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BLOW-UP OF SOLUTIONS TO SINGULAR PARABOLIC EQUATIONS WITH NONLINEAR SOURCES

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Abstract. We prove the existence of a local weak solutions for semi-linear parabolic equations with a strong singular absorption and a general source. Also, we investigate criteria for the solutions to blow up in finite time.

1. Introduction

In this article, we are interested in nonnegative solutions of the equation

$$\begin{gathered} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} = f(u,x,t) \quad \text{in } \Omega\times(0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\beta\in(0,1)$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t)>0$, i.e:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u>0, \\ 0, & \text{if } u\le 0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u=0$. Through this paper, $f : [0,\infty)\times\overline{\Omega}\times[0,\infty)\to\mathbb{R}$ will be assumed a nonnegative function satisfying the hypothesis

- (H1) $f\in\mathcal{C}^1\big([0,\infty)\times\overline{\Omega}\times[0,\infty)\big)$, $f(0,x,t)=0$, for all $(x,t)\in\Omega\times(0,\infty)$, and $f(u,x,t)\le h(u)$ for all $(x,t)\in\Omega\times(0,\infty)$, where h is a locally Lipschitz function on $[0,\infty)$, and $h(0)=0$.

In the sequel, we always consider nonnegative initial data $u_0\ne 0$.

Problem (1.1) can be considered as a limit of mathematical models describing enzymatic kinetics (see [1]), or the Langmuir-Hinshelwood model of the heterogeneous chemical catalyst (see, e.g. [22, p. 68] and [8, 20]). This problem has been studied by the authors in [4, 7, 14, 15, 18, 20, 24], and references therein. These authors have considered the existence and uniqueness, and the qualitative behavior of these solutions. For example, when $f=0$, Phillips [20] proved the existence of

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solution for the Cauchy problem associating to equation (1.1). A partial uniqueness of solution of equation (1.1) was proved by Davila and Montenegro [7], for a class of solutions with initial data $u_0(x) \geq C\,\mathrm{dist}(x, \partial\Omega)^{\mu}$, for $\mu \in (1, \frac{2}{1+\beta})$ (see also [6] the uniqueness in a different class of solutions). A beautiful result established by Winkler, [24], showed that the uniqueness of solution fails in general. One of the interesting behaviors of solutions of (1.1) is the extinction that any solution vanishes after a finite time even beginning with a positive initial data, see [20, 14] (see also [4] for a quasilinear equation of this type). It is known that this phenomenon occurs according to the presence of the nonlinear singular absorption $u^{-\beta}\chi_{\{u>0\}}$.

Equation (1.1) with source term $f(u)$ satisfying the sublinear condition, i.e: $f(u) \leq C(u+1)$, was considered by Davila and Montenegro [7]. The authors proved the existence of solution and showed that the measure of the set $\{(x,t) \in \Omega \times (0,\infty) : u(x,t) = 0\}$ is positive (see also a more general statement in [9]). In other words, the solution may exhibit the quenching behavior. Still in the sublinear case with source term $\lambda f(u)$, Montenegro [19] proved that there is a real number $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there is $t_0 > 0$ such that

$$u(x, t_0) = 0, \quad \forall x \in \Omega.$$

He called this phenomenon complete quenching.

From our knowledge, equation (1.1) with a general source term $f(u, x, t)$ has not been studied completely. Thus, we would like to investigate first the existence of solutions to equation (1.1). Furthermore, it is well known that nonlinear parabolic equations with general source $f(u, x, t)$ may cause the finite time blow-up. As mentioned above, the nonlinear absorption $u^{-\beta}\chi_{\{u>0\}}$ causes the complete quenching phenomenon. Thus, it is interesting to see when the complete quenching prevails the blow-up, and conversely. We also note that the above qualitative behavior of solutions were studied by the authors in [3, 5] for the p-Laplacian equation in one-dimension of this type. In this paper, we only consider the blowing-up solutions of (1.1). Before giving our results, it is necessary to introduce a notion of weak solution of equation (1.1).

Definition 1.1. Let $u_0 \in L^{\infty}(\Omega)$. A nonnegative function $u(x,t)$ is called a weak solution of equation (1.1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$, and $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^{\infty}(\Omega \times (0,T)) \cap \mathcal{C}([0,T); L^1(\Omega))$ satisfies equation (1.1) in the sense of distributions $\mathcal{D}'(\Omega \times (0,T))$, i.e.

$$\int_0^T \int_{\Omega} \left(-u\phi_t + \nabla u \cdot \nabla \phi + u^{-\beta}\chi_{\{u>0\}}\phi - f(u,x,t)\phi \right) \,dx\,dt = 0, \tag{1.2}$$

for all $\phi \in \mathcal{C}_c^{\infty}(\Omega \times (0,T))$.

