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COMPARISON PRINCIPLES FOR DIFFERENTIAL EQUATIONS INVOLVING CAPUTO FRACTIONAL DERIVATIVE WITH MITTAG-LEFFLER NON-SINGULAR KERNEL

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ABSTRACT. In this article we study linear and nonlinear differential equations involving the Caputo fractional derivative with Mittag-Leffler non-singular kernel of order $0 < \alpha < 1$. We first obtain a new estimate of the fractional derivative of a function at its extreme points and derive a necessary condition for the existence of a solution to the linear fractional equation. The condition obtained determines the initial condition of the associated fractional initial-value problem. Then we derive comparison principles for the linear fractional equations, and apply these principles for obtaining norm estimates of solutions and to obtain a uniqueness results. We also derive lower and upper bounds of solutions. The applicability of the new results is illustrated through several examples.

1. INTRODUCTION

Fractional differential equations have been implemented to model various problems in several fields, [15, 19, 20, 21]. The non-locality of the fractional derivative makes fractional models more practical than the usual ones, especially for systems which involve memory. In recent years there are great interests to develop new types of non-local fractional derivative with non-singular kernel, see [11, 13]. The idea is to have more types of non-local fractional derivatives, and it is the role of application that will determine which fractional model is appropriate. The theory of fractional models is effected by the type of the fractional derivative. Therefore, several papers have been devoted recently to study the new types of fractional derivatives and their applications, see [2, 3, 7] for the Caputo-Fabrizio fractional derivative and [4, 12, 14, 16, 22, 23] for the Abdon-Baleanu fractional derivative.

In this article, we analyze the solutions of a class of fractional differential equations involving the Caputo fractional derivative with Mittag-Leffler non-singular kernel of order $0 < \alpha < 1$. To the best of our knowledge this is the first theoretical study of fractional differential equations with fractional derivative of non-singular kernel. We start with the definition and main properties of the nonlocal fractional derivative with Mittag-Leffler non-singular kernel. For more details the reader is referred to [11, 12, 1].

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Definition 1.1. Let $f \in H^1(a,b)$, $a < b$, $\alpha \in (0,1)$, the left Caputo fractional derivative with Mittag-Leffler non-singular kernel is defined by

$$({}^{ABC}{}_aD^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t E_\alpha\big[-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big]f'(s)ds. \tag{1.1}$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$, and $E_\alpha[s]$ is the well known Mittag-Leffler function. The derivative is known in the literature by the Abdon-Baleanu fractional derivative.

Definition 1.2. Let $f \in H^1(a,b)$, $a < b$, $\alpha \in (0,1)$, the left Riemann-Liouville fractional derivative with Mittag-Leffler non-singular kernel is defined by

$$({}^{ABR}{}_aD^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t E_\alpha\big[-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big]f(s)ds. \tag{1.2}$$

The associated fractional integral is defined by

$$({}^{AB}{}_aI^\alpha f)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}({}_aI^\alpha f)(t), \tag{1.3}$$

where $({}_aI^\alpha f)(t)$ is the left Riemann-Liouville fractional integral of order $\alpha > 0$ defined by

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds.$$

The following statements hold:

$$({}^{ABC}{}_0D^\alpha f)(t) = ({}^{ABR}{}_0D^\alpha f)(t) - \frac{B(\alpha)}{1-\alpha}f(0)E_\alpha[-\frac{\alpha}{1-\alpha}t^\alpha], \tag{1.4}$$

$$({}^{ABR}{}_aD^\alpha\,{}^{AB}{}_aI^\alpha f)(t) = f(t), \tag{1.5}$$

$$({}^{AB}{}_aI^\alpha\,{}^{ABR}{}_aD^\alpha f)(t) = f(t). \tag{1.6}$$

The rest of the paper is organized as follows. In Section 2, we present a new estimate of the fractional derivative of a function at its extreme points. In Section 3, we develop new comparison principles for linear fractional equations and obtain a norm bound to their solutions. We also, obtain the solution for a class of linear equations in a closed form, and present a necessary condition for the existence of their solutions. In Section 4, we consider nonlinear fractional equations. We obtain a uniqueness result and derive upper and lower bounds to the solution of the problem. Finally we present some examples to illustrate the applicability of the obtained results.

