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EXISTENCE AND GLOBAL BEHAVIOR OF SOLUTIONS TO FRACTIONAL p-LAPLACIAN PARABOLIC PROBLEMS

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Abstract. First, we discuss the existence, the uniqueness and the regularity of the weak solution to the following parabolic equation involving the fractional p-Laplacian,

$$
\begin{gathered}
u_t + (-\Delta)^s_p u + g(x,u) = f(x,u) \quad \text{in } Q_T := \Omega \times (0,T), \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times (0,T), \\
u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N.
\end{gathered}
$$

Next, we deal with the asymptotic behavior of global weak solutions. Precisely, we prove under additional assumptions on f and g that global solutions converge to the unique stationary solution as $t \to \infty$.

1. Introduction and Preliminaries

In this article we study the parabolic problem involving fractional p-Laplacian,

$$
\begin{gathered}
u_t + (-\Delta)^s_p u + g(x,u) = f(x,u) \quad \text{in } Q_T := \Omega \times (0,T), \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times (0,T), \\
u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N
\end{gathered}
\tag{1.1}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ (at least C^2), $s \in (0,1)$, $1 < p < N/s$, $u_0 \in L^\infty(\Omega)$ and $f(x,z)$, $g(x,z)$ are Carathéodory functions, locally Lipschitz with respect to z uniformly in x and satisfying the following assumptions:

- (A1) $f(x,z), g(x,z) > 0$ for $t > 0$ and $f(x,0) = 0$, $g(x,0) = 0$ for a.e. $x \in \Omega$.
- (A2) For a.e. $x \in \Omega$ and $z \geq 0$, $g(x,z)$ satisfy the growth condition:

$$
g(x,z) \leq C_1 + C_2 z^{r-1}, \quad 1 < r < p_s^* := \frac{Np}{N-sp}
$$

  for some positive constants C_1 and C_2.
- (A3) $f(x,z)/z^{p-1}$ is non-increasing and $g(x,z)/z^{p-1}$ is non-decreasing in z for a.e. $x \in \Omega$.

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(A4) $\limsup_{z\to 0^+} f(x,z)/z^{p-1} > \lambda_{1,s,p}$, $\limsup_{z\to\infty} f(x,z)/z^{p-1} < \lambda_{1,s,p}$, where $\lambda_{1,s,p}$ is the first eigenvalue of $(-\Delta)^s_p$, $\limsup_{z\to 0^+} g(x,z)/z^{p-1} = 0$ and $\limsup_{z\to\infty} g(x,z)/z^{p-1} = \infty$, uniformly in $x \in \Omega$.

For instance we can take $f(x,z) = a(x)z^{q-1}$ and $g(x,z) = b(x)z^{r-1}$ where $1 < q < p$, $p < r < p_s^*$ and $a, b \in L^\infty(\Omega)$ as these functions satisfy $(A1) - (A4)$.

We recall that the fractional p-Laplacian operator $(-\Delta)^s_p$ (up to a normalizing constant) is defined as

$$(-\Delta)^s_p u(x) := \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

The systematic study of the problems involving non-local operators have found great interest in the recent years due to there occurrence in concrete real-world applications, such as, the thin obstacle problem, optimization, finance, phase transitions. Elliptic theory of linear or quasilinear non-local operators has been actively studied during last decades in the works of Caffarelli and collaborators [3, 4, 8], Kassmann [16], Silvestre [21] and many others. For further references, we refer to the surveys [24], [18] and in the nonlinear diffusion of degenerate type case (p-fractional operators type) [19]. We also refer to [10, 11, 25, 26, 27] on related existence results for nonlocal problems driven by the fractional Laplace operator.

Concerning the parabolic equation involving fractional Laplacian, the study of anomalous diffusion equation has gained interest for its occurrence in a number of phenomena in several areas of physics, finance, biology, ecology, geophysics, and many others which can be characterized as having non-Brownian scaling.

Contrary to the stationary version, there are quite few results about the corresponding evolution equations involving quasilinear and nonlocal operators. We can quote first that local existence and uniqueness of mild solutions are investigated in [17] by semi-group theory. The homogeneous Dirichlet problem for the fractional p-Laplacian evolution equation is studied also in the recent work of Vázquez where the author proved everywhere positivity of weak solutions. This striking property contrasts with the finite propagation property occurring in the local setting (p-Laplace operator). In [1], authors have studied (1.1) with the nonlinearity f depending only on x and t and prove the existence and some properties of entropy solutions. In particular, the questions related to the extinction in finite time and the finite speed of propagation are analyzed.

In this article, we investigate different issues of the existence and the regularity of energy weak solutions that in our knowledge are not discussed in former works. We also deal with the long-time behavior of weak solutions for a class of sub-homogeneous nonlinearities f and g following the approach in [5] for the local homogeneous p-Laplacian operator and in [12] for the local non-homogeneous $p(x)$-Laplacian operator. We point out that using the results in [2], authors in [13] have studied similar questions for the semilinear version of (1.1) with singular nonlinearity.

2. Preliminaries

We consider the function space

$$W^{s,p}(\mathbb{R}^N) := \Big\{ u | u : \mathbb{R}^N \to \mathbb{R}^N \text{ is measurable }, u \in L^p(\mathbb{R}^N)$$

$$\text{and } \frac{(u(x)-u(y))}{|x-y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)\Big\}.$$

$W^{s,p}(\mathbb{R}^N)$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx\, dy\Big)^{1/p}.$$

Also define the closed linear subspace $X_0(\Omega)$ of $W^{s,p}(\mathbb{R}^N)$ as

$$X_0(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) | u(x) = 0 \text{ a.e. } x \in \mathcal{C}\Omega\}$$

endowed with the norm

$$\|u\|_{X_0(\Omega)} := \Big(\frac{1}{2}\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx\, dy\Big)^{1/p}$$

where $Q = \mathbb{R}^N \times \mathbb{R}^N \backslash \mathcal{C}\Omega \times \mathcal{C}\Omega$, $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$. Then $X_0(\Omega)$ is a uniformly convex Banach space. Also $C_0^\infty(\Omega)$ is dense in $X_0(\Omega)$ and $X_0(\Omega)$ is compactly embedded in $L^r(\Omega)$ for $1 \le r < p_s^*$.

Remark 2.1. Let $t^+ = \max(t, 0)$. If $v \in X_0(\Omega)$, then

$$|v(x)-v(y)|^{p-2}(v^+(x)-v^+(y))(v(x)-v(y)) \ge |v^+(x)-v^+(y)|^p.$$

Definition 2.2. Set $d(x) :=$ dist $(x, \partial\Omega)$. Define the normed space

$$C_{d(\Omega)} := \{u \in C_0(\overline{\Omega}) : \exists c \ge 0 \text{ such that } |u(x)| \le c d(x), \forall x \in \Omega\}$$

Definition 2.3. Define the open convex subset of $C_{d(\Omega)}$,

$$C^+_{d^s(\Omega)} := \{u \in C_{d(\Omega)} : \inf_{x\in\Omega} \frac{u(x)}{d^s(x)} > 0\}.$$

Let $\phi_{1,s,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,s,p}$ of the operator $(-\Delta)^s_p$. Then $\phi_{1,s,p} \in C^+_{d^s}(\Omega)$.

We also recall the following inequalities due to Simon [22]: for all $u, v \in \mathbb{R}^N$,

$$\big||u|^{p-2}u - |v|^{p-2}v\big| \le \begin{cases} c|u-v|(|u|+|v|)^{p-2} & \text{if } p \ge 2,\\ c|u-v|^{p-1} & \text{if } p \le 2, \end{cases} \tag{2.1}$$

$$\langle |u|^{p-2}u - |v|^{p-2}v, u-v\rangle \ge \begin{cases} \tilde{c}|u-v|^p & \text{if } p \ge 2,\\ \tilde{c}\frac{|u-v|^2}{(|u|+|v|)^{2-p}} & \text{if } p \le 2, \end{cases} \tag{2.2}$$

where $c, \tilde{c}$ are positive constants and $\langle\cdot,\cdot\rangle$ is the canonical scalar product of $\mathbb{R}^N$.

