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GLOBAL STABILITY FOR INFECTIOUS DISEASE MODELS THAT INCLUDE IMMIGRATION OF INFECTED INDIVIDUALS AND DELAY IN THE INCIDENCE

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Abstract. We begin with a detailed study of a delayed SI model of disease transmission with immigration into both classes. The incidence function allows for a nonlinear dependence on the infected population, including mass action and saturating incidence as special cases. Due to the immigration of infectives, there is no disease-free equilibrium and hence no basic reproduction number. We show there is a unique endemic equilibrium and that this equilibrium is globally asymptotically stable for all parameter values. The results include vector-style delay and latency-style delay. Next, we show that previous global stability results for an SEI model and an SVI model that include immigration of infectives and non-linear incidence but not delay can be extended to systems with vector-style delay and latency-style delay.

1. Introduction

Many countries throughout the world have high numbers of both immigrants and short-term visitors. For example, according to the 2006 Canadian Census [3], 1,109,980 people immigrated to Canada, between January 1, 2001 and May 16, 2006 (the day of the census). At the time of the census, there were 6,186,950 immigrants in Canada, comprising 19.8% of the population. Additionally, there are millions of short-term visitors to Canada each year.

With so many individuals entering Canada (for example), it is inevitable that some individuals will already be infected with a given disease at the time of arrival. This makes it desirable to consider models that account for the immigration of infected individuals.

An immediate consequence of including immigration of infected individuals is that the disease-free space is no longer positively invariant. Thus, there is no disease-free equilibrium, and hence no basic reproduction number.

In this work, we study disease transmission models that include immigration of infected individuals. The models also include delayed effects. We consider both delay due to vector transmission and delay due to latency.

For vector transmitted diseases, one can view the vector as providing a connection between a susceptible individual at time t and an infected individual from an

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earlier time, say $t-\tau$. This puts a delay in the incidence term, as done by Cooke [4] and Takeuchi et al. [11], and many others since. For mass action models, this vector-style delay generally results in a term of the form $\beta S(t)I(t-\tau)$ being subtracted from the susceptible equation and a corresponding term being added to the equation for the first infected class.

Another way that delay often arises in compartmental models for disease transmission relates to a latency period of duration τ between the time when an individual becomes infected and when they become infectious. For mass action models, it is then common to have a negative term $\beta S(t)I(t)$ to represent the rate at which individuals infected at time t leave the susceptible class and a related positive term $\beta S(t-\tau)I(t-\tau)$ to represent the rate at which individuals were infected at time $t-\tau$, who are now entering the infectious class, for example.

Another feature that we include in this work is that the force of infection may have a non-linear dependence on the size of the infectious population, so that the incidence function takes the form $Sf(I)$, for some reasonable function f, described in Section 2. This form of incidence includes mass action βSI and saturating incidence $\beta\frac{SI}{1+mI}$ as special cases.

We now provide a brief review of earlier work on models that include immigration of infected individuals. In [2], the authors study a non-delayed SIS model. They show that there is a unique equilibrium, which is strictly positive. Using the Bendixson-Dulac Criterion, they show that the equilibrium is globally asymptotically stable. They also consider an SIRS model, again showing that the unique (positive) equilibrium is globally asymptotically stable. This time the proof was based on converting the system of ordinary differential equations to a scalar integral equation. In each case they first work with mass action incidence βSI, before extending their work to the case where β is a function of the total population size.

In [8] an ODE model including standard incidence was studied, finding a unique positive equilibrium. Using compound matrix techniques, they prove the unique positive equilibrium is globally asymptotically stable for a portion of the parameter space.

In [9], the authors study an ODE model of HIV infection with proportional mixing. Using a Lyapunov function they show that the unique positive equilibrium is globally asymptotically stable for a portion of the parameter space.

The paper [12] presents an ODE model of vector transmission of malaria, accounting for the vector population (mosquitoes) explicitly and including immigration of infected hosts. Using compound matrix techniques, they prove the unique positive equilibrium is globally asymptotically stable.

In [10] an SEI model was studied and in [5] an SVI model (where V stands for vaccinated) was studied. For each of these models, which were systems of ordinary differential equations, it was shown that there was a globally asymptotically stable equilibrium through the use of a Lyapunov function.

In [1], the authors studied an SVIR model that includes diffusion within the region of interest. The unique endemic equilibrium was shown to be globally asymptotically stable through the use of a Lyapunov functional.

In [7], an SEI model with continuous age-in-class structure for the infected classes is studied. The unique endemic equilibrium was shown to be globally asymptotically stable through the use of a Lyapunov functional.

The current paper is organized as follows. In Sections 2-5, an SI model with immigration of infecteds, vector-style delay and non-linear incidence is studied in full detail. In Section 6, it is shown that the same results also apply to a corresponding SI model with latency-style delay. In Section 7, the global dynamics are resolved for two SEI models, one with vector-style delay and the other with latency-style delay. In Section 8, the global dynamics are resolved for two SVI models, one with vector-style delay and the other with latency-style delay. A discussion of the results is given in Section 9.