2. Estimates of fractional derivatives at extreme points

We start with estimating the fractional derivative of a function at its extreme points, this result is analogous to the ones obtained in [5] for the Caputo and Riemann-Liouville fractional derivatives. The applicability of these results were indicated in ([6]-[10]) by establishing new comparison principles and studying various fractional diffusion models. Therefore, the current result can be used to study fractional diffusion models involving the Caputo and Riemann-Liouville fractional derivatives with Mittag-Leffler non-singular kernel, and we leave this for a future work.

Lemma 2.1. *Let a function $f \in H^1(a,b)$ attain its maximum at a point $t_0 \in [a,b]$ and $0 < \alpha < 1$. Then*

$$({}^{ABC}{}_aD^\alpha f)(t_0) \geq \frac{B(\alpha)}{1-\alpha} E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha](f(t_0)-f(a)) \geq 0\,. \tag{2.1}$$

Proof. We define the auxiliary function $g(t) = f(t_0) - f(t)$, $t \in [a,b]$. Then it follows that $g(t) \geq 0$, on $[a,b]$, $g(t_0) = g'(t_0) = 0$ and $({}^{ABC}{}_aD^\alpha g)(t) = -({}^{ABC}{}_aD^\alpha f)(t)$. Since $g \in H^1(a,b)$, then g' is integrable and integrating by parts with

$$u = E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha], \quad dv = g'(s)ds,$$

yields

$$\begin{aligned}
({}^{ABC}{}_aD^\alpha g)(t_0) &= \frac{B(\alpha)}{1-\alpha}\int_a^{t_0} E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha]g'(s)\,ds \\
&= \frac{B(\alpha)}{1-\alpha}\Big(E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha]g(s)|_a^{t_0} \\
&\quad - \int_a^{t_0}\frac{d}{ds}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha]g(s)ds\Big) \\
&= \frac{B(\alpha)}{1-\alpha}\Big(E_\alpha[0]g(t_0) - E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha]g(a) \\
&\quad - \int_a^{t_0}\frac{d}{ds}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha]g(s)ds\Big) \\
&= \frac{B(\alpha)}{1-\alpha}\Big(-E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha]g(a) \\
&\quad - \int_a^{t_0}\frac{d}{ds}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha]g(s)ds\Big).
\end{aligned} \tag{2.2}$$

We recall that for $0 < \alpha < 1$, see [17], we have

$$E_\alpha[-t^\alpha] = \int_0^\infty e^{-rt}K_\alpha(r)dr,$$

where

$$K_\alpha(r) = \frac{1}{\pi}\frac{r^{\alpha-1}\sin(\alpha\pi)}{r^{2\alpha}+2r^\alpha\cos(\alpha\pi)+1} > 0.$$

Thus,

$$\begin{aligned}
&\frac{d}{ds}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-s)^\alpha] \\
&= \frac{d}{ds}E_\alpha\Big[-\Big((\frac{\alpha}{1-\alpha})^{1/\alpha}(t_0-s)\Big)^\alpha\Big] \\
&= \frac{d}{ds}\int_0^\infty e^{-r(\frac{\alpha}{1-\alpha})^{1/\alpha}(t_0-s)}K_\alpha(r)dr = \int_0^\infty \frac{d}{ds}e^{-r(\frac{\alpha}{1-\alpha})^{1/\alpha}(t_0-s)}K_\alpha(r)dr \\
&= (\frac{\alpha}{1-\alpha})^{1/\alpha}\int_0^\infty re^{-r(\frac{\alpha}{1-\alpha})^{1/\alpha}(t_0-s)}K_\alpha(r)dr > 0,
\end{aligned} \tag{2.3}$$

which together with $g(t) \geq 0$ on $[a, b]$, will lead to the integral in (2.2) is nonnegative. We recall here that $E_\alpha[t] > 0$, $0 < \alpha < 1$, see [18], and thus

$$\begin{aligned}({}^{ABC}{}_aD^\alpha g)(t_0) &\leq \frac{B(\alpha)}{1-\alpha}\Big(- E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha]g(a)\Big)\\ &= -\frac{B(\alpha)}{1-\alpha}\ E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha](f(t_0)-f(a)) \leq 0.\end{aligned} \tag{2.4}$$

