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TWO-POINT BOUNDARY PROBLEM FOR MODELING THE JET FLOW OF THE ANTARCTIC CIRCUMPOLAR CURRENT

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Abstract. Using a functional-analytic approach for two-point boundary value problems, for a large class of oceanic vorticities, we establish the existence of solutions to a model for the jet flows of the Antarctic circumpolar current with no azimuthal variations. In our approach we rely on the stereographic projection to pass from spherical to planar coordinates.

1. Introduction

This article studies the flow of the Antarctic circumpolar current (ACC), one of the strongest and largest currents in the oceans. Because the scales that are relevant, we regard the ACC as a gyre flow – a large ocean flow driven by the prevailing wind pattern and the forces created by Earth's rotation, whose center is located on the land mass of Antarctica. The ACC encircles the Southern ice-covered continent, being with an overall length of about 24000 km the longest oceanic current, flowing clockwise from west to east around Antarctica between latitudes 45°S and 55°S, where there are no land masses to interfere with this continuous stretch of water. Relatively slow, the ACC extends from the sea surface to depths of 2000-4000 m reaching, unlike other major currents, from the surface to the bottom of the ocean. Its width exceeds at places 2000 km and overall the ACC has a very large volume transport (up to 150 times the volume of water flowing in all of the world's rivers), isolating Antarctica with a ring of cold water and being to a large extent responsible for Antarctic permanent glaciation. The ACC plays an important role in the global climate, being the major means of exchange of water between the three great ocean basins (Atlantic, Indian and Pacific). The ACC is composed of a number of high-speed coherent but narrow jets (about 40–50km wide, with typical speeds exceeding 1 m/s), separated by zones of low-speed flow (with speeds less than 20 cm/s), and remains one of the most poorly represented components of global climate models (see the discussion in [6]). Many observations of the ACC flow were gathered but the quest for realistic models is ongoing. With regard to the large-scale dynamics of the ACC, the presence of surface waves is not of relevance, even though the study of wave-current interactions in the Southern Ocean is an active area of research (see [10]), especially since large waves (with

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heights of 35 m) are frequently encountered (see the data in [29]). We point out that the Arctic and Antarctic regions have a quite different geography: the Arctic is a semi-enclosed ocean, almost completely surrounded by land, while the Antarctic region is almost a geographic opposite of the Arctic, being a land mass surrounded by an ocean. We refer to [4] for a discussion of arctic gyres.

In this article we model the jets of the ACC using a recent model for gyres [9], and, along the lines of the considerations pursued in [25, 26], considering the setting of flows that are uniform in the azimuthal direction. This physically relevant assumption has the consequence that the elliptic partial differential equation that governs the large-scale motion in [9] in spherical coordinates reduces to a second-order ordinary differential equation after suitable transformations which involve the stereographic projection. T his leads us to two-point boundary-value problem with Dirichlet boundary conditions. We investigate the existence of solutions for a larger class of oceanic vorticities than those studied in [25, 26].

2. Preliminaries

In this section we briefly describe the main features of gyre flows and we also explain how these considerations apply to the specific case of the ACC.

A gyre flow extends over very large ocean areas (measured in thousands of km^2) and has negligible vertical speeds, with the ratio of vertical speed to either of the horizontal speed components typically about 10^{-3}, so that we may realistically regard ocean gyres as shallow water flows on a rotating sphere [9]. Consider spherical coordinates, with $\theta \in [0, \pi)$ the polar angle (with $\theta = 0$ corresponding to the North Pole) and $\varphi \in [0, 2\pi)$ the angle of longitude (or azimuthal angle), see Figure 1. We recall that the Earth is rotating eastwards around the polar axis, turning counterclockwise if viewed from the North Pole star Polaris, with angular speed of about 7.29×10^{-5} radians per second (so that the Earth rotates once in about 24 hours); the radius of the practically spherical Earth being about 6378 km.