2. An SI model with immigration and vector-style delay

A human population is divided into two classes: susceptible and infectious. The sizes of the classes are denoted by S and I, respectively. The influxes of new individuals entering the population into each group (through birth and immigration) are denoted by $\Lambda_S > 0$ for the susceptible class and by $\Lambda_I > 0$ for the infectious class.

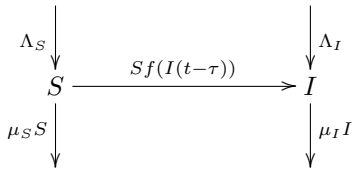
The incidence rate at which susceptibles become infectious is assumed to be linear in S but may have a non-linear dependence on the size of the infectious population, taking the form $Sf(I)$, where f is a twice differentiable function satisfying the following hypotheses:

(H1) $f(I) \geq 0$ with equality if and only if $I = 0$.
(H2) $f'(I) \geq 0$.
(H3) $f''(I) \leq 0$.

These hypotheses were also used in [5, 10].

We assume that the disease is transmitted through a vector (such as a mosquito). Following the work of Cooke [4], we assume that infected vectors become infectious after a fixed time τ, and that the number of infected humans is a good proxy for the number of infected vectors. This implies that rate of new human infections at time t is $S(t)f(I(t-\tau))$. We note that the form of incidence used here includes mass action βSI and saturating incidence $\beta\frac{SI}{1+mI}$ as special cases.

Individuals leave the susceptible and infectious classes with per capita death rates of μ_S and μ_I, respectively. We assume that $0 < \mu_S \leq \mu_I$ so that the death rate for those infected with the disease is at least as high as for those who are not infected. The transfer diagram for the model is shown below.



The corresponding system of differential equations is

$$\begin{aligned} \frac{dS}{dt} &= \Lambda_S - Sf(I(t-\tau)) - \mu_S S \\ \frac{dI}{dt} &= \Lambda_I + Sf(I(t-\tau)) - \mu_I I. \end{aligned} \tag{2.1}$$

The phase space for the system is $\mathcal{Y} = \mathbb{R}_{\geq 0} \times \mathcal{C}([-\tau, 0], \mathbb{R}_{\geq 0})$, where $\mathcal{C}([-\tau, 0], \mathbb{R}_{\geq 0})$ is the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}_{\geq 0}$.

Given a function $h : [T-\tau, T] \to \mathbb{R}_{\geq 0}$, we define the associated function $h_T : [-\tau, 0] \to \mathbb{R}_{\geq 0}$ by $h_T(\theta) = h(T+\theta)$ for all $\theta \in [-\tau, 0]$. The initial condition for (2.1) is

$$\big(S(0), I_0(\cdot)\big) = \big(\bar{S}, \phi(\cdot)\big) \in \mathcal{Y}.$$

Note that we are using the convention that if the argument for S or I is omitted, then the variable is to be evaluated at time t; if the variable is to be evaluated at any other time (such as $t-\tau$, for example) then the argument will be given explicitly.

3. Equilibria

Since $\frac{dI}{dt}\big|_{I=0} = \Lambda_I > 0$, there is no disease-free equilibrium, and so there is no basic reproduction number. There is, however, an endemic equilbrium. The proof of the following result is similar to the proof of [10, Proposition 3.1].

Proposition 3.1. *There exists a unique equilibrium $(S^*, I^*) \in \mathbb{R}^2_{\geq 0}$. Furthermore, $S^*, I^* > 0$.*

Proof. Since $\frac{dS}{dt}\big|_{S=0} = \Lambda_S > 0$ and $\frac{dI}{dt}\big|_{I=0} = \Lambda_I > 0$, it follows that there are no equilibria for which either S or I is zero. Thus, we may restrict our search for equilibria to $\mathbb{R}^2_{>0}$.

Solving $\frac{dS}{dt} + \frac{dI}{dt} = 0$, gives $S^* = \frac{1}{\mu_S}(\Lambda_S + \Lambda_I - \mu_I I^*)$, and therefore to have S^* positive, we must have $I^* < I_{\max}$, where $I_{\max} = \frac{\Lambda_S + \Lambda_I}{\mu_I}$.

Rearranging $\frac{dS}{dt} = 0$ gives $H(I^*) = 0$, where

$$H(I^*) = f(I^*) + \mu_S - \frac{\mu_S \Lambda_S}{\Lambda_S + \Lambda_I - \mu_I I^*}. \tag{3.1}$$

Recalling from (H1) that $f(0) = 0$, we note that $H(0) = \mu_S - \frac{\mu_S \Lambda_S}{\Lambda_S + \Lambda_I} > 0$. Also, $H(I^*)$ tends to negative infinity as I^* increases to $I_{\max}$. Thus, there exists at least one zero of H in the interval $(0, I_{\max})$.