3. Main results

First we consider the problem

$$\begin{gathered} u_t + (-\Delta)^s_p u = h(x,t) \quad \text{in } Q_T := \Omega \times (0,T),\\ u(x,t) = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times (0,T),\\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{3.1}$$

where $T > 0$, $h \in L^\infty(Q_T)$. Considering the initial data $u_0 \in L^\infty(\Omega)$, we study the weak solution of (3.1) defined as follows:

Definition 3.1. A weak solution of (3.1) is a function $u \in L^\infty(0,T;X_0(\Omega))$ such that $u_t \in L^2(Q_T)$ and for any $\phi \in C_0^\infty(Q_T)$,

$$\begin{aligned}&\int_{Q_T} u_t\phi\,dx\,dt + \frac{1}{2}\int_0^T\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,dx\,dy\,dt\\&= \int_{Q_T} h(x,t)\phi\,dx\,dt\end{aligned}$$

and $u(x,0)=u_0(x)$ for a.e. $x\in\Omega$.

Next, we consider the initial data u_0 in $C^+_{d^s(\Omega)}$ and study the evolution equation

$$\begin{gathered}u_t + (-\Delta)^s_p u + g(x,u) = f(x,u) \quad \text{in } Q_T := \Omega\times(0,T),\\ u(x,t) = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega\times(0,T),\\ u(x,0)=u_0(x) \quad \text{in } \Omega.\end{gathered} \tag{3.2}$$

Definition 3.2. A solution of (3.2) is a function $u \in L^\infty(0,T;X_0(\Omega))$ such that $u_t \in L^2(Q_T)$ and for any $\phi \in C_0^\infty(Q_T)$,

$$\begin{aligned}&\int_{Q_T} u_t\phi\,dx\,dt + \frac{1}{2}\int_0^T\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,dx\,dy\,dt\\&+ \int_{Q_T} g(x,u)\phi\,dx\,dt\\&= \int_{Q_T} f(x,u)\phi\,dx\,dt\end{aligned}$$

and $u(x,0)=u_0(x)$ for a.e. $x\in\Omega$.

Theorem 3.3. *Let $T>0$, $h(x,t)\in L^\infty(Q_T)$ and $u_0\in L^\infty(\Omega)$. Then there exists a unique weak solution u to the problem (3.1). Moreover $u\in C([0,T],X_0(\Omega))$ and satisfies for any $t\in[0,T]$:*

$$\begin{aligned}&\int_0^t\int_\Omega \Big(\frac{\partial u}{\partial t}\Big)^2\,dx\,ds + \frac{1}{p}\|u(t)\|^p_{X_0(\Omega)}\\&= \int_0^t\int_\Omega h(x,s)\Big(\frac{\partial u}{\partial t}\Big)\,dx\,ds + \frac{1}{p}\|u_0\|^p_{X_0(\Omega)}\end{aligned} \tag{3.3}$$

Concerning problem (3.2), we deduce the following similar result.

Theorem 3.4. *Let f,g be Carathéodory functions, locally Lipschitz with respect to second variable uniformly in $x\in\Omega$ and satisfying the assumptions $(A1),(A2)$ and $(A4)$. Let $u_0\in C^+_{d^s(\Omega)}$. Then for any $T>0$, there exists a unique weak solution u to problem (3.2). Moreover $u\in C([0,T],X_0(\Omega))$ and satisfies for any $t\in[0,T]$:*

$$\begin{aligned}&\int_0^t\int_\Omega \Big(\frac{\partial u}{\partial t}\Big)^2\,dx\,ds + \frac{1}{p}\|u(x,t)\|^p_{X_0(\Omega)}\\&= \int_\Omega F(x,u(x,t))\,dx - \int_\Omega G(x,u(x,t))\,dx + \frac{1}{p}\|u_0(x)\|^p_{X_0(\Omega)}\end{aligned} \tag{3.4}$$

where $F(x,z)=\int_0^z f(x,s)ds$ and $G(x,z)=\int_0^z g(x,s)ds$.

Next we observe that the operator $A:=(-\Delta)^s_p$, with Dirichlet boundary conditions, is m-accretive in $L^\infty(\Omega)$. Precisely, we have the following lemma.

Lemma 3.5. *Consider $\mathcal{D}(A) = \{u \in X_0(\Omega) \cap L^\infty(\Omega) : Au \in L^\infty(\Omega)\}$ as the domain of the operator A. Then A is m-accretive in $L^\infty(\Omega)$.*

Now by appealing the theory of maximal accretive operators in Banach spaces, we obtain the following results for the solutions of (3.1) and (3.2), respectively.

Theorem 3.6. *Let $T > 0$, $h \in L^\infty(Q_T)$ and let u_0 be in $\overline{\mathcal{D}(A)}^{L^\infty}$. Then*

(i) *the unique weak solution u to (3.1) obtained in Theorem 3.3 belongs to $\mathcal{C}([0,T]; \mathcal{C}_0(\overline{\Omega}))$.*

(ii) *If v is another mild solution to (3.1) with the initial datum $v_0 \in \overline{\mathcal{D}(A)}^{L^\infty}$ and the right-hand side $k(x,t) \in L^\infty(Q_T)$, then the following estimate holds:*

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \|h(s) - k(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s, \tag{3.5}$$

for $0 \le t \le T$.

(iii) *If $u_0 \in \mathcal{D}(A)$ and $h \in W^{1,1}(0,T; L^\infty(\Omega))$ then $u \in W^{1,\infty}(0,T; L^\infty(\Omega))$ and $(-\Delta)^s_p u \in L^\infty(Q_T)$, and the following estimate holds:*

$$\|\frac{\partial u}{\partial t}(\cdot,t)\|_{L^\infty(\Omega)} \le \|(-\Delta)^s_p u_0 + h(\cdot,0)\|_{L^\infty(\Omega)} + \int_0^T \|\frac{\partial h}{\partial t}(\cdot,\tau)\|_{L^\infty(\Omega)}\mathrm{d}\tau. \tag{3.6}$$

Theorem 3.7. *Assume that conditions and hypotheses on f, g in Theorem 3.4 are satisfied and $u_0 \in \overline{\mathcal{D}(A)}^{L^\infty}$. Then, the unique weak solution to (3.2) belongs to $C([0,T]; C_0(\overline{\Omega}))$ and*

(i) *there exists $\omega > 0$ such that if v is another weak solution to (3.2) with the initial datum $v_0 \in \overline{\mathcal{D}(A)}^{L^\infty}$ then the following estimate holds:*

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le e^{\omega t}\|u_0 - v_0\|_{L^\infty(\Omega)}, \quad 0 \le t \le T.$$

(ii) *If $u_0 \in \mathcal{D}(A)$ then $u \in W^{1,\infty}(0,T; L^\infty(\Omega))$ and $(-\Delta)^s_p u \in L^\infty(Q_T)$, and the following estimate holds:*

$$\|\frac{\partial u}{\partial t}(t)\|_{L^\infty(\Omega)} \le e^{\omega t}\|(-\Delta)^s_p u_0 + f(x,u_0)\|_{L^\infty(\Omega)}.$$

Next, we investigate the asymptotic behavior of global solution of (3.2), in particular the convergence to a stationary solution. For this first we study the following stationary problem corresponding to (3.2).

$$\begin{gathered} (-\Delta)^s_p u + g(x,u) = f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{3.7}$$

Definition 3.8. A function $u \in X_0(\Omega)$ is said to be a weak solution of (3.7) if $u > 0$ in Ω and

$$\begin{aligned} &\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}dx\,dy + \int_\Omega g(x,u)\phi(x)dx \\ &= \int_\Omega f(x,u)\phi(x)dx, \end{aligned}$$

for all $\phi \in X_0(\Omega)$.

Theorem 3.9. *Let f, g be Carathéodary functions, locally Lipschitz with respect to second variable uniformly in $x \in \Omega$ and satisfying the assumptions* (A1)–(A4). *Then there exists a unique weak solution u_∞ of* (3.7). *Moreover, $u_\infty \in C^+_{d^s(\Omega)}$.*

Theorem 3.10. *Assume that f satisfies* (A1)–(A4). *Then the weak solution u to* (3.2) *is defined in $(0, \infty) \times \Omega$ and*

$$u(t) \to u_\infty \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty$$

where u_∞ is the unique solution to the stationary problem (3.7).

4. Proofs of the main results

In this section we give the proofs of the results stated in Section 3. We begin with the following sequence of results.