We denote by (u', v', w') the velocity field in physical variables. If $(e_r, e_\theta, e_\varphi)$ are the unit vectors associated with a fixed point P on the rotating sphere, where e_r points upwards, e_φ points from west to east, and e_θ from north to south, then the Euler equation and the equation for the mass conservation are

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t'} + u'\frac{\partial}{\partial r'} + \frac{v'}{r'}\frac{\partial}{\partial \theta'} + \frac{w'}{r'\sin\theta}\frac{\partial}{\partial}\Big)(u', v', w') \\
&+ \frac{1}{r'}\big(-v'^2 - w'^2, u'v' - w'^2\cot\theta, u'w' + v'w'\cot\theta\big) \\
&+ 2\Omega'(-w'\sin\theta, -w'\cos\theta, u'\sin\theta + v'\cos\theta) - r'\Omega'^2(\sin^2\theta, \sin\theta\cos\theta, 0) \\
&= -\frac{1}{\rho'}\Big(\frac{\partial p'}{\partial r'}\frac{1}{r'}\frac{\partial p'}{\partial \theta}, \frac{1}{r'\sin\theta}\frac{\partial p'}{\partial \varphi}\Big) + (F'_{r'}, F'_\theta, F'_\varphi)
\end{aligned}
\tag{2.1}
$$

and

$$
\frac{1}{r'^2}\frac{\partial}{\partial r'}(r'^2 u') + \frac{1}{r'\sin\theta}\frac{\partial}{\partial \theta}(v'\sin\theta) + \frac{1}{r'\sin\theta}\frac{\partial w'}{\partial \varphi} = 0, \tag{2.2}
$$

respectively, where $p'(r', \theta, \varphi)$ is the pressure in the fluid, ρ' is the (constant) density and $(F'_{r'}, F'_\theta, F'_\varphi) = (-g', 0, 0)$ is the body–force vector, while $g' \approx 9.81 ms^{-1}$ is the (constant) gravitational acceleration of the Earth and $\Omega' \approx 7.29 \times 10^{-5}$ rad s^{-1} is the constant rate of rotation of the Earth around the polar axis.

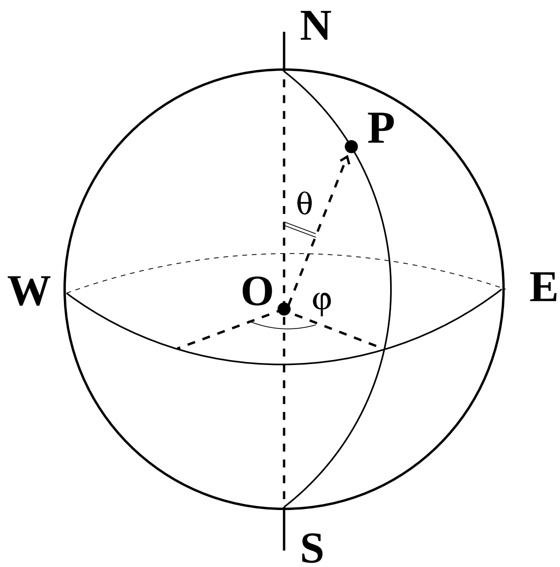

FIGURE 1. Azimuthal and polar spherical coordinates φ and θ of a point P on the spherical surface of the Earth: $\theta = 0$ and $\theta = \pi$ correspond to the North and South Pole, respectively, while $\theta = \pi/2$ corresponds to the Equator.

By defining a suitable length scale H' as the average depth of the ocean (with $H' \approx 4$km for the Southern Ocean) and the speed scale $c' = \sqrt{g'H'}$, the original physical variables can be transformed as

$$z' = H'z, (u', v', w') = c'(ku, v, w), p' = \rho' c'^2 p,$$

where k is the scaling factor, associated with the vertical component of the velocity. Typically, the ratio of vertical to horizontal speed is less than 10^{-4} (see [9]).

On setting $\varepsilon = H'/R'$, where $R' \approx 6378$ km is the radius of the Earth, the equations (2.1), (2.2) for a steady flow become

$$\begin{aligned} &\Big(\frac{k}{\varepsilon}u\frac{\partial}{\partial z} + \frac{v}{1+\varepsilon z}\frac{\partial}{\partial \theta} + \frac{w}{(1+\varepsilon z)\sin\theta}\frac{\partial}{\partial \varphi}\Big)(ku, v, w) \\ &+ \frac{1}{1+\varepsilon z}(-v^2 - w^2, kuv - w^2\cot\theta, kuw + vw\cot\theta) \\ &+ 2\frac{\Omega' R'}{c'}(-w\sin\theta, -w\cos\theta, ku\sin\theta\cos\theta, 0) \\ &- (1+\varepsilon z)\Big(\frac{\Omega' R'}{c'}\Big)^2(\sin^2\theta, \sin\theta\cos\theta, 0) \\ &= -\Big(\frac{1}{\varepsilon}\frac{\partial p}{\partial z}, \frac{1}{1+\varepsilon z}\frac{\partial p}{\partial \theta}, \frac{1}{(1+\varepsilon z)\sin\theta}\frac{\partial p}{\partial \varphi}\Big) + \frac{R'}{c'^2}(-g', 0, 0), \end{aligned} \tag{2.3}$$

$$\frac{k}{\varepsilon(1+\varepsilon z)^2}\frac{\partial}{\partial z}\{(1+\varepsilon z)^2 u\} + \frac{1}{(1+\varepsilon z)\sin\theta}\Big\{\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{\partial w}{\partial \varphi}\Big\} = 0. \tag{2.4}$$

