for some η with $0 < \eta < \eta_0$. It implies there exists a sequence $u_k \in \mathcal{S}$ with $\|u_k - (-\phi_{1,\epsilon})\| = \eta$ such that

$$\tilde{J}_s(u_k) \le \lambda_{1,\epsilon} + \frac{1}{2k^2}. \tag{3.10}$$

Consider the set $V = \{u \in \mathcal{S} : \eta - \delta \le \|u - (-\phi_{1,\epsilon})\| \le \eta + \delta\}$, where δ is chosen such that $\eta - \delta > 0$ and $\eta + \delta < \eta_0$. From (3.9) and given hypotheses, it follows that $\inf\{\tilde{J}_s(u) : u \in V\} = \lambda_{1,\epsilon}$. Now for each k, we apply Ekeland's variational principle to the functional $\tilde{J}_s$ on V to get the existence of $v_k \in V$ such that

$$\tilde{J}_s(v_k) \le \tilde{J}_s(u_k),\ \|v_k - u_k\| \le \frac{1}{k}, \tag{3.11}$$

$$\tilde{J}_s(v_k) \le \tilde{J}_s(u) + \frac{1}{k}\|u - v_k\|, \text{ for all } u \in V. \tag{3.12}$$

We claim that v_k is a Palais-Smale sequence for $\tilde{J}_s$ on $\mathcal{S}$. That is, there exists $M > 0$ such that $|\tilde{J}_s(v_k)| < M$ and $\|\tilde{J}_s'(v_k)\|_* \to 0$ as $k \to \infty$. Once this is proved then by Lemma 3.5, there exists a subsequence denoted by v_k of v_k such that $v_k \to v$ strongly in $\mathcal{W}^{\alpha,p}$. Clearly, $v \in \mathcal{S}$ and satisfies $\|v - (-\phi_{1,\epsilon})\| \le \eta + \delta < \eta_0$ and $\tilde{J}_s(v) = \lambda_{1,\epsilon}$ which contradicts (3.8).

Now, the boundedness of $\tilde{J}_s(v_k)$ follows from (3.10) and (3.11). So, we only need to prove that $\|\tilde{J}_s'(v_k)\|_* \to 0$. Let $k > \frac{1}{\delta}$ and take $w \in \mathcal{W}^{\alpha,p}$ tangent to $\mathcal{S}$ at v_k. That is, $\frac{1}{\epsilon}\int_{\Omega_\epsilon} |v_k|^{p-2}v_k w\,dx = 0$. Then by taking $u_t := \frac{\epsilon^{1/p}(v_k+tw)}{\|v_k+tw\|_{L^p(\Omega_\epsilon)}}$ for $t \in \mathbb{R}$, we obtain

$$\begin{aligned} \lim_{t\to 0}\|u_t - (-\phi_{1,\epsilon})\| &= \|v_k - (-\phi_{1,\epsilon})\| \le \|v_k - u_k\| + \|u_k - (-\phi_{1,\epsilon})\| \\ &\le \frac{1}{k} + \eta < \delta + \eta, \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to 0}\|u_t - (-\phi_{1,\epsilon})\| &= \|v_k - (-\phi_{1,\epsilon})\| \ge \|u_k - (-\phi_{1,\epsilon})\| - \|v_k - u_k\| \\ &\ge \eta - \frac{1}{k} > \eta - \delta. \end{aligned}$$

Hence, for t small enough $u_t \in V$ and replacing u by u_t in (3.12), we obtain

$$\tilde{J}_s(v_k) \le \tilde{J}_s(u_t) + \frac{1}{k}\|u_t - v_k\|.$$

Let $r(t) := \epsilon^{1/p}\|v_k + tw\|_{L^p(\Omega_\epsilon)}$, then

$$\begin{aligned}
&\frac{J_s(v_k) - J_s(v_k + tw)}{t} \\
&\le \frac{J_s(u_t) + \frac{1}{k}\|u_t - v_k\| - J_s(v_k + tw)}{t} \\
&= \frac{1}{k\, t\, r(t)}\|v_k(1 - r(t) + tw)\| + \frac{1}{t}\Big(\frac{1}{r(t)^p} - 1\Big)J(v_k + tw).
\end{aligned}$$

Now since

$$\frac{d}{dt}r(t)^p|_{t=0} = \frac{p}{\epsilon}\int_{\Omega_\epsilon}|v_k|^{p-2}v_k w = 0,$$

we obtain $\frac{r(t)^p - 1}{t} \to 0$ as $t \to 0$, and then $\frac{1-r(t)}{t} \to 0$ as $t \to 0$. Therefore, we obtain

$$|\langle J_s'(v_k), w\rangle| \le \frac{1}{k}\|w\|. \tag{3.13}$$

Since w is arbitrary in $\mathcal{W}^{\alpha,p}$, we choose a_k such that $\frac{1}{\epsilon}\int_{\Omega_\epsilon}|v_k|^{p-2}v_k(w - a_k v_k)\,dx = 0$. Replacing w by $w - a_k v_k$ in (3.13), we obtain

$$\big|\langle J_s'(v_k), w\rangle - a_k\langle J_s'(v_k), v_k\rangle\big| \le \frac{1}{k}\|w - a_k v_k\|.$$

Since $\|a_k v_k\| \le C\|w\|$, we obtain $\big|\langle J_s'(v_k), w\rangle - t_k\int_\Omega |v_k|^{p-2}v_k w\,dx\big| \le \frac{C}{k}\|w\|$, where $t_k = \langle J_s'(v_k), v_k\rangle$. Hence, $\|\tilde{J}_s'(v_k)\|_* \to 0$ as $k \to \infty$, as we required. $\square$

Proposition 3.7. *Let $\mathcal{W}^{\alpha,p}$ be a Banach Space. Let $\eta > 0$ such that $\|\phi_{1,\epsilon} - (-\phi_{1,\epsilon})\| > \eta$ and*

$$\inf\{\tilde{J}_s(u) : u \in \mathcal{S} \text{ and } \|u - (-\phi_{1,\epsilon})\| = \eta\} > \max\{\tilde{J}_s(-\phi_{1,\epsilon}), \tilde{J}_s(\phi_{1,\epsilon})\}.$$

Then $\Gamma = \{\gamma \in C([-1,1], \mathcal{S}) : \gamma(-1) = -\phi_{1,\epsilon}$ and $\gamma(1) = \phi_{1,\epsilon}\}$ is non empty and

$$c(s) = \inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]} J_s(u) \tag{3.14}$$

is a critical value of $\tilde{J}_s$. Moreover $c(s) > \lambda_{1,\epsilon}$.

Proof. We prove that Γ is non-empty. To end this, we take $\phi \in \mathcal{W}^{\alpha,p}$ such that $\phi \notin \mathbb{R}\phi_{1,\epsilon}$ and consider the path $t\phi_{1,\epsilon} + (1 - |t|)\phi$ then

$$w = \frac{\epsilon^{1/p}(t\phi_{1,\epsilon} + (1 - |t|)\phi)}{\|t\phi_{1,\epsilon} + (1 - |t|)\phi\|_{L^p(\Omega_\epsilon)}}.$$

Moreover the (PS) condition and the geometric assumption are satisfied by the Lemmas 3.5 and 3.6. Then by Proposition 3.4, $c(s)$ is a critical value of $\tilde{J}_s$. Using the definition of $c(s)$ we have $c(s) > \max\{\tilde{J}_s(-\phi_{1,\epsilon}), \tilde{J}_s(\phi_{1,\epsilon})\} = \lambda_{1,\epsilon}$. $\square$

Thus we have proved the following result.