Lemma 4.1. *Let f, g be Carathéodory functions, locally Lipschitz with respect to second variable uniformly in $x \in \Omega$ and satisfying the assumptions* (A1)–(A3). *Then there exists a non-negative and non trivial weak solution $u \in X_0(\Omega)$ to the equation in* (3.7).

Proof. Consider the energy functional J corresponding to (3.7), given by

$$J(u) = \frac{1}{p} \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dx\, dy + \int_\Omega G(x, u)\, dx - \int_\Omega F(x, u)\, dx.$$

Note that J is coercive in $X_0(\Omega)$. Indeed, by the assumption (A1), (A2) and the Sobolev embedding theorem we have

$$\begin{aligned} J(u) &= \frac{1}{p} \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dx\, dy + \int_\Omega G(x, u)\, dx - \int_\Omega F(x, u)\, dx \\ &\geq \frac{1}{p} \|u\|^p_{X_0(\Omega)} - C_1^* \|u\|_{X_0(\Omega)} - C_2^* \|u\|^q_{X_0(\Omega)} \end{aligned}$$

which tends to ∞ for $\|u\|_{X_0(\Omega)}$ large enough. Thus J is coercive. Furthermore $J \in C^1(X_0(\Omega))$ and weakly lower semi-continuous in $X_0(\Omega)$ and therefore admits a global minimizer which is a weak solution to (3.7). From (A4), we obtain easily $\inf_{X_0(\Omega)} J < 0$ and then $u \not\equiv 0$. Also $J(|u|) \leq J(u)$. Indeed, as by triangle inequality, for $x, y \in \mathbb{R}^N$, we have $\||u(x)| - |u(y)|\| \leq |u(x) - u(y)|$. Also from (A1), $F(x, |u(x)|) = F(x, u(x))$ and $G(x, |u(x)|) = G(x, u(x))$ for all $x \in \Omega$. Thus we have $J(|u|) \leq J(u)$ for $u \in X_0(\Omega)$, and hence the minimizer of J in $X_0(\Omega)$ can be assumed non-negative. This establishes the existence of a non-negative, non-trivial weak solution $u \in X_0(\Omega)$ of (3.7). $\square$

Now we prove that $u \in L^\infty(\Omega)$ and is unique. For proving these properties, first we recall the following Picone inequality (see [7]).

Lemma 4.2. *For every $a_1, a_2 \geq 0$ and $b_1, b_2 > 0$*

$$|a_1 - a_2|^p \geq |b_1 - b_2|^{p-2}(b_1 - b_2)\Big(\frac{a_1^p}{b_1^{p-1}} - \frac{a_2^p}{b_2^{p-1}}\Big).$$

The equality holds if and only if $(a_1, a_2) = k(b_1, b_2)$ for some constant k.

Proposition 4.3. *Let $u \in X_0(\Omega)$ be a weak solution of* (3.7). *Then $u \in L^\infty(\Omega)$.*

Proof. We adapt arguments from [9]. First we note that due to the homogeneity of the problem (3.7), it suffices to prove that

$$\|u^+\|_{L^\infty(\Omega)} \le 1 \text{ whenever } \|u^+\|_{L^p(\Omega)} \le \delta \text{ for some } \delta > 0. \tag{4.1}$$

A similar assertion can be established for u^- where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$. Therefore $u \in L^\infty(\Omega)$. For $k \ge 1$, set $w_k(x) = (u(x) - (1 - 2^{-k}))^+$. Then first we make the following observations about $w_k(x)$.

(i) $w_{k+1}(x) \le w_k(x)$ for all $x \in \Omega$,

(ii) $u(x) < (2^{k+1} + 1)w_k(x)$ for $x \in \{w_{k+1}(x) > 0\}$. For proving this, take $x \in \{w_{k+1}(x) > 0\}$. Then $w_k(x) = u(x) - (1 - 2^{-k})$. This implies

$$(2^{k+1} + 1)w_k(x) = u(x) + 2^{k+1}u(x) - (2^{k+1} + 1)(1 - 2^{-k}).$$

Now as for $x \in \{w_{k+1}(x) > 0\}$, $u(x) > (1 - 2^{-(k+1)}) = 1 - 2^{-k} + 2^{-(k+1)}$. This implies

$$\begin{aligned} 2^{k+1}u(x) &> 2^{k+1}(1 - 2^{-k}) + 1 \\ &> 2^{k+1}(1 - 2^{-k}) + (1 - 2^{-k}) \\ &= (2^{k+1} + 1)(1 - 2^{-k}). \end{aligned}$$

Therefore $(2^{k+1} + 1)w_k(x) = u(x) + 2^{k+1}u(x) - (2^{k+1} + 1)(1 - 2^{-k}) > u(x)$ for $x \in \{w_{k+1}(x) > 0\}$.

(iii) $\{w_{k+1} > 0\} \subset \{w_k > 2^{-(k+1)}\}$.

Now set $U_k := \|w_k\|^p_{L^p}$. Taking $v = u - (1 - 2^{-(k+1)})$ in Lemma 2.1 we obtain

$$|u(x) - u(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(u(x) - u(y)) \ge |w_{k+1}(x) - w_{k+1}(y)|^p.$$

Therefore, using (i)-(ii) above, we obtain

$$\begin{aligned} \|w_{k+1}\|^p_{X_0(\Omega)} &= \int_Q \frac{|w_{k+1}(x) - w_{k+1}(y)|^p}{|x - y|^{N+sp}} dx\, dy \\ &\le \int_Q \frac{|u(x) - u(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(u(x) - u(y))}{|x - y|^{N+sp}} dx\, dy \\ &\le \int_\Omega |f(x, u)| w_{k+1} dx \\ &\le \int_{\{w_{k+1}(x)>0\}} (C_1 + C_2|u|^{p-1}) w_{k+1} dx \\ &= C_1 \int_{\{w_{k+1}(x)>0\}} w_{k+1} dx + C_2 \int_{\{w_{k+1}(x)>0\}} |u|^{p-1} w_{k+1} dx \\ &\le C_1 |\{x \in \Omega : w_{k+1}(x) > 0\}|^{1-1/p} U_k^{1/p} \\ &\quad + C_2 \int_{\{w_{k+1}(x)>0\}} (2^{k+1} + 1)^{p-1} w_k^p dx \\ &\le C_1 |\{x \in \Omega : w_{k+1}(x) > 0\}|^{1-1/p} U_k^{1/p} + C_2 (2^{k+1} + 1)^{p-1} U_k. \end{aligned}$$

Now as for (iii) we have

$$U_k = \int_\Omega w_k^p dx \ge \int_{\{w_{k+1}>0\}} w_k^p \ge 2^{-(k+1)p} |\{x \in \Omega : w_{k+1}(x) > 0\}|.$$

Therefore,

$$\|w_{k+1}\|^p_{X_0(\Omega)} \leq (C_1 2^{(k+1)(p-1)} + C_2(2^{k+1}+1)^{p-1})U_k \leq C_3(2^{k+1}+1)^{p-1}U_k. \quad (4.2)$$

Also from Hölder's inequality we have

$$\begin{aligned} U_{k+1} &= \int_{\{w_{k+1}(x)>0\}} w^p_{k+1} dx \\ &\leq \Big(\int_{\{w_{k+1}(x)>0\}} w_{k+1}^{\frac{N}{N-sp}} \Big)^{\frac{N-sp}{N}} |\{x \in \Omega : w_{k+1}(x) > 0\}|^{sp/N} \\ &\leq C_4 \|w_{k+1}\|^p_{X_0(\Omega)} (2^{(k+1)p} U_k)^{sp/N}. \end{aligned} \quad (4.3)$$

Hence,

$$\begin{aligned} U_{k+1} &\leq C_5(2^{k+1}+1)^{p-1}U_k(2^{(k+1)p}U_k)^{sp/N} \\ &\leq C_5(2^{k+1}+1)^{p(1+\frac{sp}{N})} U_k^{1+\frac{sp}{N}} \\ &\leq C_5 C^k U_k^{1+\alpha} \end{aligned} \quad (4.4)$$

where $C > 1$ and $\alpha = \frac{sp}{N}$. This will imply that

$$\lim_{k\to\infty} U_k = 0 \quad (4.5)$$

provided that $\|u^+\|^p_{L^p(\Omega)} = U_0 \leq C^{-\frac{1}{\alpha^2}} =: \delta^p$. As $w_k(x) \to (u(x)-1)^+$ for a.e. $x \in \mathbb{R}^N$, (4.1) follows from (4.5). $\square$

Now we recall [15, Theorem 1.1] that provides the $C^\alpha(\overline{\Omega})$ regularity of weak solution of (3.7).