The scaling factor k is taken equal to ε^2 (see the discussion in [9]). Define

$$P = p + \frac{H'g'}{c'^2}z,$$

to obtain the following form of the governing equations:

$$\begin{aligned} &\Big(\varepsilon u\frac{\partial}{\partial z}+\frac{v}{1+\varepsilon z}\frac{\partial}{\partial\theta}+\frac{w}{(1+\varepsilon z)\sin\theta}\frac{\partial}{\partial\varphi}\Big)(\varepsilon^3 u,v,w)\\ &+\frac{1}{1+\varepsilon z}(-\varepsilon v^2-\varepsilon w^2,\varepsilon^2 uv-w^2\cot\theta,\varepsilon^2 uw+vw\cot\theta)\\ &+2\omega(-\varepsilon w\sin\theta,-w\cos\theta,\varepsilon^2 u\sin\theta+v\cos\theta)\\ &-(1+\varepsilon z)\omega^2(\varepsilon\sin^2\theta,\sin\theta\cos\theta,0)\\ &=-\Big(\frac{\partial P}{\partial z},\frac{1}{1+\varepsilon z}\frac{\partial P}{\partial\theta},\frac{1}{(1+\varepsilon z)\sin\theta}\frac{\partial P}{\partial\varphi}\Big), \end{aligned} \tag{2.5}$$

and

$$\frac{\varepsilon}{(1+\varepsilon z)^2}\frac{\partial}{\partial z}\{(1+\varepsilon z)^2 u\}+\frac{1}{(1+\varepsilon z)\sin\theta}\Big\{\frac{\partial}{\partial\theta}(v\sin\theta)+\frac{\partial w}{\partial\varphi}\Big\}=0, \tag{2.6}$$

where $\omega=\frac{\Omega' R'}{c'}$ is the non-dimensional form of the Coriolis parameter.

The leading-order problem in ε is obtained in the limit $\varepsilon\to 0$, being therefore given by

$$\frac{\partial\Pi}{\partial z}=0, \tag{2.7}$$

$$\Big(v\frac{\partial}{\partial\theta}+\frac{w}{\sin\theta}\frac{\partial}{\partial\varphi}\Big)v-w^2\cot\theta-2\omega w\cos\theta=-\frac{\partial\Pi}{\partial\theta}, \tag{2.8}$$

$$\Big(v\frac{\partial}{\partial\theta}+\frac{w}{\sin\theta}\frac{\partial}{\partial\varphi}\Big)w+vw\cot\theta+2\omega v\cos\theta=-\frac{1}{\sin\theta}\frac{\partial\Pi}{\partial\varphi}, \tag{2.9}$$

$$\frac{\partial}{\partial\theta}(v\sin\theta)+\frac{\partial w}{\partial\varphi}=0, \tag{2.10}$$

where

$$\Pi=P+\frac{1}{4}\omega^2\cos 2\theta.$$

Using (2.10) one can introduce the stream function ψ by

$$v=\frac{1}{\sin\theta}\psi_\varphi,\quad w=-\psi_\theta. \tag{2.11}$$

The compatibility confirm generated by the elimination of Π from (2.7)–(2.9) yields the vorticity equation

$$\begin{aligned} &\psi_\varphi\Big(\frac{1}{\sin^2\theta}\psi_{\varphi\varphi}+\psi_\varphi\cot\theta+\psi_{\theta\theta}-2\omega\cos\theta\Big)_\theta\\ &-\psi_\theta\Big(\frac{1}{\sin^2\theta}\psi_{\varphi\varphi}+\psi_\varphi\cot\theta+\psi_{\theta\theta}-2\omega\cos\theta\Big)_\varphi=0. \end{aligned} \tag{2.12}$$

Here the vorticity in the flow, at leading order, expressed in spherical coordinates, is given by the expression

$$\frac{1}{\sin^2\theta}\psi_{\varphi\varphi}+\psi_\varphi\cot\theta+\psi_{\theta\theta}.$$