Theorem 3.8. *For each $s \ge 0$, the point $(s + c(s), c(s))$, where $c(s) > \lambda_{1,\epsilon}$ is defined by the minimax formula (3.14), then the point $(s + c(s), c(s))$ belongs to Σ_p.*

It is a trivial fact that Σ_p is symmetric with respect to diagonal. The whole curve, that we obtain using Theorem 3.8 is denoted by

$$\mathcal{C} := \{(s + c(s), c(s)), (c(s), s + c(s)) : s \ge 0\}.$$

4. First nontrivial curve

We start this section by establishing that the lines $\mathbb{R}\times\{\lambda_{1,\epsilon}\}$ and $\{\lambda_{1,\epsilon}\}\times\mathbb{R}$ are isolated in Σ_p. Then we state some topological properties of the functional $\tilde{J}_s$ and some Lemmas. Finally, we prove that the curve $\mathcal{C}$ constructed in the previous section is the first non trivial curve in the spectrum Σ_p. As a consequence of this, we also obtain a variational characterization of the second eigenvalue $\lambda_{2,\epsilon}$.

Proposition 4.1. *The lines $\mathbb{R}\times\{\lambda_{1,\epsilon}\}$ and $\{\lambda_{1,\epsilon}\}\times\mathbb{R}$ are isolated in Σ_p. In other words, there exists no sequence $(a_k,b_k)\in\Sigma_p$ with $a_k>\lambda_{1,\epsilon}$ and $b_k>\lambda_{1,\epsilon}$ such that $(a_k,b_k)\to(a,b)$ with $a=\lambda_{1,\epsilon}$ or $b=\lambda_{1,\epsilon}$.*

Proof. Suppose by contradiction that there exists a sequence $(a_k,b_k)\in\Sigma_p$ with a_k, $b_k>\lambda_{1,\epsilon}$ and $(a_k,b_k)\to(a,b)$ with a or $b=\lambda_{1,\epsilon}$. Let $u_k\in\mathcal{W}^{\alpha,p}$ be a solution of

$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)^\alpha_p u_k+|u_k|^{p-2}u_k=\frac{\chi_{\Omega_\epsilon}}{\epsilon}(a_k(u_k^+)^{p-1}-b_k(u_k^-)^{p-1})\quad\text{in }\Omega,\\ \mathcal{N}_{\alpha,p}u_k=0\quad\text{in }\mathbb{R}^n\setminus\overline{\Omega},\end{gathered}\tag{4.1}$$

with $\frac{1}{\epsilon}\int_{\Omega_\epsilon}|u_k|^pdx=1$. Multiplying by u_k in (4.1) and integrate, we have

$$\begin{aligned}&\Lambda_{n,p}(1-\alpha)\int_Q\frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy+\int_\Omega|u_k|^pdx\\&=\frac{a_k}{\epsilon}\int_{\Omega_\epsilon}(u_k^+)^p\,dx-\frac{b_k}{\epsilon}\int_{\Omega_\epsilon}(u_k^-)^p\,dx\le a_k.\end{aligned}$$

Thus $\{u_k\}$ is a bounded sequence in $\mathcal{W}^{\alpha,p}$. Therefore up to a subsequence $u_k\rightharpoonup u$ weakly in $\mathcal{W}^{\alpha,p}$ and $u_k\to u$ strongly in $L^p(\Omega_\epsilon)$. Then taking limit $k\to\infty$ in the weak formulation of (4.1), we obtain

$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)^\alpha_p u+|u|^{p-2}u=\frac{\chi_{\Omega_\epsilon}}{\epsilon}(\lambda_{1,\epsilon}(u^+)^{p-1}-b(u^-)^{p-1})\quad\text{in }\Omega,\\ \mathcal{N}_{\alpha,p}u=0\quad\text{in }\mathbb{R}^n\setminus\overline{\Omega}.\end{gathered}\tag{4.2}$$

Taking u^+ as test function in (4.2) we obtain

$$\Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u,u^+)+\int_\Omega(u^+)^pdx=\frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx.\tag{4.3}$$

Observe that

$$((u(x)-u(y))(u^+(x)-u^+(y))=2u^-(x)u^+(y)+(u^+(x)-u^+(y))^2,\tag{4.4}$$

and

$$\begin{aligned}|u(x)-u(y)|^{p-2}&=(|u(x)-u(y)|^2)^{\frac{p-2}{2}}\\&=(|u^+(x)-u^+(y)|^2+|u^-(x)-u^-(y)|^2+2u^+(x)u^-(y)\\&\quad+2u^+(y)u^-(x))^{\frac{p-2}{2}}\\&\ge|u^+(x)-u^+(y)|^{p-2}.\end{aligned}\tag{4.5}$$

Using (4.4) and (4.5) in (4.3) and the definition of $\lambda_{1,\epsilon}$, we obtain

$$\begin{aligned}\frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx&\le\Lambda_{n,p}(1-\alpha)\int_Q\frac{|u^+(x)-u^+(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy+\int_\Omega(u^+)^pdx\\&\le\frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx.\end{aligned}$$

Thus

$$\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega (u^+)^p dx = \frac{\lambda_{1,\epsilon}}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx,$$

so either $u^+ \equiv 0$ or $u = \phi_{1,\epsilon}$. If $u^+ \equiv 0$ then $u \le 0$ and (4.2) implies that u is an eigenfunction with $u \le 0$ so that $u = -\phi_{1,\epsilon}$. So, in any case u_k converges to either $\phi_{1,\epsilon}$ or $-\phi_{1,\epsilon}$ in $L^p(\Omega_\epsilon)$. Thus

$$\text{either } |\{x \in \Omega_\epsilon : u_k(x) < 0\}| \to 0 \text{ or } |\{x \in \Omega_\epsilon : u_k(x) > 0\}| \to 0 \tag{4.6}$$

as $k \to \infty$. On the other hand, taking u_k^+ as test function in (4.1), we obtain

$$\Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u_k,u_k^+) + \int_\Omega |u_k|^{p-2}u_k u_k^+ = \frac{a_k}{\epsilon}\int_{\Omega_\epsilon}(u_k^+)^p. \tag{4.7}$$

Using Hölders inequality, fractional Sobolev embeddings and (4.7), we obtain

$$\begin{aligned}
&\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k^+(x)-u_k^+(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega (u_k^+)^p dx \\
&\le \Lambda_{n,p}(1-\alpha)\int_Q \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(u_k^+(x)-u_k^+(y))}{|x-y|^{n+p\alpha}}\,dx\,dy \\
&\quad + \int_\Omega |u_k|^{p-2}u_k u_k^+ dx \\
&= \Lambda_{n,p}(1-\alpha)\mathcal{H}_{\alpha,p}(u_k,u_k^+) + \int_\Omega |u_k|^{p-2}u_k u_k^+ dx \\
&= \frac{a_k}{\epsilon}\int_{\Omega_\epsilon}(u_k^+)^p\,dx \\
&\le \frac{a_k}{\epsilon}C|\{x \in \Omega_\epsilon : u_k(x) > 0\}|^{1-\frac{p}{q}}\|u_k^+\|^p
\end{aligned}$$

with a constant $C > 0$, $p < q \le p^* = \frac{np}{n-p\alpha}$. Then we have

$$|\{x \in \Omega : u_k(x) > 0\}|^{1-\frac{p}{q}} \ge \epsilon a_k^{-1}C^{-1}\min\{\Lambda_{n,p}(1-\alpha), 1\}.$$