Our first result is the existence of a local solution to (1.1).

Theorem 1.2. *Let $u_0 \in L^{\infty}(\Omega)$, and let f satisfy $(H1)$. Then, there exists a finite time $T = T(u_0) > 0$ such that equation (1.1) has a maximal weak solution u in $\Omega \times (0,T)$, i.e: for any weak solution v in $\Omega \times (0,T)$, we have*

$$v \leq u, \quad \text{in } \Omega \times (0,T).$$

Moreover, there is a positive constant $C = C(f, \|u_0\|_{\infty})$ such that

$$|\nabla u(x,\tau)|^2 \leq C u^{1-\beta}\left(\tau^{-1} + 1\right), \quad \text{for a.e. } (x,\tau) \in \Omega \times (0,T). \tag{1.3}$$

Besides, if $\nabla(u_0^{1/\gamma}) \in L^\infty(\Omega)$*, with* $\gamma = \frac{2}{1+\beta}$*, then there is a positive constant* $C = C(f, u_0)$ *such that*

$$|\nabla u(x,\tau)|^2 \le C u^{1-\beta}(x,\tau), \quad \text{for a.e. } (x,\tau) \in \Omega \times (0,T). \tag{1.4}$$

Remark 1.3. Theorem 1.2 implies that u is continuous up to the boundary. Furthermore, u is continuous up to $t = 0$ provided $\nabla(u_0^{1/\gamma}) \in L^\infty(\Omega)$ (see for example [3, 4, 5, 20]).

Remark 1.4. Similarly as in the case of p-Laplacian of the equation of this type (see [3]), when $f(u,x,t) = f(u)$, the results of Theorem 1.2 still hold s for f a locally Lipschitz function on $[0,\infty)$, instead of $f \in C^2([0,\infty))$, required in the previous works (see for example [7, 19]). For example, our existence result can take into account the function $f(u) = (u-1)^+ u$.

After that, we study the global nonexistence of solutions of (1.1), the so called finite time blowing-up solution. In this paper, we point out some criteria on initial data u_0 to guarantee the blow-up of solution in a finite time. For simplicity, we consider $f(u,x,t) = f(u)$. We will give the first result of blow-up for the superlinear case, i.e. $f(u) = u^p$, for $p > 1$. Then, it is convenient to introduce the energy functional

$$E(t) = \int_\Omega \Big(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{1-\beta} u^{1-\beta}(t) - \frac{1}{p+1} u^{p+1}(t)\Big)dx, \tag{1.5}$$

Our first criterion considers $E(0)$ negative.

Theorem 1.5. *Let* $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$*. Suppose that* $f(u) = u^p$*, for* $p > 1$*, and* $E(0) \le 0$*. Let* u *be a solution of equation* (1.1)*. Then,* u *blows up in a finite time.*

It is interesting to find out an optimal condition of nonlinear source $f(u)$ such that the explosion of solution holds. Let us remind a necessary and sufficient condition for blow-up of solutions of equation (1.1) without the singular absorption $u^{-\beta}\chi_{\{u>0\}}$,

$$\begin{gathered} \partial_t u - \Delta u = f(u) \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.6}$$

It is known that if f is a convex function on $(0,\infty)$, and

$$\int_a^\infty \frac{1}{f(s)} ds < +\infty, \tag{1.7}$$

for some $a > 0$, then the solution u of (1.6) must blow up in a finite time provided that u_0 is large enough (see also [10, 11], necessary and sufficient conditions for blow-up of solution of the porous medium equation). One can take for instance a typical weak superlinear $f(u) = (1+u)\log^p(1+u)$, which is convex and satisfies (1.7), with $u \ge 0$, $p > 1$. We also note that only condition (1.7) is not sufficient to guarantee the explosion of u in a finite time if lacking of the convexity of f, see [21, Theorem 19.15]. Here, we will demonstrate that the explosion of solution of (1.1) occurs with f as above.