Defining

$$\Psi(\theta,\varphi)=\omega\cos\theta+\psi(\theta,\varphi),$$

as the vorticity of the underlying motion of the ocean (relative to the Earth's surface and not driven by the rotation of the Earth), equation (2.12) then becomes

$$\begin{aligned} &(\Psi-\omega\cos\theta)_\varphi\Big(\frac{1}{\sin^2\theta}\psi_{\varphi\varphi}+\psi_\varphi\cot\theta+\psi_{\theta\theta}\Big)_\theta\\ &-(\Psi-\omega\cos\theta)_\theta\Big(\frac{1}{\sin^2\theta}\psi_{\varphi\varphi}+\psi_\varphi\cot\theta+\psi_{\theta\theta}\Big)_\varphi=0. \end{aligned} \tag{2.13}$$

In regions where $\nabla(\Psi-\omega\cos\theta)\neq 0$, by the rank theorem (see [1]) the solution of (2.13) can be expressed in the form

$$\frac{1}{\sin^2\theta}\,\Psi_{\varphi\varphi}+\Psi_\theta\cot\theta+\Psi_{\theta\theta}=F(\Psi-\omega\,\cos\theta)\,, \tag{2.14}$$

where $F(\Psi-\omega\,\cos\theta)$ is the oceanic vorticity, which is typically one order of magnitude larger than the planetary vorticity $2\omega\cos\theta$, generated by the Earth's rotation (see the data in [9]). The (total) vorticity of a the gyre flow is the sum of the oceanic vorticity, $F(\Psi-\omega\,\cos\theta)$, and of the planetary vorticity $2\omega\,\cos\theta$. The planetary vorticity is prescribed by the characteristics of the Earth's rotation but the oceanic vorticity can change from location to location, being dependent on specific features (for example, the prevailing wind pattern, which induces near-surface currents) of the type of ocean flow that is under consideration. The main sources of oceanic vorticity are wind force [23] and the gravitational forces due to the relative motions of the Moon, the Sun and the Earth in the form of the tidal currents – the horizontal unidirectional movements of water associated with the rise and fall of the tide. These two major types of oceanic vorticities can be regarded as non-zero constants (see the discussions in [11], [15]), with the sign (positive or negative) depending on the prevalent wind direction, and, respectively, on whether the tidal flow mode is of ebb or flood type. Furthermore, non-constant oceanic vorticities are often encountered in gyre flows. Gyres exist at all latitudes, except near the Equator (see discussions in [8, 16]).

Let us briefly explain why the ocean flow near the Equator is quite different from other latitudes. In equation (2.1), the contributions from the rotation of the Earth are

$$2\Omega'(-w'\sin\theta,-w'\cos\theta,u'\sin\theta+v'\cos\theta)-r'\Omega'^2(\sin^2\theta,\sin\theta\cos\theta,0)$$

and at the Equator $\theta=\frac{\pi}{2}$ there become

$$\Omega'(-w',0,u')-r'\Omega'^2(1,0,0).$$

We observe a vanishing of the meridional component of the Coriolis terms, which has the physical effect that the Equator works as a natural boundary, guiding the flow propagation towards the east-west direction (see [16]). Furthermore, there is a pronounced stratification in equatorial ocean regions, greater than anywhere else in the ocean (see the discussion in [9]): this manifests itself by the presence of a sharp thermocline which separates the near-surface layer from the deeper layer, both being accurately described as having constant density with a difference in density across the thermocline of about 1% (the deeper layer being denser, so that we have stably stratified setting). Furthermore, the water masses of the Equatorial undercurrents in the Pacific Ocean and in the Atlantic Ocean move from westwards in the upper layers to eastwards in the lower ones, while at the depth of about 240m we observe a motionless still water. The situation in the Indian Ocean is even more complicated, with flow direction reversal due to the monsoon seasons (see the discussion in [7]). For these reasons, in dealing with ocean flows, at the Equator, one has to account for strong currents which are depth–dependent, and this places such type of considerations outside the scope of the study [9] and of the present considerations. Though the study of wave-current interactions in flows with vorticity is a topic of great current interest (see the discussions in [5], [12]–[14], [18]–[20], [28]), at the large scales that are relevant for the ACC these are secondary effects that can be ignored.