Similarly, one can show that

$$|\{x \in \Omega : u_k(x) < 0\}|^{1-\frac{p}{q}} \ge \epsilon b_k^{-1}C^{-1}\min\{\Lambda_{n,p}(1-\alpha), 1\}.$$

Since (a_k, b_k) does not belong to the trivial lines $\lambda_{1,\epsilon} \times \mathbb{R}$ and $\mathbb{R} \times \lambda_{1,\epsilon}$ of Σ_p, by (4.1) we conclude that u_k changes sign. Hence, from the above inequalities, we obtain a contradiction with (4.6). Therefore, the trivial lines $\lambda_{1,\epsilon} \times \mathbb{R}$ and $\mathbb{R} \times \lambda_{1,\epsilon}$ are isolated in Σ_p. □

Lemma 4.2 ([6]). *Let $\mathcal{S} = \{u \in \mathcal{W}^{\alpha,p} : \frac{1}{\epsilon}\int_{\Omega_\epsilon}|u|^p\,dx = 1\}$ then*

1. *$\mathcal{S}$ is locally arcwise connected.*
2. *Any open connected subset $\mathcal{O}$ of $\mathcal{S}$ is arcwise connected.*
3. *If $\mathcal{O}'$ is any connected component of an open set $\mathcal{O} \subset \mathcal{S}$, then $\partial\mathcal{O}' \cap \mathcal{O} = \emptyset$.*

Lemma 4.3. *Let $\mathcal{O} = \{u \in \mathcal{S} : \tilde{J}_s(u) < r\}$, then any connected component of $\mathcal{O}$ contains a critical point of $\tilde{J}_s$.*

Proof. Let $\mathcal{O}_1$ be any connected component of $\mathcal{O}$, let $d = \inf\{\tilde{J}_s(u) : u \in \overline{\mathcal{O}}_1\}$, where $\overline{\mathcal{O}}_1$ denotes the closure of $\mathcal{O}_1$ in $\mathcal{W}^{\alpha,p}$. We show that there exists $u_0 \in \mathcal{W}^{\alpha,p}$ such that $\tilde{J}_s(u_0) = d$. For this let $u_k \in \mathcal{O}_1$ be a minimizing sequence such that

$\tilde{J}_s(u_k) \leq d + \frac{1}{2k^2}$. For each k, by applying Ekeland's Variational principle, we obtain a sequence $v_k \in \overline{\mathcal{O}}_1$ such that

$$\tilde{J}_s(v_k) \leq \tilde{J}_s(u_k),\ \|v_k - u_k\| \leq \frac{1}{k}, \quad \tilde{J}_s(v_k) \leq \tilde{J}_s(v) + \frac{1}{k}\|v - v_k\| \quad \forall v \in \overline{\mathcal{O}}_1.$$

For k large enough, we have

$$\tilde{J}_s(v_k) \leq \tilde{J}_s(u_k) \leq d + \frac{1}{2k^2} < r,$$

then $v_k \in \mathcal{O}$. By Lemma 4.2, we obtain $v_k \notin \partial\mathcal{O}_1$ so $v_k \in \mathcal{O}_1$. On the other hand, for t small enough and w such that $\frac{1}{\epsilon}\int_{\Omega_\epsilon} |v_k|^{p-2} v_k w\, dx = 0$, we have

$$u_t := \frac{\epsilon^{1/p}(v_k + tw)}{\|v_k + tw\|_{L^p(\Omega_\epsilon)}} \in \overline{\mathcal{O}}_1.$$

Then $\tilde{J}_s(v_k) \leq \tilde{J}_s(u_t) + \frac{1}{k}\|u_t - v_k\|$. Following the same calculation as in Lemma 3.6, we have that v_k is a Palais-Smale sequence for $\tilde{J}_s$ on $\mathcal{S}$ i.e $\tilde{J}_s(v_k)$ is bounded and $\|\tilde{J}_s(v_k)\|_* \to 0$. Again by Lemma 3.5, up to a subsequence $v_k \to u_0$ strongly in $\mathcal{W}^{\alpha,p}$ and hence $\tilde{J}_s(u_0) = d < r$ and moreover $u_0 \in \mathcal{O}$. By part 3 of Lemma 4.2, $u_0 \notin \partial\mathcal{O}_1$ so $u_0 \in \mathcal{O}_1$. Hence u_0 is a critical point of $\tilde{J}_s$, which completes the proof. □

Before proving the main Theorem 1.1, we state some Lemmas and the details of the proof can be found in [4] and [11].

Lemma 4.4 ([4, Lemma B.1]). *Let $1 \leq p \leq \infty$ and $U, V \in \mathbb{R}$ such that $U.V \leq 0$. Define the following function*

$$g(t) = |U - tV|^p + |U - V|^{p-2}(U - V)V|t|^p,\ t \in \mathbb{R}.$$

Then we have

$$g(t) \leq g(1) = |U - V|^{p-2}(U - V)U,\ t \in \mathbb{R}.$$

Lemma 4.5 ([11, Lemma 4.1]). *Let $\alpha \in (0, 1)$ and $p > 1$. For any non-negative functions $u,\ v \in \mathcal{W}^{\alpha,p}$, consider the function $\sigma_t := [(1 - t)v^p(x) + tu^p(x)]^{1/p}$ for all $t \in [0, 1]$. Then*

$$[\sigma_t]_{\alpha,p} \leq (1 - t)[v]_{\alpha,p} + t[u]_{\alpha,p}, \quad \text{for all } t \in [0, 1],$$

where $[u]_{\alpha,p}$ is defined in (2.1).

Proof of Theorem 1.1. Assume by contradiction that there exists μ such that $\lambda_{1,\epsilon} < \mu < c(s)$ and $(s + \mu, \mu) \in \Sigma_p$. Using the fact that $\{\lambda_{1,\epsilon}\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_{1,\epsilon}\}$ are isolated in Σ_p and Σ_p is closed we can choose such a point with μ minimum. In other words, $\tilde{J}_s$ has a critical value μ with $\lambda_{1,\epsilon} < \mu < c(s)$, but there is no critical value in $(\lambda_{1,\epsilon}, \mu)$. If we construct a path connecting from $\phi_{1,\epsilon}$ to $-\phi_{1,\epsilon}$ such that $\tilde{J}_s \leq \mu$, then we obtain a contradiction with the definition of $c(s)$, which wiil complete the proof.