The last inequality yields

$$-({}^{ABC}{}_aD^\alpha f)(t_0) \leq -\frac{B(\alpha)}{1-\alpha}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha](f(t_0)-f(a)) \leq 0,$$

which proves the result. □

By applying analogous steps for $-f$ we have the following result.

Lemma 2.2. *Let a function $f \in H^1(a, b)$ attain its minimum at a point $t_0 \in [a, b]$ and $0 < \alpha < 1$. Then*

$$({}^{ABC}{}_aD^\alpha f)(t_0) \leq \frac{B(\alpha)}{1-\alpha}E_\alpha[-\frac{\alpha}{1-\alpha}t_0](f(t_0)-f(a)) \leq 0\,. \tag{2.5}$$

Lemma 2.3. *Let a function $f \in H^1(a, b)$ then it holds that*

$$({}^{ABC}{}_aD^\alpha f)(a) = 0, \quad 0 < \alpha < 1. \tag{2.6}$$

Proof. Because $E_\alpha[-\frac{\alpha}{1-\alpha}(t-s)]$ is continuous on $[a, b]$, then it is in $L^2[a, b]$. Applying the Cauchy-Schwartz inequality we have

$$|({}^{ABC}{}_aD^\alpha f)(t)|^2 \leq \frac{B^2(\alpha)}{(1-\alpha)^2}\int_a^t \Big(E_\alpha[-\frac{\alpha}{1-\alpha}(t-s)^\alpha]\Big)^2 ds \int_a^t \Big(f'(s)\Big)^2 ds. \tag{2.7}$$

Since $f \in H^1(a, b)$ then f' is square integrable and it holds that $\int_a^a \big(f'(s)\big)^2 ds = 0$. The result is obtained as the first integral in (2.7) is bounded. □

3. Linear equations

We implement the results in Section 1 to obtain new comparison principles for the linear fractional differential equations of order $0 < \alpha < 1$, and to derive a necessary condition for the existence of their solutions. We then use these principles to obtain a norm bound of the solution. We also present the solution of certain linear equation by the Laplace transform.

Lemma 3.1 (Comparison Principle-1)**.** *Let a function $u \in H^1(a, b)\cap C[a, b]$ satisfies the fractional inequality*

$$P_\alpha(u) = ({}^{ABC}{}_aD^\alpha u)(t) + p(t)u(t) \leq 0,\ t > a,\ 0 < \alpha < 1, \tag{3.1}$$

where $p(t) \geq 0$ is continuous on $[a, b]$ and $p(a) \neq 0$. Then $u(t) \leq 0$, $t \geq a$.

Proof. Since $u \in H^1(a, b)$ then by Lemma 2.3 we have $({}^{ABC}{}_aD^\alpha u)(a) = 0$. By the continuity of the solution, the fractional inequality (3.1) yields $p(a)u(a) \leq 0$, and hence $u(a) \leq 0$. Assume by contradiction that the result is not true, because u is continuous on $[a, b]$ then u attains absolute maximum at $t_0 \geq a$ with $u(t_0) > 0$. Since $u(a) \leq 0$, then $t_0 > a$. Applying the result of Lemma 2.1 we have

$$({}^{ABC}{}_aD^\alpha u)(t_0) \geq \frac{B(\alpha)}{1-\alpha}E_\alpha[-\frac{\alpha}{1-\alpha}(t_0-a)^\alpha](u(t_0)-u(a)) > 0.$$