To avoid the complications associated with the use of spherical coordinates we rely on the stereographic projection of the unit sphere centred at origin from the North Pole to the equatorial plane (see Figure 2). The model (2.14) in spherical coordinates is thus transformed into an equivalent planar elliptic partial differential equation [9]: in our coordinates the stereographic projection is defined by

$$\xi = r\, e^{i\,\phi} \quad \text{with} \quad r = \cot\Big(\frac{\theta}{2}\Big) = \frac{\sin\theta}{1-\cos\theta}\,, \tag{2.15}$$

where (r, ϕ) are the polar coordinates in the equatorial plane, and it transforms (2.14) into

$$\psi_{\xi\bar{\xi}} + 2\omega\,\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} - \frac{F(\psi)}{(1+\xi\bar{\xi})^2} = 0\,.$$

The above equation is equivalent, using the Cartesian coordinates (x, y) in the complex ξ-plane, to the following semilinear elliptic partial differential equation

$$\Delta\psi + 8\omega\,\frac{1-(x^2+y^2)}{(1+x^2+y^2)^3} - \frac{4F(\psi)}{(1+x^2+y^2)^2} = 0\,, \tag{2.16}$$

where $\Delta = \partial_x^2 + \partial_y^2$ denotes the Laplace operator.

The ACC presents a considerable uniformity in the azimuthal direction and this feature is helpful to simplify the problem further. Indeed, gyres with no variation in the azimuthal direction correspond to radially symmetric solutions $\psi = \psi(r)$ of problem (2.16). In this setting, performing the change of variables

$$\psi(r) = U(s)\,, \quad s_1 < s < s_2\,, \tag{2.17}$$

with

$$r = e^{-s/2} \quad \text{for } 0 < s_1 = -2\ln(r_+) < s_2 = -2\ln(r_-)\,, \tag{2.18}$$

for $0 < r_- < r_+ < 1$, transforms the semilinear elliptic partial differential equation (2.16) to the second-order ordinary differential equation

$$U''(s) - \frac{e^s}{(1+e^s)^2}\,F(U(s)) + \frac{2\omega e^s(1-e^s)}{(1+e^s)^3} = 0\,, \quad s_1 < s < s_2\,. \tag{2.19}$$

The flow in a jet component of the ACC, between the parallels of latitude defined by an appropriate choice of $r_\pm \in (0,1)$ with $r_+/r_- \in (1,2)$, is modelled by coupling (2.19) with the boundary conditions

$$U(s_1) = U(s_2) = 0\,. \tag{2.20}$$

expressing the fact that the boundary of the jet is a streamline – since the flow is steady, this means that a particle there will be confined to the boundary at all times. We therefore propose (2.19)-(2.20) as a model for a jet component of the ACC. In this formulations the choice of the oceanic vorticity F entails different properties of the solution U, which determines the entire flow pattern.

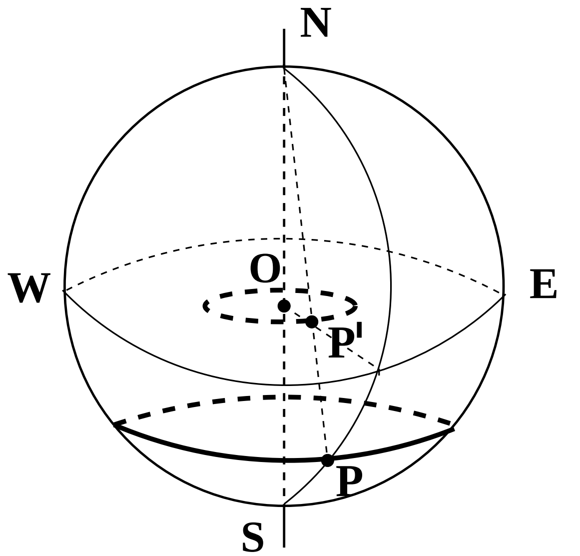

FIGURE 2. The stereographic projection $P \mapsto P'$ from the North Pole to the equatorial plane: for any point P in the Southern Hemisphere, the straight line connecting it to the North Pole intersects the equatorial plane in a point P' belonging to the interior of the circular region delimited by the Equator. The depicted thick band, delimited by two parallels of latitude, represents one of the jets of the Antarctic Circumpolar Current and is mapped bijectively into an annular planar region concentric with the Equator.