Note that $H''(I^*) = f''(I^*) - \frac{2\mu_I^2 \mu_S \Lambda_S}{(\Lambda_S + \Lambda_I - \mu_I I^*)^3}$, which is negative for $I^* \in (0, I_{\max})$. Since H is positive at 0 and concave down on $(0, I_{\max})$, it follows that the zero of H in $(0, I_{\max})$ is unique. The result follows. □

4. Local stability

Proposition 4.1. *The equilibrium (S^*, I^*) is locally asymptotically stable.*

Proof. We begin by determining the characteristic equation. Let $s(t) = S(t) - S^*$ and $i(t) = I(t) - I^*$. Then for sufficiently small s and i,

$$\begin{aligned}
\frac{ds}{dt} = \frac{dS}{dt} &= \Lambda_S - Sf(I(t-\tau)) - \mu_S S \\
&= \Lambda_S - (S^* + s) f(I^* + i(t-\tau)) - \mu_S (S^* + s) \\
&\approx \Lambda_S - (S^* + s)\left[f(I^*) + f'(I^*) i(t-\tau)\right] - \mu_S (S^* + s) \\
&= \left[\Lambda_S - S^* f(I^*) - \mu_S S^*\right] - \left[S^* f'(I^*) i(t-\tau) + s f(I^*) + \mu_S s\right] \\
&\quad - s f'(I^*) i(t-\tau) \\
&= -\left[S^* f'(I^*) i(t-\tau) + s f(I^*) + \mu_S s\right] - s f'(I^*) i(t-\tau) \\
&\approx -S^* f'(I^*) i(t-\tau) - s f(I^*) - \mu_S s,
\end{aligned} \tag{4.1}$$

where the first approximation comes from taking a first order Taylor series of f and the second approximation comes from dropping the quadratic term $s(t)f'(I^*)i(t-\tau)$. A similar calculation yields

$$\frac{di}{dt} \approx S^* f'(I^*)i(t-\tau) + sf(I^*) - \mu_I i. \tag{4.2}$$

We now use the exponential ansatz

$$\begin{bmatrix} s(t) \\ i(t) \end{bmatrix} = \mathrm{e}^{\lambda t} \begin{bmatrix} s(0) \\ i(0) \end{bmatrix}.$$

Filling into (4.1) and (4.2), and canceling $\mathrm{e}^{\lambda t}$ from each side, the equations can be re-written as

$$(M - \lambda I_{2\times 2}) \begin{bmatrix} s(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} -(f(I^*) + \mu_S) & -S^* f'(I^*)\mathrm{e}^{-\lambda\tau} \\ f(I^*) & S^* f'(I^*)\mathrm{e}^{-\lambda\tau} - \mu_I \end{bmatrix} \tag{4.3}$$

and $I_{2\times 2}$ is the 2×2 identity matrix.

There are nontrivial solutions if and only if $M - \lambda I_{2\times 2}$ is singular. Thus, the characteristic equation is

$$\begin{aligned} 0 &= \det(M - \lambda I_{2\times 2}) \\ &= (f(I^*) + \mu_S + \lambda)[(\mu_I + \lambda) - S^* f'(I^*)\mathrm{e}^{-\lambda\tau}] + f(I^*)S^* f'(I^*)\mathrm{e}^{-\lambda\tau} \\ &= \lambda^2 + (f(I^*) + \mu_S + \mu_I)\lambda + (f(I^*) + \mu_S)\mu_I - (\lambda + \mu_S)S^* f'(I^*)\mathrm{e}^{-\lambda\tau} \\ &= \lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)\,\mathrm{e}^{-\lambda\tau}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} p_1 &= f(I^*) + \mu_S + \mu_I \quad q_1 = -S^* f'(I^*) \\ p_0 &= (f(I^*) + \mu_S)\mu_I \quad q_0 = -\mu_S S^* f'(I^*). \end{aligned}$$

To assist with the upcoming calculations, we first obtain a useful inequality. Using (H1)–(H3) and following the proof of [10, Proposition 4.1] it can be shown that $f'(I^*) \le \frac{f(I^*)}{I^*}$. Then, using the fact that $\frac{dI}{dt}$ is zero at the equilibrium, we can replace $f(I^*)$ in this inequality to write $f'(I^*) \le \frac{\mu_I I^* - \Lambda_I}{S^* I^*}$. It follows that

$$\mu_I > S^* f'(I^*). \tag{4.5}$$

We now use a four step approach, part of which comes from the approach described in [13, Section 2], to show that all solutions λ of (4.4) have negative real part.

Step A: Consider the matrix M for $\tau = 0$. Note that the $(2,2)$-entry becomes $S^* f'(I^*) - \mu_I$. By (4.5), this is negative and so, for $\tau = 0$, M has the sign pattern

$$\begin{bmatrix} - & - \\ + & - \end{bmatrix}.$$

Thus, $\mathrm{trace}(M) < 0$ and $\det(M) > 0$. It follows that the eigenvalues of M both have negative real part and so (S^*, I^*) is locally asymptotically stable for $\tau = 0$.

Step B: In this step, we study the possibility of eigenvalues appearing at infinity. In particular, we show for $\tau \ge 0$, that there is an upper bound on the magnitude of any eigenvalues with positive real part, thereby precluding the possibility of eigenvalues appearing at infinity.