We have

$$({}^{ABC}{}_aD^\alpha u)(t_0) + p(t_0)u(t_0) \geq ({}^{ABC}{}_aD^\alpha u)(t_0) > 0,$$

which contradicts the fractional inequality (3.1), and completes the proof. □

Corollary 3.2 (Comparison Principle-2)**.** *Let $u_1, u_2 \in H^1(a,b) \cap C[a,b]$ be the solutions of*

$$\begin{aligned} ({}^{ABC}{}_aD^\alpha u_1)(t) + p(t)u_1(t) = g_1(t), \quad t > a,\ 0 < \alpha < 1, \\ ({}^{ABC}{}_aD^\alpha u_2)(t) + p(t)u_2(t) = g_2(t), \quad t > a,\ 0 < \alpha < 1, \end{aligned}$$

where $p(t) \geq 0, g_1(t), g_2(t)$ are continuous on $[a,b]$ and $p(a) \neq 0$. If $g_1(t) \leq g_2(t)$, then

$$u_1(t) \leq u_2(t), \quad t \geq a.$$

Proof. Let $z = u_1 - u_2$, then

$$P_\alpha(z) = ({}^{ABC}{}_aD^\alpha z)(t) + p(t)z(t) = g_1(t) - g_2(t) \leq 0, \quad t > a,\ 0 < \alpha < 1. \tag{3.2}$$

By Lemma 3.1 we have $z(t) \leq 0$, and hence the result follows. □

Lemma 3.3. *Let $u \in H^1(a,b)$ be the solution of*

$$({}^{ABC}{}_aD^\alpha u)(t) + p(t)u(t) = g(t), \quad t > a,\ 0 < \alpha < 1, \tag{3.3}$$

where $p(t) > 0$ is continuous on $[a,b]$. Then it holds that

$$\|u\|_{[a,b]} = \max_{t\in[a,b]} |u(t)| \leq M = \max_{t\in[a,b]} \{|\frac{g(t)}{p(t)}|\}.$$

Proof. We have $M \geq |\frac{g(t)}{p(t)}|$, or $Mp(t) \geq |g(t)|$ for $t \in [a,b]$. Let $v_1 = u - M$, then

$$\begin{aligned} P_\alpha(v_1) &= ({}^{ABC}{}_aD^\alpha v_1)(t) + p(t)v_1(t) = ({}^{ABC}{}_aD^\alpha u)(t) + p(t)u(t) - p(t)M \\ &= g(t) - p(t)M \leq |g(t)| - p(t)M \leq 0. \end{aligned}$$

Thus by Lemma 3.1 we have $v_1 = u - M \leq 0$, which implies

$$u \leq M. \tag{3.4}$$

Analogously, let $v_2 = -M - u$, then it holds that

$$\begin{aligned} P_\alpha(v_2) &= ({}^{ABC}{}_aD^\alpha v_2)(t) + p(t)v_2(t) \\ &= -({}^{ABC}{}_aD^\alpha u)(t) - p(t)u(t) - p(t)M \\ &= -g(t) - p(t)M \leq -g(t) - |g(t)| \leq 0. \end{aligned}$$

Thus by Lemma 3.1 we have $v_2 = -u - M \leq 0$, thus

$$u \geq -M. \tag{3.5}$$

By combining (3.4) and (3.5) we have $|u(t)| \leq M,\ t \in [a,b]$ and hence the result follows. □

Lemma 3.4. *The fractional initial value problem*

$$({}^{ABC}{}_aD^\alpha u)(t) = \lambda u + f(t), \quad t > 0,\ 0 < \alpha < 1, \tag{3.6}$$

$$u(0) = u_0. \tag{3.7}$$

has the unique solution

$$u(t) = \frac{1}{B(\alpha) - \lambda(1-\alpha)}\Big(B(\alpha)u_0E_\alpha[\omega t^\alpha] + (1-\alpha)(g(t) * f'(t) + f(0)g(t))\Big), \tag{3.8}$$

in the functional space $H^1(0,b)\cap C[0,b]$*, if and only if,* $\lambda u_0+f(0)=0$*, where* $\omega=\frac{\lambda\alpha}{B(\alpha)-\lambda(1-\alpha)}$*, and*

$$g(t)=E_\alpha[\omega t^\alpha]+\frac{\alpha}{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}*E_\alpha[wt^\alpha].$$