3. Main results

Given $0 < s_1 < s_2$, the change of variables

$$u(t) = U(s) \quad \text{with} \quad t = \frac{s - s_1}{s_2 - s_1}\,, \tag{3.1}$$

transforms the second-order differential equation (2.19) with the boundary conditions (2.20) to the equivalent two-point boundary-value problem

$$u'' = a(t)F(u) + b(t)\,, \quad 0 < t < 1\,, \tag{3.2}$$

$$u(0) = u(1) = 0\,, \tag{3.3}$$

where

$$a(t) = \frac{(s_2 - s_1)^2\, e^{(s_2-s_1)t+s_1}}{(1 + e^{(s_2-s_1)t+s_1})^2} > 0\,,$$

$$b(t) = -\frac{2\omega(s_2 - s_1)^2 e^{(s_2-s_1)t+s_1}(1 - e^{(s_2-s_1)t+s_1})}{(1 + e^{(s_2-s_1)t+s_1})^3}\,,$$

for $t \in [0,1]$. Boundary-value problem (3.2)-(3.3) is called *non-resonant* if it has a solution for every continuous forcing b, while *resonance* refers to the fact that it is solvable only for suitable continuous functions b. However, in our setting the functions a and b are fixed and of interest are various choices for the nonlinearity F. Explicit solutions for F constant and for $F(u) = -2u$ were provided in [25], while the existence of solutions for a special class of nonlinear functions F was proved in [26].

Considering a function $F : \mathbb{R} \to \mathbb{R}$ having the decomposition

$$F(u) = -\lambda u + f(u)\,, \tag{3.4}$$

for a suitable parameter λ and some function $f : \mathbb{R} \to \mathbb{R}$, the linear problem associated with (3.2)-(3.3) is

$$u'' = -\lambda a(t)u + b(t)\,, \quad 0 < t < 1\,, \tag{3.5}$$

$$u(0) = u(1) = 0\,. \tag{3.6}$$

Since $a(t) > 0$ on $(0, 1)$ has a second derivative that admits a continuous extension to $[0, 1]$, the corresponding homogeneous linear problem,

$$u'' + \lambda a(t)u = 0\,, \quad 0 < t < 1\,, \tag{3.7}$$

$$u(0) = u(1) = 0\,, \tag{3.8}$$

can be transformed into the problem

$$w'' + [\lambda A^2 + \mathfrak{A}(T)]\, w = 0\,, \quad 0 < T < 1\,, \tag{3.9}$$

$$w(0) = w(1) = 0\,, \tag{3.10}$$

by the Liouville transformation (see [24, Chapter III])

$$T = \frac{1}{A}\int_0^t \sqrt{a(\tau)}\, d\tau\,, \quad A = \int_0^1 \sqrt{a(\tau)}\, d\tau\,, \tag{3.11}$$

$$w(T) = \sqrt[4]{a(t)}\, u(t)\,, \quad \mathfrak{A}(T) = \frac{1}{\sqrt[4]{a(t)}}\frac{d^2 \sqrt[4]{a(t)}}{dT^2}\,, \tag{3.12}$$

whose inverse is given by

$$t = A\int_0^T \frac{1}{[a^*(\xi)]^2}\, d\xi\,, \quad \frac{1}{A} = \int_0^1 \frac{1}{[a^*(\xi)]^2}\, d\xi\,, \tag{3.13}$$

where $a^*(T) = \sqrt[4]{a(t)}$ is, for $T \in (0, 1)$, a positive solution of the differential equation

$$\frac{da^*}{dT^2} = \mathfrak{A}(T)a^*(T)\,.$$

On the other hand, the nonlinear boundary-value problem (3.2)-(3.3) is transformed by means of (3.11)-(3.12) into

$$w'' + [\lambda A^2 + \mathfrak{A}(T)]\, w = \mathfrak{F}(w, T) + B(T)\,, \quad 0 < T < 1\,, \tag{3.14}$$

$$w(0) = w(1) = 0\,, \tag{3.15}$$

where

$$\begin{aligned} \mathfrak{F}(w, T) &= \frac{A^2}{[a^*(T)]^3}\, f\Big(\frac{w(T)}{a^*(T)}\Big)\,, \\ B(T) &= \frac{A^2}{[a^*(T)]^3}\, b(t)\,, \end{aligned} \tag{3.16}$$

for $T \in [0, 1]$ and $w \in \mathbb{R}$. Setting $\mathfrak{F} \equiv 0$ in (3.14) yields the transformation of the inhomogeneous boundary-value problem (3.5)-(3.6) by means of (3.11)-(3.12) into

$$w'' + [\lambda A^2 + \mathfrak{A}(T)]w = B(T)\,, \quad 0 < T < 1\,, \tag{3.17}$$

$$w(0) = w(1) = 0\,. \tag{3.18}$$

3.1. **Linear models.** Multiplying (3.7) by $u(t)$ and integrating the outcome on $[0,1]$ yields