Let $u \in \mathcal{S}$ be a critical point of $\tilde{J}_s$ at level μ. Then u satisfies

$$\begin{aligned} &\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u|^{p-2}uv\,dx \\ &= \frac{(s+\mu)}{\epsilon}\int_{\Omega_\epsilon} (u^+)^{p-1}v\,dx - \frac{\mu}{\epsilon}\int_{\Omega_\epsilon} (u^-)^{p-1}v\,dx \end{aligned} \tag{4.8}$$

for all $v \in \mathcal{W}^{\alpha,p}$. Substituting $v = u^+$ in (4.8), we have

$$\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega (u^+)^p dx$$
$$= \frac{(s+\mu)}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\,dx. \tag{4.9}$$

Since, $|u^+(x)-u^+(y)|^p \le |u(x)-u(y)|^{p-2}(u(x)-u(y))(u^+(x)-u^+(y)$, we obtain

$$\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega (u^+)^p dx - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p dx \le \mu.$$

Again substituting $v = u^-$ in (4.8), we have

$$\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{n+p\alpha}}\,dx\,dy - \int_\Omega (u^-)^p dx$$
$$= -\frac{\mu}{\epsilon}\int_{\Omega_\epsilon}(u^-)^p\,dx. \tag{4.10}$$

Therefore,

$$\begin{aligned}&\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^{p-2}((u^-(x)-u^-(y))^2+2u^+(x)u^-(y))}{|x-y|^{n+p\alpha}}\,dx\,dy\\&+\int_\Omega (u^-)^p dx\\&=\frac{\mu}{\epsilon}\int_{\Omega_\epsilon}(u^-)^p\,dx\end{aligned}$$

Since $|u^-(x)-u^-(y)|^p \le |u(x)-u(y)|^{p-2}[(u^-(x)-u^-(y))^2+2u^+(x)u^-(y)]$, ti follows that

$$\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u^-|^p dx \le \mu.$$

Therefore, from all above relations, one can easily verify that

$$\tilde{J}_s(u) = \mu,\ \tilde{J}_s\Big(\frac{\epsilon^{\frac{1}{p}}u^+}{\|u^+\|_{L^p(\Omega_\epsilon)}}\Big) \le \mu, \tilde{J}_s\Big(\frac{\epsilon^{\frac{1}{p}}u^-}{\|u^-\|_{L^p(\Omega_\epsilon)}}\Big) \le \mu - s, \tilde{J}_s\Big(\frac{-\epsilon^{\frac{1}{p}}u^-}{\|u^-\|_{L^p(\Omega_\epsilon)}}\Big) \le \mu.$$

Since, u changes sign (see Proposition 3.3), the following paths are well-defined on $\mathcal{S}$:

$$\begin{gathered}u_1(t) = \frac{u^+-(1-t)u^-}{\epsilon^{\frac{-1}{p}}\|u^+-(1-t)u^-\|_{L^p(\Omega_\epsilon)}},\\ u_2(t) = \frac{[(1-t)(u^+)^p+t(u^-)^p]^{1/p}}{\epsilon^{\frac{-1}{p}}\|(1-t)(u^+)^p+t(u^-)^p\|_{L^p(\Omega_\epsilon)}},\\ u_3(t) = \frac{(1-t)u^+-u^-}{\epsilon^{\frac{-1}{p}}\|(1-t)u^+-u^-\|_{L^p(\Omega_\epsilon)}}.\end{gathered}$$

Then, using the above calculations and Lemma 4.4 for $U = u^+(x)-u^+(y)$ and $V = u^-(x)-u^-(y)$, one can easily obtain that for all $t \in [0,1]$,

$$\tilde{J}_s(u_1(t)) \le \frac{\Lambda_{n,p}(1-\alpha)\int_Q \frac{|U-V|^{p-2}(U-V)U}{|x-y|^{n+p\alpha}} + \int_\Omega (u^+)^p - \frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p}{\epsilon^{-1}\|u^+-(1-t)u^-\|^p_{L^p(\Omega_\epsilon)}}$$

$$+\frac{|1-t|^p\big[-\Lambda_{n,p}(1-\alpha)\int_Q\frac{|U-V|^{p-2}(U-V)V}{|x-y|^{n+p\alpha}}+\int_\Omega(u^-)^p\big]}{\epsilon^{-1}\|u^+-(1-t)u^-\|^p_{L^p(\Omega_\epsilon)}}=\mu,$$

by using (4.9) and (4.10). Now using Lemma 4.5 we have

$$\begin{aligned}\tilde J_s(u_2(t))&\le\frac{(1-t)\big[\Lambda_{n,p}(1-\alpha)\int_Q\frac{|u^+(x)-u^+(y)|^p}{|x-y|^{n+p\alpha}}+\int_\Omega(u^+)^p-\frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\big]}{\epsilon^{-1}\|(1-t)(u^+)^p+t(u^-)^p\|^p_{L^p(\Omega_\epsilon)}}\\&\quad+\frac{t\big[\Lambda_{n,p}(1-\alpha)\int_Q\frac{|u^-(x)-u^-(y)|^p}{|x-y|^{n+p\alpha}}+\int_\Omega(u^-)^p-\frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^-)^p\big]}{\epsilon^{-1}\|(1-t)(u^+)^p+t(u^-)^p\|^p_{L^p(\Omega_\epsilon)}}\\&\le\mu-\frac{st\int_{\Omega_\epsilon}(u^-)^p}{\epsilon^{-1}\|(1-t)(u^+)^p+t(u^-)^p\|^p_{L^p(\Omega_\epsilon)}}\le\mu.\end{aligned}$$

Again, by Lemma 4.4, for $U=u^-(y)-u^-(x)$ and $V=u^+(y)-u^+(x)$, we obtain

$$\begin{aligned}&\tilde J_s(u_3(t))\\&\le\frac{\Lambda_{n,p}(1-\alpha)\int_Q\frac{|U-V|^{p-2}(U-V)U}{|x-y|^{n+p\alpha}}+\int_\Omega(u^-)^p}{\epsilon^{-1}\|(1-t)u^+-u^-\|^p_{L^p(\Omega_\epsilon)}}\\&\quad+\frac{|1-t|^p\big[-\Lambda_{n,p}(1-\alpha)\int_Q\frac{|U-V|^{p-2}(U-V)V}{|x-y|^{n+p\alpha}}+\int_\Omega(u^+)^p-\frac{s}{\epsilon}\int_{\Omega_\epsilon}(u^+)^p\big]}{\epsilon^{-1}\|(1-t)u^+-u^-\|^p_{L^p(\Omega_\epsilon)}}\\&=\mu,\quad\text{by using (4.9) and (4.10).}\end{aligned}$$

Let $\mathcal{O}=\{v\in\mathcal{S}:\tilde J_s(v)<\mu-s\}$. Then clearly $\phi_{1,\epsilon}\in\mathcal{O}$, while $-\phi_{1,\epsilon}\in\mathcal{O}$ if $\mu-s>\lambda_{1,\epsilon}$. Moreover $\phi_{1,\epsilon}$ and $-\phi_{1,\epsilon}$ are the only possible critical points of $\tilde J_s$ in $\mathcal{O}$ because of the choice of μ. We note that

$$\tilde J_s\Big(\frac{\epsilon^{1/p}u^-}{\|u^-\|_{L^p(\Omega_\epsilon)}}\Big)\le\mu-s,$$