Theorem 1.6. *Let* $f(u,x,t) = f(u)$ *be a locally Lipschitz function on* $[0,\infty)$*. Suppose that* $f(u)$ *is a convex function on* $(0,\infty)$*, and* f *satisfies* (1.7) *for some* $a > 0$*. Then, the solution* u *of* (1.1) *blows up in a finite time if* $u_0 \in \mathcal{C}_b(\Omega)$ *is large enough.*

Our proof of Theorem 1.6 is based on the first eigenvalue method introduced by Kaplan [13]. Note that our equation contains the singular term $u^{-\beta}\chi_{\{u>0\}}$, which causes a difficulty in estimating this solution. To overcome this obstacle, we show that if u_0 is positive inside of Ω and large enough, then $u(t)$ is also positive inside of Ω for a certain large time interval. Note that the concave method used by the authors in [3] to prove the explosion of solutions for p-Laplacian equation in one dimension of this type cannot be applied to this situation. Finally, one can find a rich source of topic of explosive solutions in [12, 17, 21, 23], and references therein.

This article is organized as follows: In the next section, we prove the existence of a local solution to (1.1). To do that, we prove some gradient estimates for the approximating solutions. The last section is devoted to study of blowing-up of solutions.

The notation that will be used in this paper is the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C depends on p, β, τ.

2. Existence of a local solution

In this section, we consider a regularized equation of (1.1):

$$\begin{gathered}\partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t) \quad \text{in } \Omega \times (0,\infty),\\ u_\varepsilon = \eta \quad \text{on } \partial\Omega \times (0,\infty),\\ u_\varepsilon(0) = u_0 + \eta \quad \text{on } \Omega\end{gathered} \tag{2.1}$$

for any $0 < \eta < \varepsilon$, with $g_\varepsilon(s) = \psi_\varepsilon(s)s^{-\beta}$, $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$, and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a non-decreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \le 1$, and $\psi(s) = 1$ for $s \ge 2$. Note that g_ε is a globally Lipschitz function for any $\varepsilon > 0$. We will show that solution $u_{\varepsilon,\eta}$ of equation (2.1) tends to a solution of equation (1.1) as $\eta, \varepsilon \to 0$. In passing to the limit, we need to derive some gradient estimates for solution $u_{\varepsilon,\eta}$, see also [6, 7, 20]. Then, we have the following result.

Lemma 2.1. *Let $u_0 \in \mathcal{C}_c^\infty(\Omega)$, $u_0 \neq 0$. There exists a classical unique solution $u_{\varepsilon,\eta}$ of (2.1) in $\Omega \times (0,T)$.*

(i) *There is a constant $C > 0$ only depending on $\beta, T, f, \|u_0\|_\infty$ such that*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\tau^{-1} + 1\big), \quad \text{for any } (x,\tau) \in \Omega \times (0,T), \tag{2.2}$$

(ii) *If $\nabla(u_0^{1/\gamma}) \in L^\infty(\Omega)$, then we obtain*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau), \quad \text{for any } (x,\tau) \in \Omega \times (0,T), \tag{2.3}$$

with $C > 0$ merely depends on $\beta, T, f, \|u_0\|_\infty, \|\nabla(u_0^{1/\gamma})\|_\infty$.

Proof. (1) Fix $\varepsilon \in (0, \|u_0\|_\infty)$. For any $\eta \in (0,\varepsilon)$, the existence and uniqueness of a classical solution $u_{\varepsilon,\eta}$ of problem (2.1) is well-known (see [16]). We denote by $u = u_{\varepsilon,\eta}$ for short. Let $\Gamma(t)$ be the flat solution of the ODE:

$$\begin{gathered}\partial_t \Gamma = h(\Gamma), \quad \text{in } [0, T'],\\ \Gamma(0) = 2\|u_0\|_\infty,\end{gathered} \tag{2.4}$$

where h is the function in (H1) above, and T' is the maximal existence time of $\Gamma(t)$. Note that T' depends merely on $\|u_0\|_\infty$, see [2, Chapter 1]. It follows from the comparison principle that

$$\eta \le u \le \Gamma(t), \quad \forall t \in [0, T'].$$

Let us put $u = \phi(v) = v^\gamma$, with $\gamma = 2/(1+\beta)$. Then

$$v_t - \Delta v = \frac{\phi''}{\phi'}|\nabla v|^2 - \frac{1}{\phi'}\big(g_\varepsilon(\phi(v)) - f(\phi(v), x, t)\big). \tag{2.5}$$

For any $\tau \in (0, T'/3)$, let us consider a cut-off function $\xi(t) \in \mathcal{C}^\infty(0, \infty)$, $0 \le \xi(t) \le 1$, such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, \frac{T'}{3}], \\ 0, & \text{outside } (\frac{\tau}{2}, \frac{T'}{3} + \frac{\tau}{2}), \end{cases}$$

and $|\xi_t| \le \frac{c_0}{\tau}$, for some constant $c_0 > 0$.