Proof. Since $u\in H^1(0,b)$ we have $({}^{ABC}{}_aD^\alpha u)(0)=0$. Thus, a necessary condition for the existence of a solution to (3.6) is that

$$\lambda u_0+f(0)=0. \tag{3.9}$$

Applying the Laplace transform to (3.6) and using the fact that

$$({}^{ABC}{}_0D^\alpha u)(t)=\frac{B(\alpha)}{1-\alpha}E_\alpha[-\frac{\alpha}{1-\alpha}t^\alpha]*u'(t),$$

we have

$$\lambda L(u)+L(f(t))=\frac{B(\alpha)}{1-\alpha}L\Big(E_\alpha[-\frac{\alpha}{1-\alpha}t^\alpha]*u'(t)\Big).$$

Applying the convolution result of the Laplace transform and

$$L(E_\alpha[-\frac{\alpha}{1-\alpha}t^\alpha])=\frac{s^{\alpha-1}}{s^\alpha+\frac{\alpha}{1-\alpha}},\quad |\frac{\alpha}{1-\alpha}\frac{1}{s^\alpha}|<1,$$

leads to

$$\lambda L(u)+L(f(t))=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha-1}}{s^\alpha+\frac{\alpha}{1-\alpha}}(sL(u)-u(0)). \tag{3.10}$$

Direct calculations lead to

$$L(u)=\frac{B(\alpha)u_0}{B(\alpha)-\lambda(1-\alpha)}\frac{s^{\alpha-1}}{s^\alpha-\omega}+\frac{1-\alpha}{B(\alpha)-\lambda(1-\alpha)}\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^\alpha-\omega}L(f(t)), \tag{3.11}$$

where $\omega=\frac{\lambda\alpha}{B(\alpha)-\lambda(1-\alpha)}$. Thus,

$$\begin{aligned}
u(t)&=\frac{B(\alpha)u_0}{B(\alpha)-\lambda(1-\alpha)}L^{-1}\Big(\frac{s^{\alpha-1}}{s^\alpha-\omega}\Big)\\
&\quad+\frac{1-\alpha}{B(\alpha)-\lambda(1-\alpha)}L^{-1}\Big(\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^\alpha-\omega}L(f(t))\Big),\\
&=\frac{B(\alpha)u_0}{B(\alpha)-\lambda(1-\alpha)}E_\alpha[\omega t^\alpha]\\
&\quad+\frac{1-\alpha}{B(\alpha)-\lambda(1-\alpha)}L^{-1}\Big(\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^\alpha-\omega}L(f(t))\Big).
\end{aligned} \tag{3.12}$$

Let

$$G(s)=\frac{1}{s}\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^\alpha-\omega}=\frac{s^{\alpha-1}}{s^\alpha-\omega}+\frac{\alpha}{1-\alpha}\frac{1}{s^\alpha}\frac{s^{\alpha-1}}{s^\alpha-\omega},$$

then

$$g(t)=L^{-1}\big(G(s)\big)=E_\alpha[\omega t]+\frac{\alpha}{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}*E_\alpha[\omega t^\alpha].$$

Applying the convolution result we have

$$\begin{aligned} L^{-1}\Big(\frac{s^\alpha+\frac{\alpha}{1-\alpha}}{s^\alpha-\omega}L(f(t))\Big) &= L^{-1}\Big(G(s)sL(f(t))\Big) \\ &= L^{-1}\Big(G(s)[sL(f)-f(0)+f(0)]\Big) \\ &= L^{-1}\Big(G(s)[L(f')+f(0)]\Big) \\ &= L^{-1}\Big(G(s)L(f')+f(0)G(s)\Big) \\ &= g(t)*f'(t)+f(0)g(t). \end{aligned} \tag{3.13}$$

The result follows by substituting (3.13) in (3.12). □

Corollary 3.5. *The fractional differential equation*

$$({}^{ABC}{}_0D^\alpha u)(t)=\lambda u, \quad t>0,\ 0<\alpha<1, \tag{3.14}$$

has only the trivial solution $u=0$, in the functional space $H^1(0,b)\cap C[0,b]$.