Theorem 4.4. *There exist $\alpha = \alpha(N,p,s) \in (0,s]$ and $C = C(N,p,s,\Omega,\|u\|_{L^\infty}(\Omega))$, such that, for all weak solutions $u \in X_0(\Omega)$ of (3.7), $u \in C^\alpha(\overline{\Omega})$ and $\|u\|_{C^\alpha(\overline{\Omega})} \leq C$.*

Next, we have the following Hopf Lemma from [20, Theorems 1.4 and 1.5, p. 778].

Lemma 4.5. *Let Ω satisfy the interior ball condition and $u \in X_0(\Omega) \cap C(\overline{\Omega})$ be a non-trivial, non-negative weak super-solution of*

$$(-\Delta)^s_p u = c(x)|u|^{p-1} \quad \text{in } \Omega$$

with $c \in L^1_{\rm loc}(\Omega)$ and non-positive. Then $u > 0$ in Ω and

$$\liminf_{B_R \ni x \to x_0} \frac{u(x)}{d_R(x)^s} > 0 \quad (4.6)$$

where B_R is a ball such that $x_0 \in B_R \subset \Omega$ and $d_R(x)$ is distance from x to ∂B_R.

Writing $g(x,u) = c(x)u^{p-1}$ and using (A1) and (A4), we obtain that any non-negative and non trivial weak solution u to the equation in (3.7) is positive and satisfies $u \geq kd(x)$ for some $k > 0$. Next, using [15, Theorem 4.4], we obtain that any nonnegative and non trivial weak solution u to the equation in (3.7) belongs to $C^+_{d^s(\Omega)}$. Then it follows that any couple of non trivial and nonnegative weak solutions u, v to the equation in (3.7) satisfy u/v, $v/u \in L^\infty(\Omega)$. We use this property to prove the uniqueness of the solution of (3.7).

Theorem 4.6. *Let $u, v \in X_0(\Omega)$ be two non trivial and nonnegative weak solutions to the equation in (3.7). The $u = v$ for a.e. in Ω.*

Proof. Set $u_n = u + \frac{1}{n}$ and $v_n = v + \frac{1}{n}$ and define

$$\tilde{v}_n := \frac{u^p}{v_n^{p-1}}, \quad \tilde{u}_n := \frac{v^p}{u_n^{p-1}}.$$

First we claim that $\tilde{v}_n,\ \tilde{u}_n \in X_0(\Omega)$. Note that since $u, v > 0$ in Ω, $\tilde{v}_n,\ \tilde{u}_n > 0$ in Ω, $\tilde{v}_n, \tilde{u}_n = 0$ in $\mathbb{R}^N \setminus \Omega$ for all $n \in \mathbb{N}$. Also since $u, v \in L^\infty(\Omega)$, we have that $\tilde{v}_n, \tilde{u}_n \in L^p(\Omega)$ for all $n \in \mathbb{N}$. Also as

$$\begin{aligned}
&|\tilde{v}_n(x) - \tilde{v}_n(y)| \\
&= \Big|\frac{u^p(x)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(y)}\Big| \\
&\le \Big|\frac{u^p(x)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(x)}\Big| + \Big|\frac{u^p(y)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(y)}\Big| \\
&= \Big|\frac{u^p(x) - u^p(y)}{v_n^{p-1}(x)}\Big| + |u^p(y)|\Big|\frac{v_n^{p-1}(y) - v_n^{p-1}(x)}{v_n^{p-1}(x)v_n^{p-1}(y)}\Big| \\
&\le n^{p-1}|u_n^p(x) - u_n^p(y)| + \|u\|_{L^\infty(\Omega)}^p\Big|\frac{v_n^{p-1}(y) - v_n^{p-1}(x)}{v_n^{p-1}(x)v_n^{p-1}(y)}\Big| \\
&\le 2n^{p-1}p\|u_n\|_{L^\infty(\Omega)}^{p-1}|u(x) - u(y)| \\
&\quad + (p-1)\|u\|_{L^\infty(\Omega)}^p\Big|\frac{v_n^{p-2}(y) + v_n^{p-2}(x)}{v_n^{p-1}(x)v_n^{p-1}(y)}\Big||v_n(y) - v_n(x)| \\
&= 2n^{p-1}p\|u\|_{L^\infty(\Omega)}^{p-1}|u(x) - u(y)| \\
&\quad + (p-1)\|u\|_{L^\infty(\Omega)}^p\Big|\frac{1}{v_n^{p-1}(x)v_n(y)} + \frac{1}{v_n^{p-1}(y)v_n(x)}\Big||v(y) - v(x)| \\
&\le 2n^{p-1}p\|u\|_{L^\infty(\Omega)}^{p-1}|u(x) - u(y)| + 2n^p(p-1)\|u\|_{L^\infty(\Omega)}^p|v(y) - v(x)| \\
&\le C(n, p, \|u\|_{L^\infty(\Omega)})\,(|u(x) - u(y)| + |v(y) - v(x)|)
\end{aligned} \tag{4.7}$$

for all $(x, y) \in \mathbb{R}^{2N}$. Thus $\tilde{v}_n \in X_0(\Omega)$ for all $n \in \mathbb{N}$. Similarly $\tilde{u}_n \in X_0(\Omega)$ for all $n \in \mathbb{N}$. As u and v solve (3.7), we have

$$\langle(-\Delta)_p^s u, u - \tilde{u}_n\rangle = (f(x, u) - g(x, u))\,(u - \tilde{u}_n), \tag{4.8}$$
$$\langle(-\Delta)_p^s v, v - \tilde{v}_n\rangle = (f(x, v) - g(x, v))\,(u - \tilde{u}_n). \tag{4.9}$$

Set

$$\begin{aligned}
&L(u, v)(x, y) \\
&= |u(x) - u(y)|^p - |v(x) - v(y)|^{p-2}(v(x) - v(y))\Big(\frac{u^p(x)}{v^{p-1}(x)} - \frac{u^p(x)}{v^{p-1}(y)}\Big).
\end{aligned}$$

Now using (4.8), (4.9) and Lemma 4.2 we have the estimate

$$\begin{aligned}
0 &\le \int_Q L(u, v_n)(x, y) + L(v, u_n)(x, y)\,dx\,dy \\
&= \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} - \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x - y|^{N+sp}} \\
&\quad \times \Big(\frac{u^p(x)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(y)}\Big)dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} - \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} \\
&\times \Big(\frac{v^p(x)}{u_n^{p-1}(x)} - \frac{v^p(y)}{u_n^{p-1}(y)}\Big)\, dx\, dy \\
&= \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} - \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \\
&\times \Big(\frac{v^p(x)}{u_n^{p-1}(x)} - \frac{v^p(y)}{u_n^{p-1}(y)}\Big) dx\, dy \\
&+\int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} - \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} \\
&\times \Big(\frac{u^p(x)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(y)}\Big) dx\, dy \\
&= \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \Big(\frac{u^p(x)}{u^{p-1}(x)} - \frac{u^p(y)}{u^{p-1}(y)}\Big) dx\, dy \\
&- \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \Big(\frac{v^p(x)}{u_n^{p-1}(x)} - \frac{v^p(y)}{u_n^{p-1}(y)}\Big) dx\, dy \\
&+ \int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} \Big(\frac{v^p(x)}{v^{p-1}(x)} - \frac{v^p(y)}{v^{p-1}(y)}\Big) dx\, dy \\
&- \int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} \Big(\frac{u^p(x)}{v_n^{p-1}(x)} - \frac{u^p(y)}{v_n^{p-1}(y)}\Big) dx\, dy \\
&= \int_\Omega (f(x,u)-g(x,u))\,(u-\tilde{u}_n)\, dx + \int_\Omega (f(x,v)-g(x,v))\,(v-\tilde{v}_n)\, dx.
\end{aligned} \tag{4.10}
$$

Also using the Monotone convergence theorem we estimate the right-hand side of (4.10) for large n as follows.