Then, we set $w = \xi(t)|\nabla v|^2$. If $\max_{\Omega \times [0,T]} w = 0$, then $\nabla v(\tau) = 0$, so estimate (2.2) is trivial.

If not, there is a point $(x_0, t_0) \in \Omega \times (0, 2T'/3)$ such that $\max_{\Omega \times [0,T']} w = w(x_0, t_0)$. Thus, we have at (x_0, t_0):

$$w_t = 0, \quad \nabla w = 0, \quad \Delta w \le 0. \tag{2.6}$$

This implies

$$0 \le w_t - \Delta w = \xi_t|\nabla v|^2 + 2\xi(t)\big(\nabla v.\nabla v_t - \nabla v.\nabla(\Delta v)\big) - 2\xi(t)|D^2 v|^2,$$

or

$$0 \le \xi_t|\nabla v|^2 + 2\xi(t)\nabla v \cdot \nabla(v_t - \Delta v). \tag{2.7}$$

A combination of (2.5) and (2.7) provides us with

$$0 \le \xi_t|\nabla v|^2 + 2\xi(t)\nabla v \cdot \nabla\big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) - f(\phi(v), x, t)}{\phi'}\big).$$

Since $\xi(t_0) > 0$, we obtain

$$0 \le \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + \nabla v \cdot \nabla\big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) - f(\phi(v), x, t)}{\phi'}\big). \tag{2.8}$$

At the moment, we estimate the terms on the right hand side of (2.8). First of all, we have from (2.6) that $\nabla(|\nabla v(x_0, t_0)|^2) = 0$, so

$$\nabla v \cdot \nabla\big(\frac{\phi''}{\phi'}|\nabla v|^2\big) = \nabla v \cdot \nabla\big(\frac{\phi''}{\phi'}\big)|\nabla v|^2 = (\gamma - 1)(2\gamma - 3)v^{-2}|\nabla v|^4. \tag{2.9}$$

Next, we have

$$\begin{aligned} &\nabla v.\nabla\Big(\frac{f(\phi, x_0, t_0)}{\phi'}\Big) \\ &= \frac{D_x f(\phi, x_0, t_0)}{\phi'}\nabla v + D_u f(\phi, x_0, t_0)|\nabla v|^2 - f(\phi, x_0, t_0)\frac{\phi''}{\phi'^2}|\nabla v|^2 \\ &= \frac{1}{\gamma}D_x f(\phi, x_0, t_0)v^{1-\gamma}\nabla v + D_u f(\phi, x_0, t_0)|\nabla v|^2 \\ &\quad - (\frac{\gamma - 1}{\gamma})f(\phi, x_0, t_0)v^{-\gamma}|\nabla v|^2. \end{aligned} \tag{2.10}$$

Since $f \ge 0$, and $\gamma > 1$, it follows from (2.10) that

$$\nabla v \cdot \nabla\Big(\frac{f(\phi, x_0, t_0)}{\phi'}\Big) \le \frac{1}{\gamma}|D_x f(\phi, x_0, t_0)|v^{1-\gamma}|\nabla v| + |D_u f(\phi, x_0, t_0)||\nabla v|^2. \tag{2.11}$$

Concerning the last term, we have

$$\nabla v\cdot\nabla\Big(\frac{g_\varepsilon(\phi)}{\phi'}\Big)=(g'_\varepsilon-g_\varepsilon\frac{\phi''}{\phi'^2})|\nabla v|^2=\Big(\psi'_\varepsilon(\phi)v^{-\beta}-(\beta+\frac{\gamma-1}{\gamma})\psi_\varepsilon(\phi)v^{-(1+\beta)\gamma}\Big)|\nabla v|^2.$$

Since $\psi'_\varepsilon\geq 0$, and $0\leq\psi_\varepsilon\leq 1$, we obtain

$$-\nabla v\cdot\nabla\left(\frac{g(\phi)}{\phi'}\right)\leq(\beta+\frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2. \tag{2.12}$$

By inserting (2.9), (2.11) and (2.12) into (2.8), we obtain

$$\begin{aligned}(\gamma-1)v^{-2}|\nabla v|^4 &\leq \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2+(\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2\\ &\quad+|D_uf||\nabla v|^2+\frac{1}{\gamma}v^{1-\gamma}|D_xf||\nabla v|.\end{aligned} \tag{2.13}$$