Let $K > \max\{|p_1| + |q_1| + 1, |p_0| + |q_0|\}$. Suppose λ is a solution of (4.4) with $\mathrm{Re}(\lambda) > 0$ and $|\lambda| > K$. Let $Z = \lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)\mathrm{e}^{-\lambda\tau}$. Then,

$$\begin{aligned}
|Z| &\geq |\lambda^2| - |p_1\lambda + p_0 + (q_1\lambda + q_0)\mathrm{e}^{-\lambda\tau}| \\
&\geq |\lambda^2| - |p_1\lambda| - |p_0| - |q_1\lambda||\mathrm{e}^{-\lambda\tau}| - |q_0||\mathrm{e}^{-\lambda\tau}| \\
&\geq |\lambda^2| - |p_1||\lambda| - |p_0| - |q_1||\lambda| - |q_0| \\
&= |\lambda|(|\lambda| - |p_1| - |q_1|) - (|p_0| + |q_0|) \\
&> |\lambda|(K - |p_1| - |q_1|) - (|p_0| + |q_0|) \\
&> |\lambda| - (|p_0| + |q_0|) \\
&> K - (|p_0| + |q_0|) > 0.
\end{aligned}$$

Thus, $Z \neq 0$ and so λ is not a solution to the characteristic equation. This means that each solution of the characteristic equation either has negative real part or has a magnitude of at most K. Combining this with the result of Step A, it follows that the only possible loss of (local) stability that can happen as τ increases from 0, is that eigenvalues could cross from the left half-plane to the right half-plane (but only with magnitude less than K). In Steps C and D, we rule out that possibility.

Step C: For any $\tau \geq 0$, filling $\lambda = 0$ into the expression on the right-hand side of the characteristic equation (4.4) gives $p_0 + q_0 = (f(I^*) + \mu_S)\mu_I - \mu_S S^* f'(I^*)$, which is positive by (4.5). Thus, $\lambda = 0$ is never a solution to (4.4).

Step D: Suppose $\tau > 0$ and suppose $\lambda = \omega i$ (with $\omega \neq 0$) is a solution of (4.4). Replacing λ in (4.4) and separating the real and imaginary parts gives

$$\begin{aligned}
\omega^2 - p_0 &= q_1\omega \sin\omega\tau + q_0 \cos\omega\tau, \\
p_1\omega &= q_0 \sin\omega\tau - q_1\omega\cos\omega\tau.
\end{aligned}$$

Squaring both equations, adding the results and letting $z = \omega^2 > 0$, gives

$$z^2 + \left(p_1^2 - 2p_0 - q_1^2\right) z + p_0^2 - q_0^2 = 0. \tag{4.6}$$

We now show that the constant and linear terms on the left-hand side of (4.6) are positive. Since z is positive, it will then follow that (4.6) has no valid solutions, and so ωi must not be a characteristic root for the equilibrium.

It follows from (4.5) that $\mu_S\mu_I > \mu_S S^* f'(I^*)$. This means that p_0 is further from 0 than q_0 is, and so $p_0^2 - q_0^2 > 0$. That is, the constant term on the left-hand side of (4.6) is positive.

Also, $p_1^2 - 2p_0 - q_1^2 = \left(f(I^*)^2 + \mu_S\right)^2 + \mu_I^2 - \left(S^* f'(I^*)\right)^2 > \mu_I^2 - \left(S^* f'(I^*)\right)^2 > 0$. Thus, the linear coefficient on the left-hand side of (4.6) is positive. Therefore, (4.6) has no positive roots and so ωi cannot be a solution to (4.4).

Combining the results of Steps A, B, C, D, it follows that all roots of (4.4) have negative real part. Thus, the equilibrium (S^*, I^*) is locally asymptotically stable for all $\tau \geq 0$. □

5. Global stability

Theorem 5.1. *The equilibrium (S^*, I^*) is globally asymptotically stable on the set $\mathcal{Y}$.*

Proof. Let

$$\begin{aligned} g(x) &= x - 1 - \ln(x), \\ \mathcal{V} &= S^* g\Big(\frac{S}{S^*}\Big) + I^* g\Big(\frac{I}{I^*}\Big), \\ \mathcal{W} &= \int_0^\tau g\Big(\frac{f(I(t-\sigma))}{f(I^*)}\Big)\mu_I \sigma, \\ \mathcal{U} &= \mathcal{V} + S^* f(I^*)\mathcal{W}. \end{aligned} \tag{5.1}$$

Note that $\frac{dS}{dt}|_{S=0} = \Lambda_S > 0$ and $\frac{dI}{dt}|_{I=0} = \Lambda_I > 0$ implies that $S(t), I(t) > 0$ for all $t > 0$. Thus, we may assume that $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{U}$ are well-defined and finite for all $t > \tau$.