$$
\begin{aligned}
&\int_\Omega (f(x,u)-g(x,u))\,(u-\tilde{u}_n)\, dx + \int_\Omega (f(x,v)-g(x,v))\,(v-\tilde{v}_n)\, dx \\
&= \int_\Omega (f(x,u)-g(x,u))\, u\, dx - \int_\Omega (f(x,u)-g(x,u))\, \frac{v^p}{(u+\frac{1}{n})^{p-1}} dx \\
&\quad + \int_\Omega (f(x,v)-g(x,v))\, v\, dx - \int_\Omega (f(x,v)-g(x,v))\, \frac{u^p}{(v+\frac{1}{n})^{p-1}} dx \\
&\quad + o_n(1) \\
&= \int_\Omega (f(x,u)-g(x,u))\, u\, dx - \int_\Omega (f(x,u)-g(x,u))\, \frac{v^p}{u^{p-1}} dx \\
&\quad + \int_\Omega (f(x,v)-g(x,v))\, v\, dx - \int_\Omega (f(x,v)-g(x,v))\, \frac{u^p}{v^{p-1}} dx + o_n(1) \\
&= \int_\Omega \Big(\frac{f(x,u)-g(x,u)}{u^{p-1}} - \frac{f(x,v)-g(x,v)}{v^{p-1}}\Big)(u^p - v^p)\, dx + o_n(1) \\
&\le o_n(1).
\end{aligned}
$$

Thus from this inequality and (4.10), and passing to the limit as $n \to \infty$ together with $u/v, v/u \in L^\infty(\Omega)$, we infer that

$$\int_Q (L(u,v)+L(v,u))dx = 0.$$

Using Lemma 4.2 this implies $ku(x)=v(x)$ for a.e. $x\in\Omega$ for some $k>0$. Assume that $k\neq 1$. Then, without loss of generality, we can take $k<1$. Therefore, using (A3),

$$\begin{aligned}(-\Delta)_p^s(ku) &= k^{p-1}(-\Delta)_p^s u = k^{p-1}(f(x,u)-g(x,u))\\ &< f(x,ku)-g(x,ku) = (-\Delta)_p^s(v)\end{aligned}$$

from which we obtain a contradiction. Hence $k=1$ and $u=v$. □

Now we proceed to prove Theorem 3.3. First, we consider the following stationary problem.

$$\begin{gathered} u+\lambda(-\Delta)_p^s u = \tilde{g} \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega \end{gathered} \tag{4.11}$$

where $\lambda>0$ and $\tilde{g}\in L^\infty(\Omega)$. We have the following existence result for the problem (4.11).

Lemma 4.7. *For any $\lambda>0$, (4.11) admits a unique weak solution u in the sense that $u\in X_0(\Omega)$ satisfies*

$$\int_\Omega u\varphi\,dx + \lambda\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\varphi(x)-\varphi(y))\,dx\,dy = \int_\Omega \tilde{g}\varphi\,dx,$$

for all $\varphi\in X_0(\Omega)$. Moreover, $u\in C_0(\overline{\Omega})$.

Proof. The proof follows using the similar arguments as above. Precisely, for the existence of a weak solution we can argue as in the proof of Lemma 4.1. From the weak comparison principle, we obtain $\|u\|_{L^\infty(\Omega)}\leq\|\tilde{g}\|_{L^\infty(\Omega)}$ and from Theorem 4.4, $u\in C_0(\overline{\Omega})$. The uniqueness of the weak solution is a consequence of the monotonicity of the operator $(-\Delta)_p^s$. □

Proof of Theorem 3.3. Let $\mathcal{N}\in\mathbb{N}$ and $T>0$. We set $\Delta_t=\frac{T}{\mathcal{N}}$. For $0\leq n\leq\mathcal{N}$, we define $t_n=n\Delta_t$. We perform the proof along four steps.

Step 1. Approximation of h. For $n\in\{1,\dots,\mathcal{N}\}$, we define for $t\in[t_{n-1},t_n)$ and $x\in\Omega$,

$$h_{\Delta_t}(t,x)=h^n(x):=\frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s,x)ds.$$

Then by Jensen's Inequality for any $1<q<\infty$,

$$\begin{aligned}\|h_{\Delta_t}\|^q_{L^q(Q_T)} &= \Delta_t\sum_{n=1}^{\mathcal{N}}\|h^n\|^q_{L^q} = \Delta_t\sum_{n=1}^{\mathcal{N}}\|\frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s,x)ds\|^q_{L^q}\\ &\leq C(\Omega,T)\|h\|^q_{L^\infty(Q_T)}.\end{aligned}$$

Thus $h_{\Delta_t}\in L^q(Q_T)$. Also note that $h_{\Delta_t}\to h$ in $L^q(Q_T)$.

Step 2. Approximation of (3.1). We define the iterative scheme: $u^0=u_0$ and for $1\leq n\leq\mathcal{N}$, u^n is solution of

$$\begin{gathered}\frac{u^n-u^{n-1}}{\Delta_t}+(-\Delta)_p^s u^n = h^n \quad \text{in } \Omega,\\ u^n=0 \quad \text{on } \mathbb{R}^N\backslash\Omega.\end{gathered} \tag{4.12}$$

Note that the sequence $(u^n)_{n\in\{1,\dots,\mathcal{N}\}}$ is well defined. Indeed, we apply Lemma 4.7 with $g=\Delta_t h^1+u^0\in L^\infty(\Omega)$ to prove the existence of $u^1\in X_0(\Omega)\cap L^\infty(\Omega)$. Inductively we obtain the existence of (u^n), for any $n=2,\dots,\mathcal{N}$. Defining the functions u_{Δ_t} and $\tilde u_{\Delta_t}$, for $n=1,\dots,\mathcal{N}$ and $t\in[t_{n-1},t_n)$ as

$$u_{\Delta_t}(t)=u^n \quad\text{and}\quad \tilde u_{\Delta_t}(t)=\frac{(t-t_{n-1})}{\Delta_t}(u^n-u^{n-1})+u^{n-1} \tag{4.13}$$

we obtain

$$\frac{\partial \tilde u_{\Delta_t}}{\partial t}+(-\Delta)^s_p u_{\Delta_t}=h_{\Delta_t}\quad\text{in } Q_T. \tag{4.14}$$

Step 3. A priori estimates for u_{Δ_t} and $\tilde u_{\Delta_t}$. Multiplying the equation in (4.12) by (u^n-u^{n-1}) and summing from $n=1$ to $N'\le\mathcal{N}$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{N'}\Delta_t\int_\Omega\Big(\frac{u^n-u^{n-1}}{\Delta_t}\Big)^2dx\\
&+\sum_{n=1}^{N'}\int_Q\frac{|u^n(x)-u^n(y)|^{p-2}(u^n(x)-u^n(y))}{|x-y|^{N+sp}}\big((u^n-u^{n-1})(x)\big)\,dx\,dy\\
&=\sum_{n=1}^{N'}\int_\Omega h^n(u^n-u^{n-1})\,dx.
\end{aligned}\tag{4.15}$$

Hence by Young's inequality and using the convexity property

$$\begin{aligned}
\frac1p(\|u^n\|^p_{X_0}-\|u^{n-1}\|^p_{X_0})\le \frac12\int_Q&\frac{|u^n(x)-u^n(y)|^{p-2}(u^n(x)-u^n(y))}{|x-y|^{N+sp}}\\
&\times\big((u^n-u^{n-1})(x)-(u^n-u^{n-1})(y)\big)\,dx\,dy
\end{aligned}\tag{4.16}$$

we obtain

$$\begin{aligned}
&\frac12\sum_{n=1}^{N'}\Delta_t\int_\Omega\Big(\frac{u^n-u^{n-1}}{\Delta_t}\Big)^2dx\\
&+\sum_{n=1}^{N'}\int_Q\frac1p\Big(\frac{|u^n(x)-u^n(y)|^p}{|x-y|^{N+sp}}-\frac{|u^{n-1}(x)-u^{n-1}(y)|^p}{|x-y|^{N+sp}}\Big)\,dx\,dy\\
&\le\frac{C(\Omega,T)}{2}\|h\|^2_{L^\infty(Q_T)}.
\end{aligned}$$

This implies that

$$\Big(\frac{\partial\tilde u_{\Delta t}}{\partial t}\Big)_{\Delta t}\text{ is bounded in } L^2(Q_T)\text{ uniformly in }\Delta_t, \tag{4.17}$$

$$\begin{gathered}(u_{\Delta_t})\text{ and }(\tilde u_{\Delta_t})\text{ are bounded in } L^\infty(0,T,X_0(\Omega))\cap L^\infty(Q_T)\\ \text{and uniformly in }\Delta_t.\end{gathered} \tag{4.18}$$

Furthermore, we have

$$\|u_{\Delta_t}-\tilde u_{\Delta_t}\|_{L^\infty(0,T:L^2(\Omega))}\le\max_{n=1,\dots,N}\|u^n-u^{n-1}\|_{L^2(\Omega)}\le C\Delta_t^{1/2}. \tag{4.19}$$

Therefore for $\Delta_t\to0$, there exist $u,v\in L^\infty(0,T,X_0(\Omega))\cap L^\infty(Q_T)$ such that (up to a subsequence)

$$\tilde u_{\Delta_t}\overset{*}{\rightharpoonup}u\text{ in }L^\infty(0,T,X_0(\Omega)),\quad u_{\Delta_t}\overset{*}{\rightharpoonup}v\text{ in }L^\infty(0,T,X_0(\Omega)), \tag{4.20}$$

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q_T). \tag{4.21}$$

It follows from (4.19) that $u \equiv v$.