Now, we multiply both sides of (2.13) by v^2 to get

$$\begin{aligned}(\gamma-1)|\nabla v|^4 &\leq \frac{1}{2}\xi^{-1}|\xi_t|v^2|\nabla v|^2+(\beta+1-\frac{1}{\gamma})|\nabla v|^2+v^2|D_uf||\nabla v|^2\\ &\quad+\frac{1}{\gamma}v^{3-\gamma}|D_xf||\nabla v|.\end{aligned} \tag{2.14}$$

If $|\nabla v(x_0,t_0)|\leq 1$, then $w(x_0,t_0)\leq 1$. This leads to $w(x,\tau)\leq 1$, thereby proves

$$|\nabla u(x,\tau)|^2\leq\frac{4}{(1+\beta)^2}u^{1-\beta}(x,\tau).$$

Then, estimate (2.2) follows immediately.

If not, we have $|\nabla v(x_0,t_0)|>1$, it follows then from (2.14)

$$\begin{aligned}(\gamma-1)|\nabla v|^4 &\leq \frac{1}{2}\xi^{-1}|\xi_t|v^2|\nabla v|^2+(\beta+1-\frac{1}{\gamma})|\nabla v|^2+v^2|D_uf||\nabla v|^2\\ &\quad+\frac{1}{\gamma}v^{3-\gamma}|D_xf||\nabla v|^2.\end{aligned}$$

By simplifying the term $|\nabla v|^2$ both sides of the last inequality, we obtain

$$(\gamma-1)|\nabla v|^2\leq\frac{1}{2}\xi^{-1}|\xi_t|v^2+(\beta+1-\frac{1}{\gamma})+v^2|D_uf|+\frac{1}{\gamma}v^{3-\gamma}|D_xf|.$$

Multiplying both sides of the above inequality by $\xi(t_0)$ yields

$$(\gamma-1)\xi(t_0)|\nabla v|^2\ leq\frac{1}{2}|\xi_t|v^2+\xi(t_0)\Big((\beta+1-\frac{1}{\gamma})+v^2|D_uf|+\frac{1}{\gamma}v^{3-\gamma}|D_xf|\Big). \tag{2.15}$$

Recall that $w(x_0,t_0)=\xi(t_0)|\nabla v(x_0,t_0)|^2$, $0\leq\xi(t)\leq 1$, and $|\xi_t|\leq\tau^{-1}$. It follows from (2.15) that there is a constant $C=C(\beta)>0$ such that

$$w(x_0,t_0)\leq C\big(\tau^{-1}v^2+v^2|D_uf|+v^{3-\gamma}|D_xf|+1\big).$$

Since $w(x_0,t_0)\geq w(x,\tau)=|\nabla v(x,\tau)|^2$, we obtain

$$|\nabla v(x,\tau)|^2\leq C\big(\tau^{-1}v^2+v^2|D_uf|+v^{3-\gamma}|D_xf|+1\big)$$

Moreover, we have

$$v^\gamma(x,t)=u(x,t)\leq\Gamma(T'),\quad\text{for any }(x,t)\in\Omega\times[0,T'].$$

Then

$$|\nabla v(x,\tau)|^2\leq C\Big(\tau^{-1}\Gamma^{1+\beta}(T')+\Gamma^{1+\beta}(T')\Theta(D_uf,\Gamma(T'))$$

$$+\Gamma^{\frac{1+3\beta}{2}}(T')\Theta(D_xf,\Gamma(T'))+1\Big),$$

with $\Theta(g,r)=\max_{0\le s\le r}\{|g(s)|\}$, or

$$\begin{aligned}|\nabla u(x,\tau)|^2 \le C_1u^{1-\beta}\Big(&\tau^{-1}\Gamma^{1+\beta}(T')+\Gamma^{1+\beta}(T')\Theta(D_uf,\Gamma(T'))\\&+\Gamma^{\frac{1+3\beta}{2}}(T')\Theta(D_xf,\Gamma(T'))+1\Big).\end{aligned}$$

Thus, (i) follows by choosing $T=T'/3$.