We begin by calculating $\frac{d\mathcal{V}}{dt}$. Using $\Lambda_S = S^* f(I^*) + \mu_S S^*$ and $\mu_I = \frac{\Lambda_I + S^* f(I^*)}{I^*}$, we have

$$\begin{aligned} \frac{d\mathcal{V}}{dt} &= \Big(1 - \frac{S^*}{S}\Big)[\Lambda_S - Sf(I(t-\tau)) - \mu_S S] + \Big(1 - \frac{I^*}{I}\Big)[\Lambda_I + Sf(I(t-\tau)) - \mu_I I] \\ &= \Big(1 - \frac{S^*}{S}\Big)\left[S^* f(I^*) - Sf(I(t-\tau)) + \mu_S\left(S^* - S\right)\right] \\ &\quad + \Big(1 - \frac{I^*}{I}\Big)\Big[\Lambda_I + Sf(I(t-\tau)) - \Big(\frac{\Lambda_I + S^* f(I^*)}{I^*}\Big) I\Big] \\ &= -\mu_S \frac{(S-S^*)^2}{S} + S^* f(I^*)\Big(1 - \frac{S^*}{S}\Big)\Big(1 - \frac{Sf(I(t-\tau))}{S^* f(I^*)}\Big) \\ &\quad - \Lambda_I \frac{(I-I^*)^2}{II^*} + S^* f(I^*)\Big(1 - \frac{I^*}{I}\Big)\Big(\frac{Sf(I(t-\tau))}{S^* f(I^*)} - \frac{I}{I^*}\Big) \\ &= -\mu_S \frac{(S-S^*)^2}{S} - \Lambda_I \frac{(I-I^*)^2}{II^*} + S^* f(I^*) C, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} C &= 2 + \frac{f(I(t-\tau))}{f(I^*)} - \frac{S^*}{S} - \frac{I}{I^*} - \frac{SI^* f(I(t-\tau))}{S^* I f(I^*)} \\ &= g\Big(\frac{f(I(t-\tau))}{f(I^*)}\Big) - g\Big(\frac{S^*}{S}\Big) - g\Big(\frac{I}{I^*}\Big) - g\Big(\frac{SI^* f(I(t-\tau))}{S^* I f(I^*)}\Big). \end{aligned} \tag{5.3}$$

(This last expression can be checked by using the definition of g to obtain the previous line.) Also,

$$\begin{aligned} \frac{d\mathcal{W}}{dt} &= \frac{d}{dt}\int_0^\tau g\Big(\frac{f(I(t-\sigma))}{f(I^*)}\Big)\mu_I \sigma \\ &= \int_0^\tau \frac{d}{dt} g\Big(\frac{f(I(t-\sigma))}{f(I^*)}\Big)\mu_I \sigma \\ &= -\int_0^\tau \frac{d}{d\sigma} g\Big(\frac{f(I(t-\sigma))}{f(I^*)}\Big)\mu_I \sigma \\ &= g\Big(\frac{f(I)}{f(I^*)}\Big) - g\Big(\frac{f(I(t-\tau))}{f(I^*)}\Big). \end{aligned} \tag{5.4}$$

To find $\frac{d\mathcal{U}}{dt}$, we combine (5.2), (5.3) and (5.4), to obtain

$$\frac{d\mathcal{U}}{dt} = -\mu_S \frac{(S-S^*)^2}{S} - \Lambda_I \frac{(I-I^*)^2}{II^*} + S^* f(I^*)\mu_I, \tag{5.5}$$

where

$$D = g\Big(\frac{f(I)}{f(I^*)}\Big) - g\Big(\frac{S^*}{S}\Big) - g\Big(\frac{I}{I^*}\Big) - g\Big(\frac{SI^*f(I(t-\tau))}{S^*If(I^*)}\Big). \tag{5.6}$$

By Proposition A.1 from [10], the hypotheses (H1)–(H3) ensure that $g\Big(\frac{f(I)}{f(I^*)}\Big) - g\Big(\frac{I}{I^*}\Big) \le 0$ for all $I > 0$. Thus,

$$\mu_I \le -g\Big(\frac{S^*}{S}\Big) - g\Big(\frac{SI^*f(I(t-\tau))}{S^*If(I^*)}\Big) \le 0.$$

Therefore, $\frac{d\mathcal{U}}{dt} \le 0$, with equality only if $(S, I) = (S^*, I^*)$. Thus, by Lyapunov's Direct Method, the equilibrium is globally asymptotically stable. □

6. An SI model with immigration and latency-style delay

It is noteworthy that the model studied in Sections 2 to 5, which includes delay due to vector transmission, is equivalent to a model that includes delay due to latency. This can be seen by defining $Y(t) = S(t+\tau)$. Then, (2.1) becomes

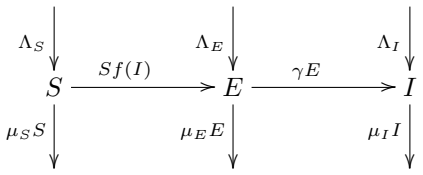
$$\begin{aligned} \frac{dY(t)}{dt} &= \Lambda_S - Y(t)f(I(t)) - \mu_S Y(t) \\ \frac{dI(t)}{dt} &= \Lambda_I + Y(t-\tau)f(I(t-\tau)) - \mu_I I(t), \end{aligned} \tag{6.1}$$

which is a similar model, but with a latency-style delay. It follows that our results also apply to (6.1).

Theorem 6.1. *The equilibrium (S^*, I^*) is locally and globally asymptotically stable under the flow described by* (6.1).

7. Two SEI models with immigration and delay

In [10], a model including susceptible, exposed and infectious classes, with immigration of infected individuals, but no delay, was studied. The transfer diagram for the model is shown below.



In this model, Λ_E gives the influx of individuals entering the system into the exposed class, μ_E is the per capita death rate of the exposed class and $\frac{1}{\gamma}$ is the average time spent in the exposed class before moving to the infectious class; other parameters have the same meaning as in the SI-model introduced in Section 2.