Step 4. u satisfies (3.1). Plugging (4.17), (4.18) and since $X_0(\Omega) \hookrightarrow L^2(\Omega)$ compactly, the Aubin-Simon's result implies that $\{u_{\Delta_t}\}$ is compact in $C([0,T];L^2(\Omega))$. Now using interpolation we obtain, up to a subsequence

$$\tilde{u}_{\Delta_t} \to u \in C([0,T], L^q(\Omega)), \quad \text{for all } q > 1. \tag{4.22}$$

and hence, from (4.19), we have

$$u_{\Delta_t} \to u \in L^\infty([0,T], L^q(\Omega)), \quad \text{for all } q > 1. \tag{4.23}$$

Multiplying (4.14) by $(u_{\Delta_t} - u)$ we obtain

$$\int_0^T\int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}(u_{\Delta_t} - u)\,dx\,dt + \int_0^T \langle (-\Delta)_p^s u_{\Delta_t}, u_{\Delta_t} - u\rangle\,dt = \int_0^T\int_\Omega h_{\Delta_t}(u_{\Delta_t} - u)\,dx\,dt.$$

Rearranging the terms in the above equation and using (4.19)-(4.20) we have

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}\Big)(\tilde{u}_{\Delta_t} - u)\,dx\,dt \\
&+ \int_0^T \langle (-\Delta)_p^s u_{\Delta_t} - (-\Delta)_p^s u, u_{\Delta_t} - u\rangle dt = o_{\Delta_t}(1).
\end{aligned}$$

Thus we obtain

$$\frac{1}{2}\int_\Omega |\tilde{u}_{\Delta_t}(T) - u(T)|^2 dx + \int_0^T \langle (-\Delta)_p^s u_{\Delta_t} - (-\Delta)_p^s u, u_{\Delta_t} - u\rangle = o_{\Delta_t}(1).$$

Using (4.22), we obtain

$$\int_0^T \langle (-\Delta)_p^s u_{\Delta_t} - (-\Delta)_p^s u, u_{\Delta_t} - u\rangle dt = o_{\Delta_t}(1).$$

This implies

$$\begin{aligned}
&\int_0^T\int_Q \Big(\big(|u_{\Delta_t}(x) - u_{\Delta_t}(y)|^{p-2}(u_{\Delta_t}(x) - u_{\Delta_t}(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))\big) \\
&\times \big(u_{\Delta_t}(x) - u_{\Delta_t}(y) - u(x) + u(y)\big)\Big)\frac{1}{|x-y|^{N+sp}}\,dx\,dy\,dt = o_{\Delta_t}(1).
\end{aligned}$$

Thus by (2.2), for $p \ge 2$ we conclude that

$$\int_0^T\int_Q \frac{|u_{\Delta_t}(x) - u_{\Delta_t}(y) - u(x) + u(y)|^p}{|x-y|^{N+sp}} = o_{\Delta_t}(1).$$

Also for $1 < p \le 2$, (2.2) together with the Hölder inequality in $\mathbb{R}^2$ imply

$$\int_0^T\int_Q \frac{|u_{\Delta_t}(x) - u_{\Delta_t}(y)|^2}{(|u_{\Delta_t}(x) - u_{\Delta_t}(y)|^p + |u(x) - u(y)|^p)^{\frac{2-p}{p}}\,|x-y|^{N+sp}} = o_{\Delta_t}(1).$$

Therefore using Hölder's inequality we obtain

$$\begin{aligned}
0 &\le \int_0^T\int_Q \frac{|(u_{\Delta_t}(x) - u_{\Delta_t}(y) - u(x) + u(y))|^p}{|x-y|^{N+sp}}\,dx\,dy\,dt \\
&= \int_0^T\int_Q \Big(\big|\big(u_{\Delta_t}(x) - u_{\Delta_t}(y) - u(x) + u(y)\big)\big|^p
\end{aligned}$$

$$
\begin{aligned}
&\times \left(|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^p+|u(x)-u(y)|^p\right)^{\frac{2-p}{2}}\Big) \\
&\times \frac{1}{\left(|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^p+|u(x)-u(y)|^p\right)^{\frac{2-p}{2}}|x-y|^{(N+sp)(\frac{p}{2}+\frac{2-p}{2})}}\,dx\,dy\,dt \\
\le& \Big(\int_0^T\int_Q \frac{|u_{\Delta_t}(x)-u_{\Delta_t}(y)+u(y)|^2}{(|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^p+|u(x)-u(y)|^p)^{\frac{(2-p)}{p}}|x-y|^{N+sp}}\Big)^{p/2} \\
&\times \Big(\int_0^T\int_Q \frac{(|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^p+|u(x)-u(y)|^p)}{|x-y|^{N+sp}}\Big)^{\frac{2-p}{2}} \\
\le& \Big(\int_0^T\int_Q \frac{|u_{\Delta_t}(x)-u_{\Delta_t}(y)+u(y)|^2}{(|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^p+|u(x)-u(y)|^p)^{\frac{(2-p)}{p}}|x-y|^{N+sp}}\Big)^{p/2} \\
&\times \left(\|u_{\Delta_t}\|^p_{X_0(\Omega)}+\|u\|^p_{X_0(\Omega)}\right)^{\frac{2-p}{2}} = o_{\Delta_t}(1).
\end{aligned}
$$

Thus in both cases we have

$$
\int_0^T\int_Q \frac{|(u_{\Delta_t}(x)-u_{\Delta_t}(y)-u(x)+u(y))|^p}{|x-y|^{N+sp}}\,dx\,dy\,dt \to 0.
$$

This implies u_{Δ_t} converges to u in $L^p(0,T,X_0(\Omega))$. Therefore, for $\phi\in C_0^\infty(Q)$,

$$
\begin{aligned}
&\int_0^T\int_Q \frac{|u_{\Delta_t}(x)-u_{\Delta_t}(y)|^{p-2}(u_{\Delta_t}(x)-u_{\Delta_t}(y))(\phi(x)-\phi(y))\,dx\,dy\,dt}{|x-y|^{N+sp}} \\
&\to \int_0^T\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))\,dx\,dy\,dt}{|x-y|^{N+sp}}
\end{aligned}
$$

Hence, we conclude passing to the limit, in the distribution sense, in equation (4.14) that u is a weak solution of (3.1). Also u is the unique weak solution of (3.1). Indeed, assume that there exists v a weak solution of (3.1). Then, we have for any arbitrary $t_0\in(0,T]$

$$
\int_0^{t_0}\int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)(x,t)\,dx\,dt+\int_0^{t_0}\langle(-\Delta)^s_p u-(-\Delta)^s_p v,u-v\rangle dt=0.
$$