(ii) The proof of estimate (2.3) is similar to the one of estimate (2.2). We just make a slight change by considering a cut-off function $\overline{\xi}(t)\in\mathcal{C}^\infty(\mathbb{R})$ (instead of $\xi(t)$ above), such that $0\le\overline{\xi}(t)\le 1$, $\overline{\xi}_t(t)\le 0$, and

$$\overline{\xi}(t)=\begin{cases}1, & \text{if } t\le T'/3,\\ 0, & \text{if } t\ge 2T'/3.\end{cases}$$

Then, we observe that either $w(x,t)$ attains its maximum at the initial data, i.e.

$$\max_{(x,t)\in I\times[0,2T_0]} w(x,t)=w(x_0,0)=\overline{\xi}(0)|\nabla v(x_0,0)|^2\le\|\nabla(u_0^{1/\gamma})\|_\infty^2,$$

for some $x_0\in\Omega$, which implies

$$|\nabla u(x,\tau)|^2\le\gamma^2\|\nabla(u_0^{1/\gamma})\|_\infty^2u^{1-\beta}(x,\tau),\quad\text{for all } x\in\Omega. \tag{2.16}$$

Thus, we obtain estimate (2.3) immediately; or there is a point $(x_0,t_0)\in\Omega\times(0,2T'/3)$ such that

$$\max_{(x,t)\in\Omega\times[0,T']} w(x,t)=w(x_0,t_0)$$

Then, we repeat the proof of (i) for this case until (2.13) to get

$$\begin{aligned}(\gamma-1)v^{-2}|\nabla v|^4\le{}&\frac{1}{2}\overline{\xi}^{-1}\overline{\xi}_t|\nabla v|^2+(\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2\\&+|D_uf||\nabla v|^2+\frac{1}{\gamma}v^{1-\gamma}|D_xf||\nabla v|.\end{aligned}$$

Since $\overline{\xi}_t(t)\le 0$, from the above inequality we have

$$(\gamma-1)v^{-2}|\nabla v|^4\le(\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2+|D_uf||\nabla v|^2+\frac{1}{\gamma}v^{1-\gamma}|D_xf||\nabla v|.$$

By repeating the proof of (i) after this inequality, we obtain

$$\begin{aligned}|\nabla u(x,\tau)|^2\le{}&Cu^{1-\beta}(x,\tau)\big(\Gamma^{1+\beta}(T')\Theta(D_uf,\Gamma(T'))\\&+\Gamma^{\frac{1+3\beta}{2}}(T')\Theta(D_xf,\Gamma(T'))+1\big),\end{aligned}\tag{2.17}$$

with $C=C(\beta)>0$. Combining (2.16) and (2.17) yields estimate (2.3), and completes the proof. □

The proof of Theorem 1.2 is similar to the one in [4] (see also [6]). It applies Lemma 2.1 to pass to the limit as $\eta\to 0$ and $\varepsilon\to 0$. We let the reader to do it.

3. Non-global existence of solutions

In this section, we study the non-global existence of solutions to equation (1.1).

Proof of Theorem 1.5. By multiplying by u (resp. u_t) in equation (1.1), we have the integral equations

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx = -\int_\Omega \big(|\nabla u(x,t)|^2 + u^{1-\beta}(x,t) - u^{q+1}(x,t)\big)dx, \tag{3.1}$$

and

$$\begin{aligned}&\int_0^t\int_\Omega |u_t|^2 dxds + \int_\Omega \big(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{1-\beta}u^{1-\beta}(t) - \frac{1}{q+1}u^{q+1}(t)\big)dx \\ &= \int_\Omega \big(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{1-\beta}u_0^{1-\beta} - \frac{1}{q+1}u_0^{q+1}\big)dx,\end{aligned} \tag{3.2}$$

see [21]. By combining (3.1) and (3.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx = -2E(t) + \frac{1+\beta}{1-\beta}\int_\Omega u^{1-\beta}(x,t)dx + \frac{q-1}{q+1}\int_\Omega u^{q+1}(x,t)dx.$$

Since $E(0) \le 0$, (3.2) implies $E(t) \le 0$, for any $t > 0$. It follows then from the last inequality that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx \ge \frac{q-1}{q+1}\int_\Omega u^{q+1}dx. \tag{3.3}$$

By Holder's inequality,

$$\int_\Omega u^2 dx \le \Big(\int_\Omega u^{q+1}dx\Big)^{\frac{2}{q+1}} |\Omega|^{\frac{q-1}{q+1}}. \tag{3.4}$$

From (3.3) and (3.4), we obtain $y'(t) \ge Cy^{\frac{q+1}{2}}(t)$, with

$$y(t) = \int_\Omega u^2(x,t)dx, \quad C = \frac{2(q-1)}{(q+1)|\Omega|^{\frac{q-1}{2}}}.$$

This inequality implies that $y(t) \to +\infty$ as $t \to T_0^-$, with $T_0 = \frac{4\|u_0\|_{L^2(\Omega)}^{1-q}}{(q+1)|\Omega|^{\frac{q-1}{2}}}$. □

Next, we prove Theorem 1.6. Since our proof below is just a local argument, it suffices to consider initial data $u_0(x) = c\Phi(x)$, with $c > 0$, and Φ is the first eigenfunction of the Dirichlet problem

$$\begin{aligned}-\Delta\Phi &= \lambda_1\Phi \quad \text{in } \Omega, \\ \Phi(x) &= 0, \quad \text{on } \partial\Omega.\end{aligned} \tag{3.5}$$

We have the following result.