The system of differential equations for the model is

$$\begin{aligned} \frac{dS}{dt} &= \Lambda_S - Sf(I) - \mu_S S \\ \frac{dE}{dt} &= \Lambda_E + Sf(I) - (\mu_E + \gamma)E \\ \frac{dI}{dt} &= \Lambda_I + \gamma E - \mu_I I, \end{aligned} \tag{7.1}$$

with $\Lambda_E, \Lambda_I \geq 0$ and $\Lambda_E + \Lambda_I, \Lambda_S, \mu_S, \mu_E, \mu_I, \gamma > 0$. Also, the force of infection f is assumed to satisfy the hypotheses (H1)–(H3).

In [10], it is shown that there is a unique equilibrium $X^* = (S^*, E^*, I^*) \in \mathbb{R}^3_{>0}$. Using the Lyapunov function

$$\mathcal{V} = S^* g\Big(\frac{S}{S^*}\Big) + E^* g\Big(\frac{E}{E^*}\Big) + \frac{S^* f(I^*)}{\gamma E^*} I^* g\Big(\frac{I}{I^*}\Big), \tag{7.2}$$

it is shown that X^* is globally asymptotically stable. In doing so, the authors of [10] obtain the inequality

$$\begin{aligned} D_{(7.1)}\mathcal{V} \leq -\mu_S \frac{(S-S^*)^2}{S} - \Lambda_E \frac{(E-E^*)^2}{E^* E} - \frac{S^* f(I^*)}{\gamma E^*} \Lambda_I \frac{(I-I^*)^2}{I^* I} \\ - S^* f(I^*) \Big[g\Big(\frac{S^*}{S}\Big) + g\Big(\frac{S E^* f(I)}{S^* E f(I^*)}\Big) + g\Big(\frac{E I^*}{E^* I}\Big)\Big]. \end{aligned} \tag{7.3}$$

Notation. For the remainder of this paper, we use notation similar to the previous equation, where $D_{(7.1)}\mathcal{V}$ gives the derivative of $\mathcal{V}$ with respect to time as the arguments of $\mathcal{V}(S, E, I)$ change according to the differential equation (7.1).

We now wish to consider delayed versions of (7.1). Consider vector-style delay in the transmission term:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda_S - S f(I(t-\tau)) - \mu_S S \\ \frac{dE}{dt} &= \Lambda_E + S f(I(t-\tau)) - (\mu_E + \gamma) E \\ \frac{dI}{dt} &= \Lambda_I + \gamma E - \mu_I I, \end{aligned} \tag{7.4}$$

and latency-style delay in the transmission term

$$\begin{aligned} \frac{dS}{dt} &= \Lambda_S - S f(I) - \mu_S S \\ \frac{dE}{dt} &= \Lambda_E + S(t-\tau) f(I(t-\tau)) - (\mu_E + \gamma) E \\ \frac{dI}{dt} &= \Lambda_I + \gamma E - \mu_I I, \end{aligned} \tag{7.5}$$

where $\tau > 0$ in each case. Note that for each of (7.4) and (7.5), the unique equilibrium is $X^* = (S^*, E^*, I^*)$, the same as for the ordinary differential equation (7.1).

Also note that by making the substitution $Y(t) = S(t+\tau)$ (and then changing Y to S), (7.5) can be shown to be equivalent to (7.4), similar to how (6.1) was shown to be equivalent to (2.1). Thus, we will focus on the stability of just (7.5).

To do this, we first state a version of Theorem 5.1 from [6], which gives conditions under which a Lyapunov function $\mathcal{V}$ for an ordinary differential equation can be extended to a Lyapunov functional $\mathcal{U}$ for a related delay differential equation that includes latency-style delay. In the terminology of [6], we are adding delay to the transmission term $q(X(t)) = S(t) f(I(t))$; we also have $x_j = E$, $A = 1$ and $L = \frac{E^*}{E}$. Then [6, Theorem 5.1] and its proof give the following result.

Theorem 7.1. *If*

$$D_{(7.1)}\mathcal{V} + A q(X^*) g\Big(\frac{q(X(t))}{q(X^*)} L\Big) \leq 0,$$

then

$$\mathcal{U} = \mathcal{V} + Aq(X^*) \int_0^\tau g\Big(\frac{X(t-\sigma)}{X^*}\Big) d\sigma$$

is a Lyapunov functional for (7.5) *satisfying*

$$D_{(7.5)}\mathcal{U} = D_{(7.1)}\mathcal{V} + Aq(X^*)\Big[g\Big(\frac{X(t)}{X^*}L\Big) - g\Big(\frac{X(t-\tau)}{X^*}L\Big)\Big].$$

Filling the expressions for q, A and L into the theorem conditions, we see that it is necessary to have $D_{(7.1)}\mathcal{V} + S^* f(I^*) g\left(\frac{Sf(I)E^*}{S^* f(I^*)E}\right)$ less than or equal to 0. Using (7.3) to replace $D_{(7.1)}\mathcal{V}$, we see that the condition is satisfied. Thus, Theorem 7.1 implies