Since $(-\Delta)^s_p$ is monotone, this together with $u(0)=v(0)$ imply

$$
\begin{aligned}
\frac12\int_\Omega (u(t_0)-v(t_0))^2dx &= \int_0^{t_0}\frac{\partial}{\partial t}\int_\Omega \frac12(u-v)^2dx\,dt \\
&= \int_{(0,t_0)\times\Omega}\frac{\partial(u-v)}{\partial t}(u-v)\,dx\,dt \le 0
\end{aligned}
$$

from which it follows that $u\equiv v$. Next we claim that $u\in C([0,T];X_0(\Omega))$ and satisfies (3.3). Using (4.22) and the compact embedding of $X_0(\Omega)$ into $L^p(\Omega)$, it is easy to check that $u(\cdot,t)\in X_0(\Omega)$ and the map $[0,T]\ni t\to u(\cdot,t)\in X_0(\Omega)$, is weakly continuous. Therefore, $\|u(\cdot,t_0)\|_{X_0(\Omega)}\le \liminf_{t\to t_0}\|u(\cdot,t_0)\|_{X_0(\Omega)}$. Now multiplying (4.12) by u^n-u^{n-1}, taking integration over $\mathbb{R}^N$ both sides, summing from $1 \le n = N''$ to $N' \le \mathcal{N}$, and using (4.16) we obtain

$$
\begin{aligned}
&\Delta_t \sum_{n=N''}^{n=N'} \Big(\frac{u^n - u^{n-1}}{\Delta_t}\Big)^2 + \frac{1}{p}\left(\|u^{N'}\|_{X_0(\Omega)} - \|u^{N''-1}\|_{X_0(\Omega)}\right) \\
&\le \sum_{n=N''}^{n=N'} \Delta_t \int_\Omega h_{\Delta_t}\Big(\frac{u^n - u^{n-1}}{\Delta_t}\Big)\,dx.
\end{aligned}
\tag{4.24}
$$

For any $t \in [t_0, T]$, choose N'' and N' such that $N''\Delta_t \to t$ and $N'\Delta_t \to t_0$. Then (4.24) gives

$$
\begin{aligned}
&\int_{t_0}^t \int_\Omega \Big(\frac{\partial u}{\partial t}\Big)^2 dx\,dt + \frac{1}{p}\|u(\cdot,t)\|_{X_0(\Omega)} \\
&\le \int_{t_0}^t \int_\Omega h\Big(\frac{\partial u}{\partial t}\Big)\,dx\,dt + \frac{1}{p}\|u(\cdot,t_0)\|_{X_0(\Omega)}.
\end{aligned}
\tag{4.25}
$$

Now from the above inequality and (4.23) we infer that

$$
\limsup_{t\to t_0^+} \|u(\cdot,t)\|_{X_0(\Omega)} \le \|u(\cdot,t_0)\|_{X_0(\Omega)},
$$

and hence the map $[0,T] \ni t \to u(\cdot,t) \in X_0(\Omega)$ is right continuous. Now for proving the left continuity, take $0 < k \le t - t_0$ and multiply (3.1) by $\tau_k(u)(s) = \frac{u(x,s+k)-u(x,s)}{k}$ and integrate over $(t_0,t)\times\mathbb{R}^N$. Using (4.16), we obtain

$$
\begin{aligned}
&\int_{t_0}^t \int_\Omega \tau_k(u)\frac{\partial u}{\partial t}\,dxd\theta + \frac{1}{pk}\int_{t_0}^t \|u(\theta+k)\|^p_{X_0(\Omega)} - \|u(\theta)\|^p_{X_0(\Omega)}d\theta \\
&\ge \int_{t_0}^t \int_\Omega \tau_k(u)h\,dxd\theta.
\end{aligned}
\tag{4.26}
$$

It follows that

$$
\begin{aligned}
&\int_{t_0}^t \int_\Omega \tau_k(u)\frac{\partial u}{\partial t}\,dxd\theta + \frac{1}{pk}\Big(\int_t^{t+k} \|u(\theta)\|^p_{X_0(\Omega)}d\theta - \int_{t_0}^{t_0+k} \|u(\theta)\|^p_{X_0(\Omega)}d\theta\Big) \\
&\ge \int_{t_0}^t \int_\Omega \tau_k(u)h\,dxd\theta.
\end{aligned}
\tag{4.27}
$$

By the right continuity of $t \mapsto u(\cdot,t)$, as $k \to 0^+$, we have

$$
\begin{aligned}
\frac{1}{pk}\int_t^{t+k} \|u(\theta)\|^p_{X_0(\Omega)}d\theta &\to \frac{1}{p}\|u(t)\|^p_{X_0(\Omega)}, \\
\frac{1}{pk}\int_{t_0}^{t_0+k} \|u(\theta)\|^p_{X_0(\Omega)}d\theta &\to \frac{1}{p}\|u(t_0)\|^p_{X_0(\Omega)}.
\end{aligned}
$$

Hence as $k \to 0^+$, (4.27) becomes

$$
\int_{t_0}^t \int_\Omega \Big(\frac{\partial u}{\partial t}\Big)^2 dxd\theta + \frac{1}{p}\|u(\cdot,t)\|^p_{X_0(\Omega)} \ge \int_{t_0}^t \int_\Omega h\frac{\partial u}{\partial t}\,dx\,ds + \frac{1}{p}\|u(\cdot,t_0)\|^p_{X_0(\Omega)}. \tag{4.28}
$$

From the above inequality, we deduce that we have the equality in (4.25) and hence the claim. This completes the proof of the Theorem 3.3. $\square$

Proof of Theorem 3.9. First we show that there exists a sub-solution $\underline{u}$ and a super solution $\bar{u}$ of (3.7) such that $\underline{u}, \bar{u} \in C^+_{d^s(\Omega)}$. Since f and g satisfy $(A4)$ and using the fact that $\phi_{1,s,p} \in C_{d^s}(\Omega)$, we can choose $\epsilon > 0$ small enough such that

$$(-\Delta)^s_p(\epsilon\phi_{1,s,p}) = \lambda_{1,s,p}\epsilon^{p-1}\phi^{p-1}_{1,s,p} \leq f(x, \epsilon\phi_{1,s,p}) - g(x, \epsilon\phi_{1,s,p}). \tag{4.29}$$

and $\epsilon\phi_{1,s,p} \leq u_0$. Also let w be the solution of the following problem

$$\begin{gathered}(-\Delta)^s_p w = \beta w^{p-1} + C \quad \text{in } \Omega\\ w > 0 \quad \text{in } \Omega\\ w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered}$$

where $\limsup_{\theta\to\infty} \frac{f(x,\theta)}{\theta^{p-1}} \leq \beta < \lambda_{1,s,p}$ and $C > 0$. Then arguing as in the proof of Proposition 4.3, we obtain $w \in L^\infty(\Omega)$ and hence by [15, Theorem 1.1], $w \in C^\alpha(\overline{\Omega})$ with $\alpha \in (0, s]$. Furthermore from [15, Theorem 4.4] we have for some constant $C_0 > 0$, $|w(x)| \leq C_0 d^s(x)$ a.e. $x \in \Omega$. Then, $w \in C_{d^s}(\Omega)$ and from the Hopf lemma (see Lemma 4.5), we obtain $w \in C^+_{d^s}(\Omega)$. Again using the fact that f and g satisfy $(A4)$, we have that for some constant $C' > 0$,

$$f(x, \theta) - g(x, \theta) \leq \beta\theta + C'.$$

Then for $M > 0$ large enough, we obtain

$$(-\Delta)^s_p(Mw) = \beta(Mw)^{p-1} + CM^{p-1} \geq f(x, Mw) - g(x, Mw) \tag{4.30}$$

and $u_0 \leq Mw$. Then $\underline{u} := \epsilon\phi_{1,s,p}$ and $\bar{u} := Mw$ are the required sub-solution and the super-solution of (3.7), respectively, such that $\underline{u}, \bar{u} \in C^+_{d^s(\Omega)}$. We define the sequence (u^n) by the iterative scheme: $u^0 = u_0$ and

$$\begin{gathered}u^n + (-\Delta)^s_p u^n + Ku^n = u^{n-1} + f(x, u^{n-1}) - g(x, u^{n-1}) + Ku^{n-1} \quad \text{in } \Omega,\\ u^n = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.\end{gathered}$$

where $K > 0$ is chosen such that the map $t \mapsto Kt + f(x,t) - g(x,t)$ is nondecreasing in $[0, \|\bar{u}\|_{X_0}]$, for a.e. $x \in \Omega$. Then the existence of a weak solution $u_\infty \in [\underline{u}, \overline{u}]$ to (3.7) is obtained by the standard arguments of the monotone iteration method. Also we have $\underline{u} \leq u_\infty \leq \bar{u}$ in Ω and $u_\infty \in C^+_{d^s}(\Omega)$. The uniqueness of the solution to (3.7) follows from Theorem 4.6. □