Theorem 3.1. *Let $f(u,x,t) = f(u)$ be a locally Lipschitz function on $[0,\infty)$ such that $f(0) = 0$. Suppose that $f(u)$ is a convex function on $(0,\infty)$, and f satisfies (1.7) for some $a > 0$. Let $u_0(x) = c\Phi(x)$, where $c > 0$ is large enough. Then, solution u must blow up in a finite time.*

We first modify a result by Davila and Montenegro [7] to show that $u(t)$ is positive inside of Ω for a certain large time interval $(0,T)$.

Lemma 3.2. *Suppose that $u_0(x) = C\Phi^\mu(x)$, for $C > 1$, and for some $\mu \in (1, \frac{2}{1+\beta})$. Then, we have*

$$u(x,t) \ge Ce^{-At}\Phi^\mu(x), \quad \forall (x,t) \in \Omega \times (0, T_{A,C}), \tag{3.6}$$

where $A > 0$ is chosen later, and $T_{A,C} = \log(C)/A$.

Proof. For any $\varepsilon > 0$, let u_ε be a unique solution of the equation

$$\begin{gathered} \partial_t u - \Delta u + g_\varepsilon(u) = f(u,x,t) \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{3.7}$$

obtained by passing to the limit as $\eta \to 0$ in (1.1). Note that u_ε converges to u, uniformly on any compact set in $\Omega \times (0,T)$, see [3]. Thus, it suffices to prove that for any $\varepsilon > 0$,

$$u_\varepsilon(x,t) \ge Ce^{-At}\Phi^\mu(x), \quad \forall (x,t) \in \Omega \times (0, T_{A,C}).$$

Put $w = Ce^{-At}\Phi^\mu(x)$. We show that w is a sub-solution of (3.7) for $A > 0$ large enough. In fact, we have

$$\begin{aligned} &\partial_t w - \Delta w + g_\varepsilon(v) - f(w) \le \partial_t w - \Delta w + w^{-\beta}\chi_{\{w>0\}} \\ &= -CAe^{-At}\Phi^\mu - C\mu e^{-At}\Phi^{\mu-1}\Delta\Phi - C\mu(\mu-1)e^{-At}\Phi^{\mu-2}|\nabla\Phi|^2 \\ &\quad + C^{-\beta}e^{A\beta t}\Phi^{-\beta\mu}\chi_{\{\Phi>0\}} \\ &= C(-A+\lambda_1\mu)e^{-At}\Phi^\mu + Ce^{-At}\Phi^{-\beta\mu}\Big(-\mu(\mu-1)\Phi^{\mu(\beta+1)-2}|\nabla\Phi|^2 \\ &\quad + C^{-\beta-1}e^{A\beta t+At}\chi_{\{\Phi>0\}}\Big). \end{aligned}$$

Note that for any $t \in (0, T_{A,C})$, we obtain $C^{-\beta-1}e^{A\beta t+At} \le 1$. This leads to

$$\begin{aligned} &\partial_t w - \Delta w + g_\varepsilon(w) - f(w) \\ &\le Ce^{-At}\Big((-A+\lambda_1\mu)\Phi^\mu + \Phi^{-\beta\mu}\big(-\mu(\mu-1)\Phi^{\mu(\beta+1)-2}|\nabla\Phi|^2 + 1\big)\Big). \end{aligned} \tag{3.8}$$

It is clear that $(-A+\lambda_1\mu)\Phi^\mu \le 0$ in $\Omega \times (0, T_{A,C})$, if $A > 2\lambda_1$.

Let $\omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$, for any $\delta > 0$. Obviously, we have

$$\big(-\mu(\mu-1)\Phi^{\mu(\beta+1)-2}|\nabla\Phi|^2 + 1\big) < 0, \quad \text{for any } x \in \omega_\delta, \tag{3.9}$$

if $\delta > 0$ is small enough because of $\mu(1+\beta) - 2 < 0$.