$$\lambda \int_0^1 a(t)u^2(t)\,dt = \int_0^1 [u'(t)]^2\,dt\,,$$

which shows that all eigenvalues λ of the weighted Sturm-Liouville problem (3.7)-(3.8) are strictly positive (since $u' \equiv 0$ forces $u \equiv 0$). On the other hand, the Liouville transformation ensures that all these Dirichlet eigenvalues are all simple (that is, the corresponding eigenspace is one-dimensional), countable in number and accumulating at $+\infty$ (see [24]); we denote them by $\{\lambda_n\}_{n\geq 1}$, in increasing order. Moreover, given $B \in L^2[0,1]$, let $\mathbb{S}(B) \in H^2(0,1)$ be the unique solution of $u'' = B$ in $(0,1)$, with $u(0) = u(1) = 0$; we have that

$$(\mathbb{S}(B))(T) = \int_0^T (T - T')B(T')\,dT'\,, \quad T \in [0,1]\,.$$

In general, functions in the Sobolev space $H^2(0,1)$ are continuously differentiable on $[0,1]$ and thus, in particular, they have a trace on the boundary. The problem (3.14)-(3.15) with $\mathcal{F} \equiv 0$ is equivalent to finding a solution $w \in H^1(0,1)$ of the functional equation $w = \mathbb{S}(B - [\lambda A^2 + \mathcal{A}]w)$. Since the operator $\mathbb{T} : H^1(0,1) \to H^1(0,1)$ defined by $\mathbb{T}w = -\mathbb{S}([\lambda A^2 + \mathcal{A}]w)$ is compact, the Fredholm alternative (see [2], Chapter 8) yields that problem (3.17)-(3.18) has a solution for every $B \in L^2[0,1]$ if and only if the only solution of (3.9)-(3.10) is $w \equiv 0$, that is, if and only if λ is not a Dirichlet eigenvalue of (3.7)-(3.8); this solution being unique. On the other hand, if $\lambda > 0$ is a Dirichlet eigenvalue with corresponding eigenfunction w_0, then problem (3.17)-(3.18) has a solution if and only if

$$\int_0^1 B(T)w_0(T)\,dT = 0\,, \tag{3.19}$$

relation that, in view of (3.16) and (3.11)-(3.12), we can recast as

$$\int_0^1 b(t)u_0(t)\,dt = 0\,, \tag{3.20}$$

in terms of the corresponding eigenfunction $u_0(t)$ of (3.7)-(3.8). If the orthogonality condition (3.19) is satisfied, the solution to (3.17)-(3.18) is not unique, as any two solutions differ by a solution of (3.9)-(3.10); equivalently, if the orthogonality condition (3.20) is satisfied, then the solution to (3.5)-(3.6) is not unique since any two solutions differ by a solution of (3.7)-(3.8).

Remark 3.1. The considerations in [25] show, by finding an explicit set of fundamental solutions, that $\lambda = 0$ and $\lambda = 2$ are not eigenvalues for (3.7)-(3.8). Note that we can deal with the case of constant F by merely taking $F \equiv 0$ (which corresponds to $F(u) = -\lambda u$ with $\lambda = 0$) and adding a suitable multiple of the function a to the forcing term n. □

Let us now prove the following result.

Theorem 3.2. *For any linear oceanic vorticity of the type $F(u) = -\lambda u$ with $\lambda \leq 2$, there exists a uniquely determined stream function that arises as a solution to the problem* (3.2)-(3.3).

Proof. Using the variational characterization of the first smallest eigenvalue for (3.7)-(3.8), along the lines of the considerations made in [5, Section 3.1] for a similar type of problem, we find that

$$\lambda_1 = \inf_{u \in H_0^1(0,1): u \not\equiv 0} \Big\{ \frac{\int_0^1 [u'(t)]^2 \, dt}{\int_0^1 a(t) u^2(t) \, dt} \Big\} , \tag{3.21}$$

where $H_0^1(0,1)$ is the Hilbert space $\{u \in H^1(0,1) : \ u(0) = u(1) = 0\}$. Since $s_2 - s_1 = \ln(r_+/r_-) < 1$, we have that

$$2a(t) = (s_2 - s_1)^2 \frac{2\, e^{(s_2-s_1)t+s_1}}{(1 + e^{(s_2-s_1)t+s_1})^2} \le (s_2 - s_1)^2 < 1 , \quad t \in [0,1] .$$