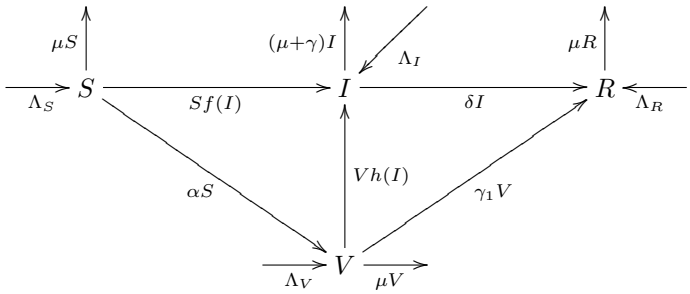
$$\begin{aligned} D_{(7.5)}\mathcal{U} \le -\mu_S \frac{(S-S^*)^2}{S} - \Lambda_E \frac{(E-E^*)^2}{E^*E} - \frac{S^* f(I^*)}{\gamma E^*}\Lambda_I \frac{(I-I^*)^2}{I^*I} \\ - S^* f(I^*)\Big[g\Big(\frac{S^*}{S}\Big) + g\Big(\frac{S(t-\tau)f(I(t-\tau))E^*}{S^* f(I^*)E}\Big) + g\Big(\frac{EI^*}{E^*I}\Big)\Big]. \end{aligned}$$

It follows from Lyapunov's Direct Method that the equilibrium X^* is globally asymptotically stable under the flow described by (7.5). Due to the equivalence of (7.4) and (7.5), X^* is also globally asymptotically stable under the flow described by (7.4).

Theorem 7.2. *The equilibrium X^* is globally asymptotically stable under the flow described by* (7.4) *and also under the flow described by* (7.5).

8. Two vaccination models with immigration and delay

In [5], a model including susceptible, vaccinated, infectious and recovered classes, with immigration of infected individuals, but no delay, was studied. The transfer diagram is:



In this model, Λ_V and Λ_R give the influxes of individuals entering the system into the vaccinated and recovered classes, μ is the per capita death rate for death that is not related to the disease, γ is the per capita disease-related death rate, α is the per capita vaccination rate, γ_1 is the per capita rate at which individuals in the vaccinated class receive permanent immunity, δ is the per capita recovery rate,

and $h(I)$ is the force of infection for vaccinated individuals. Other parameters have the same meaning as in the SI-model introduced in Section 2.

The corresponding differential equations are:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda_S - Sf(I) - (\mu+\alpha)S \\
\frac{dV}{dt} &= \Lambda_V + \alpha S - Vh(I) - (\mu+\gamma_1)V \\
\frac{dI}{dt} &= \Lambda_I + Sf(I) + Vh(I) - (\mu+\gamma+\delta)I \\
\frac{dR}{dt} &= \Lambda_R + \gamma_1 V + \delta I - \mu R,
\end{aligned} \tag{8.1}$$

with $\Lambda_S, \Lambda_V, \Lambda_I, \Lambda_R, \mu > 0$ and $\alpha, \delta, \gamma, \gamma_1 > 0$. Also, the functions f and h are assumed to satisfy the hypotheses (H1)–(H3) and to satisfy $h(I) \le f(I)$ for all $I \ge 0$.

The variable R does not appear in the first three equations and so it is sufficient to only study those three equations.

In [5], it is shown that there is a unique equilibrium $Z^* = (S^*, V^*, I^*) \in \mathbb{R}^3_{>0}$. Using the Lyapunov function

$$\mathcal{V} = S^* g\Big(\frac{S}{S^*}\Big) + V^* g\Big(\frac{V}{V^*}\Big) + I^* g\Big(\frac{I}{I^*}\Big), \tag{8.2}$$

it is shown that Z^* is globally asymptotically stable. In doing so, the authors of [5] obtain the inequality

$$\begin{aligned}
D_{(8.1)}\mathcal{V} \le &-(\mu+\alpha)S^* g\Big(\frac{S^*}{S}\Big) - \mu S^* g\Big(\frac{S}{S^*}\Big) - \Lambda_I \frac{(I-I^*)^2}{II^*} \\
&- S^* f(I^*)\Big[g\Big(\frac{S^*}{S}\Big) + g\Big(\frac{Sf(I)I^*}{S^* f(I^*)I}\Big)\Big] \\
&- (\mu+\gamma_1)V^* g\Big(\frac{V}{V^*}\Big) - \alpha S^* g\Big(\frac{SV^*}{S^*V}\Big) - V^* h(I^*) g\Big(\frac{Vh(I)I^*}{V^* h(I^*)I}\Big).
\end{aligned} \tag{8.3}$$

We now wish to consider delayed versions of (8.1). For the sake of brevity, we omit the equation for $\frac{dR}{dt}$. Consider vector-style delay in the transmission terms:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda_S - Sf(I(t-\tau)) - (\mu+\alpha)S \\
\frac{dV}{dt} &= \Lambda_V + \alpha S - Vh(I(t-\tau)) - (\mu+\gamma_1)V \\
\frac{dI}{dt} &= \Lambda_I + Sf(I(t-\tau)) + Vh(I(t-\tau)) - (\mu+\gamma+\delta)I.
\end{aligned} \tag{8.4}$$

and latency-style delay in the transmission terms:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda_S - Sf(I) - (\mu+\alpha)S \\
\frac{dV}{dt} &= \Lambda_V + \alpha S - Vh(I) - (\mu+\gamma_1)V \\
\frac{dI}{dt} &= \Lambda_I + S(t-\tau)f(I(t-\tau)) + V(t-\tau)h(I(t-\tau)) - (\mu+\gamma+\delta)I.
\end{aligned} \tag{8.5}$$

where $\tau > 0$ in each case. We have also assumed that the delay associated with susceptible individuals is the same as the delay associated with vaccinated individuals.