Proof of Theorem 3.4. Now we proceed as in the proof of Theorem 3.3. Set $\Delta_t := \frac{T}{\mathcal{N}}$, $\mathcal{N} \in \mathbb{N}$ and let $\underline{u}$ and $\bar{u}$ be as defined in the proof of Theorem 3.9. We define the sequence $\{u^n\} \in X_0(\Omega)$ as the solutions to the iterative scheme: $u^0 = u_0$ and

$$u^n + \Delta_t((-\Delta)^s_p u^n + Ku^n) = u^{k-1} + \Delta_t(f(x, u^{n-1}) - g(x, u^{n-1}) + Ku^{n-1}) \quad \text{in } \Omega,$$

The existence of $u^n \in C^+_{d^s}(\Omega)$, for any $n \geq 1$ follows from Lemma 4.7 and the Hopf Lemma. Note that from Theorem 3.9, we have $\underline{u} \leq u_0 \leq \bar{u}$, a.e. in Ω. We claim that $\underline{u} \leq u^k \leq \bar{u}$. Indeed for $k = 1$, we have

$$\underline{u} - u^1 + \Delta_t((-\Delta)^s_p\underline{u} - (-\Delta)^s_p u^1) \leq \underline{u} - u^0 + \Delta_t(f(x, \underline{u}) - f(x, u^0) - (g(x, \underline{u}) - g(x, u^0))).$$

Therefore,

$$\underline{u} - u^1 + \Delta_t((-\Delta)^s_p\underline{u} - (-\Delta)^s_p u^1 + K(\underline{u} - u^1)) \leq 0.$$

Thus by comparison principle given in [15, Theorem 2.10], we have $\underline{u} \leq u^1$. Similarly we prove $u^1 \leq \bar{u}$. The rest of the claim follows by induction. Now we define u_{Δ_t} and $\tilde{u}_{\Delta_t}$ as in the proof of Theorem 3.3, and

$$h_{\Delta_t}(t,x) := f(x, u_{\Delta_t}(t-\Delta_t, x)) - g(x, u_{\Delta_t}(t-\Delta_t, x)).$$

Then clearly as $\underline{u} \leq u_{\Delta_t} \leq \bar{u}$, $h_{\Delta_t}(t,x) \in L^\infty(Q)$.

Therefore following a similar arguments as in the proof of Theorem 3.3, for $\Delta_t \to 0$, there exist $u \in L^\infty(0,T,X_0(\Omega))$ such that (up to a subsequence)

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T,X_0(\Omega)) \text{ and } L^\infty(Q_T), \tag{4.31}$$

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T). \tag{4.32}$$

Again using a similar arguments as in the proof of Theorem 3.3, we have

$$\tilde{u}_{\Delta_t} \to u \in C([0,T], L^q(\Omega)) \text{ and } \tilde{u}_{\Delta_t} \to u \in L^\infty([0,T], L^q(\Omega)), \tag{4.33}$$

for all $q > 1$. Also using the Lipschitz continuity of f and g we have

$$\begin{aligned} &\|h_{\Delta_t}(\cdot,t) - (f-g)(\cdot, u(\cdot,t))\|_{L^2(\Omega)} \\ &= \|(f-g)(\cdot, u_{\Delta_t}(\cdot, t-\Delta_t)) - (f-g)(\cdot, u(\cdot,t))\|_{L^2(\Omega)} \\ &\leq C\|u_{\Delta_t}(\cdot, t-\Delta_t) - u(\cdot,t)\|_{L^2(\Omega)} \end{aligned} \tag{4.34}$$

Thus (4.33)-(4.34), we deduce that $h_{\Delta_t}(x,t) \to f(x,u(x))$ in $L^\infty(0,T;L^2(\Omega))$. The rest of the proof follows using step 4 of the Theorem 3.3. $\square$

Now we study the regularity of the solutions of (3.1) and (3.2) given in Theorem 3.6 and Theorem 3.7.

Proof of Lemma 3.5. Let $h_1, h_2 \in L^\infty(\Omega)$ and $u, v \in X_0(\Omega)$, respectively, be the solutions to

$$\begin{gathered} u + (-\Delta)^s_p(u) = h_1 \quad \text{in } \Omega, \\ v + (-\Delta)^s_p(v) = h_2 \quad \text{in } \Omega \end{gathered}$$

For $w \in L^\infty(\Omega)$, define $w^+(x) = \max\{w(x), 0\}$. Setting

$$\Omega_+ = \{x \in \Omega : (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})^+(x) > 0\},$$

and noting that for $x \in \Omega_+$ and $y \in \mathbb{R}^N \setminus \Omega_+$, $u(x) - u(y) \geq v(x) - v(y)$, we obtain

$$\begin{aligned} &\langle (-\Delta)^s_p u - (-\Delta)^s_p v, (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})^+ \rangle \\ &= \int_{\Omega_+} \Big(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy \Big) (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\ &\quad - \int_{\Omega_+} \Big(\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} dy \Big) (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\ &= \int_{\Omega_+} \Big(\int_{\Omega_+} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy \Big) \\ &\quad \times (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\ &\quad + \int_{\Omega_+} \Big(\int_{\mathbb{R}^N \setminus \Omega_+} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy \Big) \\ &\quad \times (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \end{aligned}$$

$$
\begin{aligned}
&- \int_{\Omega_+} \Big(\int_{\Omega_+} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} dy \Big) \\
&(u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\
&- \int_{\Omega_+} \Big(\int_{\mathbb{R}^N \setminus \Omega_+} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} dy \Big) \\
&\times (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\
\geq& \int_{\Omega_+} \Big(\int_{\Omega_+} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy \Big) (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\
&- \int_{\Omega_+} \Big(\int_{\Omega_+} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+sp}} dy \Big) (u - v - \|h_1 - h_2\|_{L^\infty(\Omega)})(x)\, dx \\
=& \int_{\Omega_+} \int_{\Omega_+} \Big(\big(|u(x)-u(y)|^{p-2}(u(x)-u(y)) - |v(x)-v(y)|^{p-2}(v(x)-v(y)) \big) \\
&\times \big((u-v)(x) - (u-v)(y) \big) \Big) \frac{1}{2|x-y|^{N+sp}}\, dy\, dx \geq 0.
\end{aligned}
$$

This implies the m-accretivity of A in $L^\infty(\Omega)$. $\square$

Now Theorem 3.6 and Theorem 3.7 follow using the approach as in [6, Theorem 4.2 and 4.4]. Next we prove the asymptotic behavior of the solution of (3.2) as given in the Theorem 3.10.

Proof of Theorem 3.10. Let $\underline{u}$ and $\bar{u}$ be the sub and super solutions respectively to (3.7) as constructed in the proof of the Theorem 3.9 such that $\underline{u} \leq u_0 \leq \bar{u}$. Let u_1 and u_2 be the unique and global solution to (3.2) with the initial data $\underline{u}$ and $\bar{u}$ respectively. Note that using the approach as in proof of [5, Theorem 0.15] we have $\underline{u}, \bar{u} \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$ and $\underline{u} \leq u_1(t) \leq u(t) \leq u_2(t) \leq \bar{u}$ and $t \mapsto u_1(t)$ ($t \mapsto u_2(t)$) is non-decreasing (non-increasing respectively) and converges a.e. to u_1^∞ (u_2^∞ respectively), as $t \to \infty$. Now from the semi-group theory we have

$$
\begin{aligned}
u_1^\infty &= \lim_{t' \to \infty} S(t'+t)(\underline{u}) = S(t)\big(\lim_{t' \to \infty} S(t')\underline{u} \big) = S(t) u_1^\infty \\
u_2^\infty &= \lim_{t' \to \infty} S(t'+t)(\bar{u}) = S(t)\big(\lim_{t' \to \infty} S(t')\bar{u} \big) = S(t) u_2^\infty
\end{aligned}
$$

where $S(t)$ is the semi-group on $L^\infty(\Omega)$ generated by the given evolution equation. This implies that u_1^∞ and u_2^∞ are the stationary solutions to (3.2). From the uniqueness of the solution in Theorem 3.9 we obtain that $u_1^\infty = u_\infty = u_2^\infty$. Thus $u(t) \to u_\infty$ in $L^\infty(\mathbb{R}^N)$. $\square$

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