$\epsilon^{1/p}u^-/\|u^-\|_{L^p(\Omega_\epsilon)}$ does not change sign and vanishes on a set of positive measure, it is not a critical point of $\tilde J_s$. Therefore, there exists a C^1 path $\eta:[-\delta,\delta]\to\mathcal{S}$ with $\eta(0)=\epsilon^{1/p}u^-/\|u^-\|_{L^p(\Omega_\epsilon)}$ and $\frac{d}{dt}\tilde J_s(\eta(t))|_{t=0}\neq0$. Using this path we can move from $\epsilon^{1/p}u^-/\|u^-\|_{L^p(\Omega_\epsilon)}$ to a point v with $\tilde J_s(v)<\mu-s$. Taking a connected component of $\mathcal{O}$ containing v and applying Lemma 4.3 we have that either $\phi_{1,\epsilon}$ or $-\phi_{1,\epsilon}$ is in this component. Let us assume that it is $\phi_{1,\epsilon}$. So we continue by a path $u_4(t)$ from $\epsilon^{1/p}u^-/\|u^-\|_{L^p(\Omega_\epsilon)}$ to $\phi_{1,\epsilon}$ which is at level less than μ. Then the path $-u_4(t)$ connects $-\epsilon^{1/p}u^-/\|u^-\|_{L^p(\Omega_\epsilon)}$ to $-\phi_{1,\epsilon}$. We observe that

$$|\tilde J_s(u)-\tilde J_s(-u)|\le s.$$

Then it follows that

$$\tilde J_s(-u_4(t))\le\tilde J_s(u_4(t))+s\le\mu-s+s=\mu\quad\text{for all } t.$$

Connecting $u_1(t)$, $u_2(t)$ and $u_4(t)$, we obtain a path from u to $\phi_{1,\epsilon}$ and joining $u_3(t)$ and $-u_4(t)$ we obtain a path from u to $-\phi_{1,\epsilon}$. These yields a path $\gamma(t)$ on $\mathcal{S}$ joining from $\phi_{1,\epsilon}$ to $-\phi_{1,\epsilon}$ such that $\tilde J_s(\gamma(t))\le\mu$ for all t, which concludes the proof. $\square$

As a consequence of Theorem 1.1, we give a variational characterization of the second value of (1.2).

Corollary 4.6. *The second eigenvalue λ_2 of* (1.2) *has the variational characterization given by*
$$\lambda_2 := \inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\Big(\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |u|^p\,dx\Big),$$
where Γ is as in Proposition 3.7.

Proof. Taking $s=0$ in Theorem 1.1 and using (3.14) we have $c(0)=\lambda_2$. □

5. Properties of the curve $\mathcal{C}$

In this section, we prove that the curve $\mathcal{C}$ is Lipschitz continuous, has a certain asymptotic behavior and is strictly decreasing. For $A\subset\Omega_\epsilon$, define the eigenvalue problem
$$\begin{gathered}\Lambda_{n,p}(1-\alpha)(-\Delta)_p^\alpha u + |u|^{p-2}u = \frac{\chi_A}{\epsilon}(\lambda|u|^{p-2}u)\quad \text{in } \Omega,\\ \mathcal{N}_{\alpha,p}u = 0\quad \text{in } \mathbb{R}^n\setminus\overline{\Omega},\end{gathered}\tag{5.1}$$
Let $\lambda_{1,\epsilon}(A)$ denotes the first eigenvalue of (5.1), then
$$\begin{aligned}\lambda_{1,\epsilon}(A) = \inf_{u\in\mathcal{W}^{\alpha,p}}\Big\{&\Lambda_{n,p}(1-\alpha)\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy\\ &+\int_\Omega |u|^p\,dx : \int_A |u|^p\,dx = \epsilon\Big\}.\end{aligned}$$

Lemma 5.1. *Let A, B be two bounded open sets in Ω_ϵ, with $A\subsetneq B$ and B is connected then $\lambda_{1,\epsilon}(A) > \lambda_{1,\epsilon}(B)$.*

Proof. Clearly from the definition of $\lambda_{1,\epsilon}$, we have $\lambda_{1,\epsilon}(A)\ge\lambda_{1,\epsilon}(B)$. Let if possible equality holds and let $\phi_{1,\epsilon}$ be a non-negative normalized eigenfunction associated to $\lambda_{1,\epsilon}(A)$ such that $\phi_{1,\epsilon}$ is equal to zero outside A. Therefore, from the definition of $\lambda_{1,\epsilon}(A)$, we have
$$\begin{aligned}&\Lambda_{n,p}(1-\alpha)\int_Q \frac{|\phi_{1,\epsilon}(x)-\phi_{1,\epsilon}(y)|^p}{|x-y|^{n+p\alpha}}\,dx\,dy + \int_\Omega |\phi_{1,\epsilon}|^p dx\\ &= \frac{\lambda_1(A)}{\epsilon}\int_A \phi_{1,\epsilon}^p\,dx = \frac{\lambda_1(B)}{\epsilon}\int_B \phi_{1,\epsilon}^p\,dx.\end{aligned}$$
It implies $\phi_{1,\epsilon}$ is an eigenfunction associated to $\lambda_{1,\epsilon}(B)$. But this is impossible since B is connected and $\phi_{1,\epsilon}$ vanishes on $B\setminus A\neq\emptyset$. □

Proposition 5.2. *The curve $s\to(s+c(s),c(s))$, $s\in\mathbb{R}^+$ is Lipschitz continuous and strictly decreasing (in the sense that $s_1<s_2$ implies $s_1+c(s_1)<s_2+c(s_2)$ and $c(s_1)>c(s_2)$).*

Proof. Let $s_1<s_2$ then $\tilde J_{s_1}(u) > \tilde J_{s_2}(u)$ for all $u\in\mathcal{S}$. So we have $c(s_1)>c(s_2)$. Now for every $\eta>0$ there exists $\gamma\in\Gamma$ such that $\max_{u\in\gamma[-1,1]}\tilde J_{s_2}(u)\le c(s_2)+\eta$, and so
$$0\le c(s_1)-c(s_2)\le \max_{u\in\gamma[-1,1]}\tilde J_{s_1}(u) - \max_{u\in\gamma[-1,1]}\tilde J_{s_2}(u)+\eta.$$
Let $u_0\in\gamma[-1,1]$ such that $\max_{u\in\gamma[-1,1]}\tilde J_{s_1}(u)=\tilde J_{s_1}(u_0)$. Then
$$0\le c(s_1)-c(s_2)\le \tilde J_{s_1}(u_0)-\tilde J_{s_2}(u_0)+\eta\le s_2-s_1+\eta,$$
as $\eta>0$ is arbitrary so the curve $\mathcal{C}$ is Lipschitz continuous with constant ≤ 1.

Next, to prove that the curve is decreasing, it suffices to argue for $s > 0$. Let $0 < s_1 < s_2$ then it implies $c(s_1) > c(s_2)$. On the other hand, since $(s_1+c(s_1), c(s_1))$, $(s_2 + c(s_2), c(s_2)) \in \Sigma_p$, Theorem 1.1 implies that $s_1 + c(s_1) < s_2 + c(s_2)$, which completes the proof. □

As $c(s)$ is decreasing and positive so the limit of $c(s)$ exists as $s \to \infty$.