Recall that for (7.4) and (7.5), the unique equilibrium is $Z^* = (S^*, E^*, I^*)$, the same as for the ordinary differential equation (7.1). By making the substitutions $Y_S(t) = S(t+\tau)$ and $Y_V(t) = V(t+\tau)$, it can be shown that (8.5) is equivalent to (8.4). Thus, we will focus only on the stability of (8.5).

By using [6, Theorem 5.1] twice (or Theorem 7.1 once), the Lyapunov function $\mathcal{V}$ given in (8.2) that resolves the global stability of Z^* for (8.1) can be extended to the Lyapunov functional $\mathcal{U}$ given by

$$\begin{aligned}\mathcal{U} = \mathcal{V} &+ S^* f(I^*) \int_0^\tau g\Big(\frac{S(t-\sigma)f(I(t-\sigma))}{S^* f(I^*)}\Big)d\sigma \\ &+ V^* h(I^*) \int_0^\tau g\Big(\frac{V(t-\sigma)h(I(t-\sigma))}{V^* h(I^*)}\Big)d\sigma.\end{aligned}$$

By following the approach used in Section 7, and described in detail in [6], we obtain

$$\begin{aligned}D_{(8.5)}\mathcal{U} =& D_{(8.1)}\mathcal{V} + S^* f(I^*)\Big[g\Big(\frac{Sf(I)I^*}{S^* f(I^*)I}\Big) - g\Big(\frac{S(t-\tau)f(I(t-\tau))I^*}{S^* f(I^*)I}\Big)\Big] \\ &+ V^* f(I^*)\Big[g\Big(\frac{Vh(I)I^*}{V^* h(I^*)I}\Big) - g\Big(\frac{V(t-\tau)h(I(t-\tau))I^*}{V^* h(I^*)I}\Big)\Big] \\ \le& -(\mu+\alpha)S^* g\Big(\frac{S^*}{S}\Big) - \mu S^* g\Big(\frac{S}{S^*}\Big) - \Lambda_I \frac{(I-I^*)^2}{II^*} \\ &- S^* f(I^*)\Big[g\Big(\frac{S^*}{S}\Big) + g\Big(\frac{S(t-\tau)f(I(t-\tau))I^*}{S^* f(I^*)I}\Big)\Big] \\ &- (\mu+\gamma_1)V^* g\Big(\frac{V}{V^*}\Big) - \alpha S^* g\Big(\frac{SV^*}{S^* V}\Big) \\ &- V^* h(I^*) g\Big(\frac{V(t-\tau)h(I(t-\tau))I^*}{V^* h(I^*)I}\Big) \le 0\end{aligned}$$

with equality if and only if $(S, V, I) = (S^*, V^*, I^*)$. By Lyapunov's Direct Method the equilibrium is globally asymptotically stable under the flow described by (8.5). Due to the equivalence of (8.4) and (8.5), it follows that the equilibrium Z^* is also globally asymptotically stable under the flow described by (8.4).

Theorem 8.1. *The equilibrium Z^* is globally asymptotically stable under the flow described by* (8.4) *and also under the flow described by* (8.5).

9. Discussion

We have studied SI, SEI and SVI models of disease spread that include immigration of infected individuals and each of

- delay due to vector transmission
- delay due to a period of latency.

Due to the immigration of infecteds, there is no disease-free equilibrium for any of these models. If one of the systems were somehow in a disease-free state, then infected individuals would enter the population through immigration and so the system would no longer be in a disease-free state. Significantly, since there is no disease-free equilibrium, there is no basic reproduction number.

For the SI model with vector-style delay, we provide a detailed analysis, showing that for all parameter values there is a unique positive equilibrium and that it is both locally and globally asymptotically stable. Thus, the level of disease in the population will asymptotically approach the equilibrium level. We then show that this vector-style model is equivalent to an SI model with latency-style delay so that our results also apply to this new model.

Previous works [5, 10] have studied SEI and SVI models with immigration of infected individuals, but without delay. Those works used Lyapunov functions to show that the unique positive equilibrium was globally asymptotically stable. We show that these Lyapunov functions can be extended to Lyapunov functionals, showing that the equilibrium is still globally asymptotically stable if the system is modified to include either vector-style or latency-style delay.

It is apparent that in order to eliminate disease in a connected world, it is necessary either to screen travelers perfectly for infection (so that there is no immigration of infecteds) or to treat disease elimination as a global problem. Due to the high level of interconnectedness of today's world, the global approach to elimination seems more likely to be successful.

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