Fix $\delta > 0$ such that (3.9) holds. On the set $\Omega\backslash\omega_\delta$, we choose $A > 0$ large enough such that

$$(-A+\lambda_1\mu)\Phi^\mu + \Phi^{-\beta\mu}\big(-\mu(\mu-1)\Phi^{\mu(\beta+1)-2}|\nabla\Phi|^2 + 1\big) < 0. \tag{3.10}$$

A combination of (3.8), (3.9), and (3.10) implies that w is a sub-solution of equation (3.7); thereby it proves

$$w \le u_\varepsilon, \quad \text{in } \Omega \times (0, T_{A,C}).$$

which completes the proof. ◻

Remark 3.3. Note that A is chosen independently of C, see (3.10) again. If we fix $A > 0$ such that (3.10) holds, then $T_{A,C} = T_C$ is as large as $\log C$.

Now we have sufficient information to complete the proof of the above theorem.

Proof of theorem 3.1. Fix $\mu \in (1, \frac{2}{1+\beta})$. Since Φ is continuous on $\overline{\Omega}$, we have

$$u_0(x) = c\Phi(x) \geq c\eta_0 \Phi^{\mu}(x), \quad \text{in } \Omega,$$

with $\eta_0 = \big(\max_{x\in\Omega}\{\Phi(x)\}\big)^{1-\mu} > 0$. By applying Lemma 3.2, we obtain

$$u(x,t) \geq C_0 e^{-At}\Phi^{\mu}(x), \quad \forall (x,t) \in \Omega \times (0, T_{C_0}),$$

with $C_0 = c\eta_0$, and $T_{C_0} = \log(C_0)/A$. Multiply both sides of (1.1) by Φ yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(x,t)\Phi(x)dx &= \int_\Omega f(u(x,t))\Phi(x)dx - \lambda_1 \int_\Omega u(x,t)\Phi(x)dx \\ &\quad - \int_\Omega u^{-\beta}\chi_{\{u>0\}}(x,t)\Phi(x)dx. \end{aligned}$$

Thanks to Lemma 3.2, we obtain that for any $t \in (0, T_{C_0})$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(x,t)\Phi(x)dx &\geq \int_\Omega f(u)\Phi(x)dx - \lambda_1 \int_\Omega u\Phi(x)dx \\ &\quad - C_0^{-\beta}e^{A\beta t}\int_\Omega \Phi^{1-\mu\beta}dx. \end{aligned} \tag{3.11}$$

Note that $C_0^{-\beta}e^{A\beta t} \leq 1$, for any $t \in (0, T_{C_0})$. By the convexity of f and (3.11), we obtain

$$z'(t) \geq f(z(t)) - \lambda_1 z(t) - \int_\Omega \Phi^{1-\mu\beta}dx, \quad \text{for } t \in (0, T_{C_0}), \tag{3.12}$$

with $z(t) = \int_\Omega u(x,t)\Phi(x)dx$. Since f is a convex function, it follows from (1.7) that $\frac{f(s)}{s} \to +\infty$ as $s \to +\infty$. Thus, there is a constant $s_0 > 0$ such that

$$\frac{1}{2}f(s) \geq \lambda_1 s + \int_\Omega \Phi^{1-\mu\beta}dx, \quad \forall s > s_0. \tag{3.13}$$

Since c is sufficiently large, we have $z(0) = c\int_\Omega \Phi^2(x)dx > \max\{a, s_0\}$. It follows from (3.12) and (3.13) that $z(t) \geq z(0)$ and

$$z'(t) \geq \frac{1}{2}f(z(t)), \quad \text{for any } t \in (0, T_{C_0}).$$

Therefore,

$$\frac{1}{2}t \leq \int_0^t \frac{z'(t)dt}{f(z(t))} = \int_{z(0)}^{z(t)} \frac{dz}{f(z)} \leq \int_a^{+\infty} \frac{dz}{f(z)}, \quad \text{for any } t \in (0, T_{C_0}).$$

This implies

$$\frac{T_{C_0}}{2} \leq \int_a^{+\infty} \frac{dz}{f(z)}\,. \tag{3.14}$$

The right-hand side of (3.14) is bounded by a constant, while T_{C_0} is as large as $\log C_0 = \log(c\eta_0)$ (see Remark 3.3). Then, we obtain a contradiction if c is large enough. This completes the proof. □

Remark 3.4. It is not difficult to show that the blow-up result in Theorem 1.6 still holds if u_0 is assumed to be positive and large enough in a ball $B(x_0, r_0) \Subset \Omega$.

Note that the result in Theorem 3.1 still holds if f is only assumed to be a convex function on (a, ∞), for some $a > 0$.

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