Theorem 5.3. *If $n \geq p\alpha$, then the limit of $c(s)$ as $s \to \infty$ is $\lambda_{1,\epsilon}$.*

Proof. For $n \geq p\alpha$, we can choose a function $\phi \in \mathcal{W}^{\alpha,p}$ such that there does not exist $r \in \mathbb{R}$ such that $\phi(x) \leq r\phi_{1,\epsilon}(x)$ a.e. in Ω_ϵ (it suffices to take $\phi \in \mathcal{W}^{\alpha,p}$ such that it is unbounded from above in a neighborhood of some point $0 \neq x \in \Omega_\epsilon$). Suppose that the result is not true then there exists $\delta > 0$ such that $\max_{u\in\gamma[-1,1]} \tilde{J}_s(u) \geq \lambda_{1,\epsilon} + \delta$ for all $\gamma \in \Gamma$ and all $s \geq 0$. Consider a path $\gamma \in \Gamma$ by

$$\gamma(t) = \frac{\epsilon^{1/p}(t\phi_{1,\epsilon} + (1 - |t|)\phi)}{\|t\phi_{1,\epsilon} + (1 - |t|)\phi\|_{L^p(\Omega_\epsilon)}} \quad \text{for all } t \in [-1, 1].$$

Now, for every $s > 0$, let $t_s \in [-1, 1]$ satisfy $\max_{t\in[-1,1]} \tilde{J}_s(\gamma(t)) = \tilde{J}_s(\gamma(t_s))$. Let $v_{t_s} = t_s\phi_{1,\epsilon} + (1 - |t_s|)\phi$. Then we have

$$\tilde{J}_s(v_{t_s}) \geq \frac{(\lambda_{1,\epsilon} + \delta)}{\epsilon} \int_{\Omega_\epsilon} |v_{t_s}|^p. \tag{5.2}$$

Letting $s \to \infty$, we can assume a subsequence $t_s \to \tilde{t} \in [-1, 1]$. Then v_{t_s} is bounded in $\mathcal{W}^{\alpha,p}$. So, from last inequality we obtain $\int_{\Omega_\epsilon} (v_{t_s}^+)^p \, dx \to 0$ as $s \to \infty$, which forces

$$\int_{\Omega_\epsilon} ((\tilde{t}\phi_{1,\epsilon} + (1 - |\tilde{t}|)\phi)^+)^p \, dx = 0.$$

Hence, $\tilde{t}\phi_{1,\epsilon} + (1 - |\tilde{t}|)\phi \leq 0$. By the choice of ϕ, $\tilde{t}$ must be equal to -1. Passing to the limit in (5.2), we obtain

$$\begin{aligned} \frac{\lambda_{1,\epsilon}}{\epsilon} \int_{\Omega_\epsilon} \phi_{1,\epsilon}^p dx &= \Lambda_{n,p}(1 - \alpha) \int_Q \frac{|\phi_{1,\epsilon}(x) - \phi_{1,\epsilon}(y)|^p}{|x - y|^{n+p\alpha}} \, dx \, dy + \int_\Omega |\phi_{1,\epsilon}|^p \, dx \\ &\geq \frac{(\lambda_{1,\epsilon} + \delta)}{\epsilon} \int_{\Omega_\epsilon} |v_{t_s}|^p. \end{aligned}$$

We arrive at a contradiction that $\delta \leq 0$. Hence $c(s) \to \lambda_{1,\epsilon}$ as $s \to \infty$. □

6. Non resonance between (λ_1, λ_1) and $\mathcal{C}$

In this section, we study the non-resonance problem with respect to the Fučik spectrum for $p = 2$ case.

Lemma 6.1. *Let $(a, b) \in \mathcal{C}$, and let $m(x)$, $b(x) \in L^\infty(\Omega)$ satisfying*

$$\lambda_{1,\epsilon} \leq m(x) \leq a, \quad \lambda_{1,\epsilon} \leq b(x) \leq b. \tag{6.1}$$

Assume that

$$\lambda_{1,\epsilon} < m(x) \text{ and } \lambda_{1,\epsilon} < b(x) \text{ on subsets of positive measure of } \Omega_\epsilon. \tag{6.2}$$

Then any non-trivial solution u of

$$\begin{gathered} \Lambda_{n,2}(1 - \alpha)(-\Delta)^\alpha u + u = \frac{\chi_{\Omega_\epsilon}}{\epsilon}(m(x)u^+ - b(x)u^-) \quad \text{in } \Omega, \\ \mathcal{N}_{\alpha,2}u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{gathered} \tag{6.3}$$

changes sign in Ω_ϵ *and*

$$m(x) = a \text{ a.e. on } \{x \in \Omega_\epsilon : u(x) > 0\} \quad b(x) = b \text{ a.e. on } \{x \in \Omega_\epsilon : u(x) < 0\}.$$

Proof. Let u be a nontrivial solution of (6.3). Replacing u by $-u$ if necessary, we can assume that the point (a, b) in $\mathcal{C}$ is such that $a \geq b$. We first claim that u changes sign in Ω_ϵ. Suppose by contradiction that this is not true, first consider the case $u \geq 0$, (case $u \leq 0$ can be proved similarly). Then u solves

$$\Lambda_{n,2}(1-\alpha)(-\Delta)^\alpha u + u = \frac{\chi_{\Omega_\epsilon}}{\epsilon} m(x) u^+ \quad \text{in } \Omega, \quad \mathcal{N}_{\alpha,2} u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

This means that u is an eigenfunction of the problem with weight $m(x)$ corresponding to the eigenvalue equal to one. From the definition of the first eigenvalue of the problem with weight $m(x) \geq \lambda_{1,\epsilon}$, we have

$$\begin{aligned} &\lambda_{1,\epsilon}(m(x)) \\ &= \inf_{0 \not\equiv u \in \mathcal{W}^{\alpha,2}} \Big\{ \frac{\Lambda_{n,2}(1-\alpha) \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy + \int_\Omega |u|^2(x) \, dx}{\frac{1}{\epsilon} \int_{\Omega_\epsilon} m(x) |u|^2 \, dx} : \Big\} = 1. \end{aligned} \tag{6.4}$$

From (6.1), (6.1) and (6.4), we have

$$\begin{aligned} 1 &= \frac{\Lambda_{n,2}(1-\alpha) \int_Q |\phi_{1,\epsilon}(x) - \phi_{1,\epsilon}(y)|^2 |x-y|^{-(n+2\alpha)} \, dx \, dy + \int_\Omega |\phi_{1,\epsilon}|^2(x) \, dx}{\lambda_{1,\epsilon}} \\ &> \frac{\Lambda_{n,2}(1-\alpha) \int_Q |\phi_{1,\epsilon}(x) - \phi_{1,\epsilon}(y)|^2 |x-y|^{-(n+2\alpha)} \, dx \, dy + \int_\Omega |\phi_{1,\epsilon}|^2(x) \, dx}{\frac{1}{\epsilon} \int_{\Omega_\epsilon} m(x) |\phi_{1,\epsilon}|^2 \, dx} \\ &\geq 1, \end{aligned}$$

which is a contradiction. Hence, u changes sign on Ω_ϵ.