On the other hand, if $t_0 \in (0,1)$ is the point in $[0,1]$ where the maximum of $t \mapsto u^2(t)$ is attained for $u \in H_0^1(0,1)$, $u \not\equiv 0$, then

$$u^2(t_0) = \Big(\int_0^{t_0} u'(t) \, dt \Big)^2 \le \Big(\int_0^1 |u'(t)| \, dt \Big)^2 \le \int_0^1 [u'(t)]^2 \, dt .$$

Consequently

$$\frac{2}{(s_2 - s_1)^2} \int_0^1 a(t) u^2(t) \, dt < \int_0^1 u^2(t) \, dt \le u^2(t_0) \le \int_0^1 [u'(t)]^2 \, dt ,$$

which yields $\lambda_1 \ge \frac{2}{(s_2-s_1)^2} > 2$. This prevents resonance for any linear function $F(u) = -\lambda u$ with $\lambda \le 2$, in view of the considerations that precede the statement, and the proof is complete. ☐

3.2. **Nonlinear models.** For a small nonlinear perturbation of a non-resonant linear state of the form $u \mapsto -\lambda u$ (that is, with λ not an eigenvalue), the existence of solutions of (3.2)-(3.3) is established by our next result.

Theorem 3.3. *Assume that F is of the form* (3.4)*, with $\lambda \in \mathbb{R}$ not an eigenvalue of the Dirichlet problem* (3.7)-(3.8)*, and with the continuous function $f : \mathbb{R} \to \mathbb{R}$ uniformly bounded. Then there exists a solution to* (3.2)-(3.3)*.*

Proof. Let $C[0,1]$ be the Banach space of all continuous functions $u : [0,1] \to \mathbb{R}$, endowed with the norm $\|u\| = \sup_{t\in[0,1]}\{|u(t)|\}$ and let $C_0^2[0,1]$ be the Banach space of all twice continuously differentiable functions $u : [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$, endowed with the norm obtained by taking the maximum over $[0,1]$ of the absolute values of the derivatives of u of order $k \le 2$ (that is, $\max\{\|u\|, \|u'\|, \|u''\|\}$). The assumption that λ is not a Dirichlet eigenvalue ensures (along the lines of the arguments presented in [3]) that the linear operator $L : C_0^2[0,1] \to C[0,1]$ defined by $Lu = u'' + \lambda a u$ is ivertible. Its inverse $L^{-1} : C[0,1] \to C_0^2[0,1]$, expressible by means of a Green's function, takes bounded subsets of $C[0,1]$ to bounded subsets of $C_0^2[0,1]$. Note that a solution of (3.2)-(3.3) is a fixed point of the operator $L^{-1}(a + f(u))$ in $C[0,1]$. Since f is uniformly bounded, we can find a closed ball in $C[0,1]$, centered at the origin, that is mapped into itself by the compact operator $u \mapsto L^{-1}(a + f(u))$; the compactness being a consequence of the Arzelà-Ascoli theorem (see [3]). The existence of a fixed point $u \in C[0,1]$ follows now from Schauder's theorem (see [3]), and a glance at the range of L^{-1} confirms that actually $u \in C_0^2[0,1]$. ☐

Remark 3.4. The requirements of Theorem 3.3 are only sufficient. Indeed, in [26] we showed that in the case $\lambda = 0$, solutions to (3.2)-(3.3) exist for continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which we can find constants m_0, $M_0 > 0$ with $uf(u) + m_0|u| \geq 0$ for $|u| \geq M_0$, and this setting comprises the example $f(u) = u$ which does not enter into the framework of Theorem 3.3. On the other hand, for $F(u) = -\lambda u$ with $\lambda > 0$ different from the discrete set $\{\lambda_n\}_{n\geq 1}$ of the Dirichlet eigenvalues of (3.7)-(3.8), the existence of a solution to (3.2)-(3.3) follows either from Theorem 3.2 or from Theorem 3.3 while the result in [26] is not applicable.

Remark 3.5. If F is of the form (3.4), with $\lambda \in \mathbb{R}$ an eigenvalue of the Dirichlet problem (3.7)-(3.8), the discussion preceding Theorem 3.2 shows that linear resonance will occur if $f \equiv 0$, in which case (3.20) is the necessary and sufficient condition for the existence of solutions to (3.2)-(3.3). The issue of the existence of nonlinear perturbations $f \not\equiv 0$ which ensure the solvability of (3.2)-(3.3) has been addressed in the research literature (see the discussion in [22]) but the results that we are aware of are of limited practical interest since they involve as a crucial constraint the validity of a constraint of the type (3.20) with an eigenfunction $u_0(t)$ that is not available in explicit form. □

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