Proof. Applying Lemma 3.4 with $f(t)=0$, yields

$$u(t)=\frac{1}{B(\alpha)-\lambda(1-\alpha)}B(\alpha)u_0E_\alpha[\omega t^\alpha].$$

The necessary condition for the existence of solution yields that $u_0=0$, and hence the result. □

4. Nonlinear equations

In this section we apply the obtained comparison principles to establish a uniqueness result for a nonlinear fractional differential equation and to estimate its solution.

Lemma 4.1. *Consider the nonlinear fractional differential equation*

$$({}^{ABC}{}_aD^\alpha u)(t)=f(t,u), \quad t>a,\ 0<\alpha<1, \tag{4.1}$$

where $f(t,u)$ is a smooth function. If $f(t,u)$ is non-increasing with respect to u then the above equation has at most one solution $u\in H^1(a,b)$.

Proof. Let $u_1,u_2\in H^1(a,b)$ be two solutions of the above equation and let $z=u_1-u_2$. Then

$$({}^{ABC}{}_aD^\alpha z)(t)=f(t,u_1)-f(t,u_2).$$

Applying the mean value theorem we have

$$f(t,u_1)-f(t,u_2)=\frac{\partial f}{\partial u}(u^*)(u_1-u_2),$$

for some u^* between u_1 and u_2. Thus,

$$({}^{ABC}{}_aD^\alpha z)(t)-\frac{\partial f}{\partial u}(u^*)z=0. \tag{4.2}$$

Since $-\frac{\partial f}{\partial u}(u^*)>0$, then $z(t)\le 0$, by Lemma 3.1. Also,(4.2) holds true for $-z$ and thus $-z\le 0$, by virtue of Lemma 3.1. Thus, $z=0$ which proves that $u_1=u_2$. □

Lemma 4.2. *Consider the nonlinear fractional differential equation*

$$({}^{ABC}{}_aD^{\alpha}u)(t) = f(t,u), \quad t > a,\ 0 < \alpha < 1, \tag{4.3}$$

where $f(t,u)$ is a smooth function. Assume that

$$\lambda_2 u + h_2(t) \le f(t,u) \le \lambda_1 u + h_1(t), \quad \textit{for all } t \in (a,b), u \in H^1(a,b),$$

where $\lambda_1, \lambda_2 < 0$. Let v_1 and v_2 be the solutions of

$$({}^{ABC}{}_aD^{\alpha}v_1)(t) = \lambda_1 v_1 + h_1(t), \quad t > a,\ 0 < \alpha < 1, \tag{4.4}$$

and

$$({}^{ABC}{}_aD^{\alpha}v_2)(t) = \lambda_2 v_2 + h_2(t), \quad t > a,\ 0 < \alpha < 1\,. \tag{4.5}$$

Then $v_2(t) \le u(t) \le v_1(t)$, $t \ge a$.

Proof. We shall prove that $u(t) \le v_1(t)$ and by applying analogous steps one can show that $v_2(t) \le u(t)$. By subtracting (4.4) from (4.3) we have

$$\begin{aligned}\big({}^{ABC}{}_aD^{\alpha}(u - v_1)\big)(t) &= f(t,u) - \lambda_1 v_1 - h_1(t)\\ &\le \lambda_1 u + h_1(t) - \lambda_1 v_1 - h_1(t) = \lambda_1(u - v_1).\end{aligned}$$

Let $z = u - v_1$. Then

$$({}^{ABC}{}_aD^{\alpha}z)(t) - \lambda_1 z(t) \le 0.$$

Since $\lambda_1 > 0$, it follows that $z \le 0$, by Lemma 3.1, which completes the proof. □

We now present some examples to illustrate the obtained results.