Let suppose by contradiction that either

$$|\{x \in \Omega_\epsilon : m(x) < a \text{ and } u(x) > 0\}| > 0 \tag{6.5}$$

or

$$|\{x \in \Omega_\epsilon : b(x) < b \text{ and } u(x) < 0\}| > 0. \tag{6.6}$$

Suppose that (6.5) holds (a similar argument will hold for (6.6)). Put $a - b = s \geq 0$. Then $b = c(s)$, where $c(s)$ is given by (3.14). We show that there exists a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \tilde{J}_s(u) < b, \tag{6.7}$$

which gives a contradiction with the definition of $c(s)$, prove the last part of the Lemma.

To construct γ we show that there exists of a function $v \in \mathcal{W}^{\alpha,2}$ such that it changes sign and satisfies

$$\begin{aligned} &\frac{\Lambda_{n,2}(1-\alpha) \int_Q |v^+(x) - v^+(y)|^2 |x-y|^{-(n+2\alpha)} \, dx \, dy + \int_\Omega (v^+)^2 \, dx}{\frac{1}{\epsilon} \int_{\Omega_\epsilon} (v^+)^2 \, dx} < a, \\ &\frac{\Lambda_{n,2}(1-\alpha) \int_Q |v^-(x) - v^-(y)|^2 |x-y|^{-(n+2\alpha)} \, dx \, dy + \int_\Omega (v^-)^2 \, dx}{\frac{1}{\epsilon} \int_{\Omega_\epsilon} (v^-)^2 \, dx} < b. \end{aligned} \tag{6.8}$$

Let $\mathcal{O}_1$ be a component of $\{x \in \Omega_\epsilon : u(x) > 0\}$ such that $|\{x \in \mathcal{O}_1 : m(x) < a\}| > 0$ and $\mathcal{O}_2$ be a component of $\{x \in \Omega_\epsilon : u(x) < 0\}$ such that $|\{x \in \mathcal{O}_2 : b(x) < b\}| > 0$. Define the eigenvalue problem

$$\begin{gathered} \Lambda_{n,2}(1-\alpha)(-\Delta)^\alpha u + u = \frac{\chi_{\mathcal{O}_i}}{\epsilon}(\lambda u) \quad \text{in } \Omega, \\ \mathcal{N}_{\alpha,2} u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \quad i = 1,2. \end{gathered} \tag{6.9}$$

Let $\lambda_{1,\epsilon}(\mathcal{O}_i)$ denote the first eigenvalue of (6.9). Next, we claim that

$$\lambda_{1,\epsilon}(\mathcal{O}_1) < a \quad \text{and} \quad \lambda_{1,\epsilon}(\mathcal{O}_2) < b, \tag{6.10}$$

where $\lambda_{1,\epsilon}(\mathcal{O}_i)$ denotes the first eigenvalue of $\Lambda_{n,2}(1-\alpha)(-\Delta)^\alpha u + u$ on $\mathcal{W}^{\alpha,2}$ and

$$\begin{aligned} \lambda_{1,\epsilon}(\mathcal{O}_1) &= \frac{\Lambda_{n,2}(1-\alpha)\int_Q |u(x)-u(y)|^2 |x-y|^{-(n+2\alpha)}\,dx\,dy + \int_\Omega |u|^2\,dx}{\frac{1}{\epsilon}\int_{\mathcal{O}_1} |u|^2\,dx} \\ &< a\frac{\Lambda_{n,2}(1-\alpha)\int_Q |u(x)-u(y)|^2 |x-y|^{-(n+2\alpha)}\,dx\,dy + \int_\Omega |u|^2\,dx}{\frac{1}{\epsilon}\int_{\mathcal{O}_1} m(x)|u|^2\,dx} = a, \end{aligned}$$

since $|x \in \mathcal{O}_1 : m(x) < a| > 0$. This implies $\lambda_{1,\epsilon}(\mathcal{O}_1) < a$. The other inequality can be proved similarly. Now with some modification on the sets $\mathcal{O}_1$ and $\mathcal{O}_2$, we construct the sets $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ such that $\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_2 = \emptyset$ and $\lambda_{1,\epsilon}(\tilde{\mathcal{O}}_1) < a$ and $\lambda_{1,\epsilon}(\tilde{\mathcal{O}}_2) < b$. For $\nu \geq 0$, $\mathcal{O}_1(\nu) = \{x \in \mathcal{O}_1 : \operatorname{dist}(x, (\Omega_\epsilon)^c) > \nu\}$. By Lemma 5.1, we have $\lambda_{1,\epsilon}(\mathcal{O}_1(\nu)) \geq \lambda_{1,\epsilon}(\mathcal{O}_1))$ and moreover $\lambda_{1,\epsilon}(\mathcal{O}_1(\nu)) \to \lambda_{1,\epsilon}(\mathcal{O}_1))$ as $\nu \to 0$. Then there exists $\nu_0 > 0$ such that

$$\lambda_{1,\epsilon}(\mathcal{O}_1(\nu)) < a \quad \text{for all } 0 \leq \nu \leq \nu_0. \tag{6.11}$$

Let $x_0 \in \partial\mathcal{O}_2 \cap \Omega_\epsilon$ (not empty as $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$), choose $0 < \nu < \min\{\nu_0, \operatorname{dist}(x_0, \Omega_\epsilon^c)\}$ and $\tilde{\mathcal{O}}_1 = \mathcal{O}_1(\nu)$ and $\tilde{\mathcal{O}}_2 = \mathcal{O}_2 \cup B(x_0, \frac{\nu}{2})$. Then $\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_2 = \emptyset$ and by (6.11), $\lambda_{1,\epsilon}(\tilde{\mathcal{O}}_1) < a$. Since $\tilde{\mathcal{O}}_2$ is connected, by (6.10) and Lemma 5.1, we obtain $\lambda_1(\tilde{\mathcal{O}}_2) < b$. Now, we define $v = v_1 - v_2$, where v_i are the eigenfunctions associated to $\lambda_{i,\epsilon}(\tilde{\mathcal{O}}_i)$. Then v satisfies (6.8).

Thus there exist $v \in \mathcal{W}^{\alpha,2}$ which changes sign, satisfies condition (6.8). Moreover we have

$$\begin{aligned} \tilde{J}_s\Big(\frac{\epsilon^{\frac{1}{2}} v}{\|v\|_{L^2(\Omega_\epsilon)}}\Big) &= \frac{\Lambda_{n,2}(1-\alpha)\int_Q \frac{|v^+(x)-v^+(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy + \int_\Omega (v^+)^2}{\frac{1}{\epsilon}\|v\|^2_{L^2(\Omega_\epsilon)}} - s\frac{\int_{\Omega_\epsilon}(v^+)^2\,dx}{\|v\|^2_{L^2(\Omega_\epsilon)}} \\ &\quad + \frac{\Lambda_{n,2}(1-\alpha)\int_Q \frac{|v^-(x)-v^-(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy + \int_\Omega (v^-)^2\,dx}{\frac{1}{\epsilon}\|v\|^2_{L^2(\Omega_\epsilon)}} \\ &\quad + 4\frac{\Lambda_{n,2}(1-\alpha)\int_Q \frac{v^+(x)v^-(y)}{|x-y|^{n+2\alpha}}\,dx\,dy}{\frac{1}{\epsilon}\|v\|^2_{L^2(\Omega_\epsilon)}} \\ &< (a-s)\frac{\int_{\Omega_\epsilon}(v^+)^2\,dx}{\|v\|^2_{L^2(\Omega_\epsilon)}} + b\frac{\int_{\Omega_\epsilon}(v^-)^2\,dx}{\|v\|^2_{L^2(\Omega_\epsilon)}} = b. \end{aligned}$$

$$\tilde{J}_s\Big(\frac{\epsilon^{\frac{1}{2}} v^+}{\|v^+\|_{L^2(\Omega_\epsilon)}}\Big) < a - s = b, \quad \tilde{J}_s\Big(\frac{\epsilon^{\frac{1}{2}} v^-}{\|v^-\|_{L^2(\Omega_\epsilon)}}\Big) < b - s.$$

Using Lemma 4.3, we have that there exists a critical point in the connected component of the set $\mathcal{O} = \{u \in \mathcal{S} : \tilde{J}_s(u) < b - s\}$. As the point $(a, b) \in \mathcal{C}$, the only possible critical point is $\phi_{1,\epsilon}$, then we can construct a path from $\phi_{1,\epsilon}$ to $-\phi_{1,\epsilon}$ exactly in the same manner as in Theorem 1.1 only by taking v in place of u. Thus we have construct a path satisfying (6.7), and hence the result follows. $\square$

Corollary 6.2. *Let $(a,b)\in\mathcal{C}$ and let $m(x)$, $b(x)\in L^\infty(\Omega)$ satisfying $\lambda_{1,\epsilon}\le m(x)\le a$ a.e., $\lambda_{1,\epsilon}\le b(x)\le b$ a.e. Assume that $\lambda_{1,\epsilon}<m(x)$ and $\lambda_{1,\epsilon}<b(x)$ on subsets of positive measure on Ω_ϵ. If either $m(x)<a$ a.e. in Ω_ϵ or $b(x)<b$ a.e. in Ω_ϵ. Then* (6.3) *has only the trivial solution.*

Proof. By Lemma 6.1, any non-trival solution of (6.3) changes sign and $m(x)=a$ a.e. on $\{x\in\Omega_\epsilon : u(x)>0\}$ or $b(x)=b$ a.e. on $\{x\in\Omega_\epsilon : u(x)<0\}$. So, by our hypotheses, (6.3) has only trivial solution. $\square$

Now, we study the non-resonance between (λ_1,λ_1) and $\mathcal{C}$,

$$\begin{gathered}\Lambda_{n,2}(1-\alpha)(-\Delta)^\alpha u+u=\frac{\chi_{\Omega_\epsilon}f(x,u)}{\epsilon}\quad\text{in }\Omega,\\ \mathcal{N}_{\alpha,2}u=0\quad\text{in }\mathbb{R}^n\setminus\overline{\Omega},\end{gathered}\tag{6.12}$$

where $f(x,u)/u$ lies asymptotically between $(\lambda_{1,\epsilon},\lambda_{1,\epsilon})$ and $(a,b)\in\mathcal{C}$.

Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a function satisfying $L^\infty(\Omega)$ Caratheodory conditions. Given a point $(a,b)\in\mathcal{C}$, we assume that

$$\gamma_\pm(x)\le\liminf_{s\to\pm\infty}\frac{f(x,s)}{s}\le\limsup_{s\to\pm\infty}\frac{f(x,s)}{s}\le\Gamma_\pm(x)\tag{6.13}$$

holds uniformly with respect to x, where $\gamma_\pm(x)$ and $\Gamma_\pm(x)$ are $L^\infty(\Omega)$ functions which satisfy

$$\begin{gathered}\lambda_{1,\epsilon}\le\gamma_+(x)\le\Gamma_+(x)\le a\quad\text{a.e. in }\Omega_\epsilon\\ \lambda_{1,\epsilon}\le\gamma_-(x)\le\Gamma_-(x)\le b\quad\text{a.e. in }\Omega_\epsilon.\end{gathered}\tag{6.14}$$

The function $F(x,s)=\int_0^s f(x,t)dt$, we also satisfies

$$\delta_\pm(x)\le\liminf_{s\to\pm\infty}\frac{2F(x,s)}{|s|^2}\le\limsup_{s\to\pm\infty}\frac{2F(x,s)}{|s|^2}\le\Delta_\pm(x)\tag{6.15}$$

uniformly with respect to x, where $\delta_\pm(x)$ and $\Delta_\pm(x)$ are $L^\infty(\Omega)$ functions which satisfy

$$\begin{gathered}\lambda_{1,\epsilon}\le\delta_+(x)\le\Delta_+(x)\le a\text{ a.e. in }\Omega_\epsilon,\quad\lambda_{1,\epsilon}\le\delta_-(x)\le\Delta_-(x)\le b\text{ a.e. in }\Omega_\epsilon,\\ \delta_+(x)>\lambda_{1,\epsilon}\text{ and }\delta_-(x)>\lambda_{1,\epsilon}\text{ on subsets of positive measure},\\ \text{either }\Delta_+(x)<a\text{ a.e. in }\Omega_\epsilon\text{ or }\Delta_-(x)<b\text{ a.e. in }\Omega_\epsilon.\end{gathered}\tag{6.16}$$

Theorem 6.3. *Let* (6.13), (6.14), (6.15) *and* (6.16) *hold and $(a,b)\in\mathcal{C}$. Then* (6.12) *admits at least one solution u in $\mathcal{W}^{\alpha,2}$.*

Define the energy functional $\Psi:\mathcal{W}^{\alpha,2}\to\mathbb{R}$ as

$$\Psi(u)=\frac{\Lambda_{n,2}(1-\alpha)}{2}\int_Q\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy+\frac12\int_\Omega|u|^2\,dx-\frac1\epsilon\int_{\Omega_\epsilon}F(x,u)\,dx$$

Then Ψ is a C^1 functional on $\mathcal{W}^{\alpha,2}$ and for all $v\in\mathcal{W}^{\alpha,2}$,

$$\langle\Psi'(u),v\rangle=\Lambda_{n,2}(1-\alpha)\int_Q\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2\alpha}}\,dx\,dy+\int_\Omega uv\,dx$$

$$- \frac{1}{\epsilon} \int_{\Omega_\epsilon} f(x,u) v \, dx$$

and critical points of Ψ are exactly the weak solutions of (6.12).

Next, we state some Lemmas, whose proofs can be found in [14, Lemma 5.2 and 5.3].

Lemma 6.4. *Ψ satisfies the (PS) condition on $\mathcal{W}^{\alpha,2}$.*

Lemma 6.5. *There exists $R > 0$ such that*

$$\max\{\Psi(R\phi_{1,\epsilon}), \Psi(-R\phi_{1,\epsilon})\} < \max_{u \in \gamma[-1,1]} \Psi(u),$$

for any $\gamma \in \Gamma_1 := \{\gamma \in C([-1,1], \mathcal{S}) : \gamma(\pm 1) = \pm R\phi_{1,\epsilon}\}$.

Proof of Theorem 6.3. Lemmas 6.4 and 6.5 complete the proof. □

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