Example 4.3. Consider the nonlinear fractional initial value problem

$$\begin{gathered}({}^{ABC}{}_0D^{\alpha}u)(t) = e^{-u} - 2, \quad t > 0,\ 0 < \alpha < 1,\\ u(0) = -\ln(2).\end{gathered} \tag{4.6}$$

Since $e^{-u} - 2 \ge -u - 1$, letting v be the solution of

$$({}^{ABC}{}_0D^{\alpha}v)(t) = -v - 1,\ t > 0,\ 0 < \alpha < 1, \tag{4.7}$$

we have $v(t) \le u(t)$ by Lemma 4.2. The solution of (4.7) is given by (3.8) with $\lambda = -1$, and $f(t) = -1$. Thus,

$$\begin{aligned}&u(t) \ge v(t)\\ &= -\frac{1}{B(\alpha) + 1 - \alpha}\Big(B(\alpha)E_{\alpha}[wt^{\alpha}] + (1-\alpha)(E_{\alpha}[wt^{\alpha}] + \frac{\alpha}{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)} * E_{\alpha}[wt^{\alpha}]\Big),\end{aligned}$$

where $\omega = -\frac{\alpha}{B(\alpha)+1-\alpha}$. We recall that (4.7) has a solution only if $v(0) = -1$.

Example 4.4. Consider the nonlinear fractional initial value problem

$$\begin{gathered}({}^{ABC}{}_0D^{\alpha}u)(t) = e^{-u} - \frac{1}{2}u^2, \quad t > 0,\ 0 < \alpha < 1,\\ u(0) = u_0,\end{gathered} \tag{4.8}$$

where u_0 is the unique solution of $e^{-u_0} = \frac{1}{2}u_0^2$. By the Taylor series expansion of $f(u) = e^{-u}$, one can easily show that $e^{-u} - \frac{1}{2}u^2 \le 1 - u$. Let v be the solution of

$$({}^{ABC}{}_0D^{\alpha}v)(t) = -v + 1, \quad t > 0,\ 0 < \alpha < 1, \tag{4.9}$$

then $v(t) \ge u(t)$ by virtue of Lemma 4.2. The solution of (4.9) is given by (3.8) with $\lambda = -1$, and $f(t) = 1$. Thus,

$$u(t) \le v(t)$$

$$= \frac{1}{B(\alpha)+1-\alpha}\big(B(\alpha)E_\alpha[wt^\alpha] + (1-\alpha)(E_\alpha[wt^\alpha] + \frac{\alpha}{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)} * E_\alpha[wt^\alpha]\big),$$

where $\omega = -\frac{\alpha}{B(\alpha)+1-\alpha}$. We recall that (4.9) has a solution only if $v(0) = 1$. Moreover, applying the result of Lemma 3.3 we have $\|v\| \leq 1$, and hence $\|u\| \leq 1$.

Example 4.5. Consider the nonlinear fractional initial value problem

$$\begin{gathered} ({}^{ABC}{}_0D^\alpha u)(t) = -e^u(3+\cos(u)) + 4e^{-t}, \quad t>0,\ 0<\alpha<1, \\ u(0) = 0. \end{gathered} \tag{4.10}$$

Let $h(u) = -e^u(3+\cos(u))$, since $h''(u) = e^u(-3+2\sin(u)) \leq 0$, by the Taylor series expansion method one can easily show that $h(u) \leq h(0) + h'(0)u = -4-4u$. Let v be the solution of

$$\begin{gathered} ({}^{ABC}{}_0D^\alpha v)(t) = -4v - 4 + 4e^{-t}, \quad t>0,\ 0<\alpha<1, \\ v(0) = 0, \end{gathered} \tag{4.11}$$

then $v(t) \geq u(t)$ by Lemma 4.2. The solution of (4.11) is given by (3.8) where $\lambda = -4$, and $f(t) = -4+4e^{-t}$. Applying Lemma 3.3 we have

$$\|u\| \leq \|v\| \leq |\frac{-4+4e^{-t}}{4}| = 1 - e^{-t}, \quad t>0.$$

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