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CONDITIONAL STABILITY OF A SOLUTION OF A DIFFERENCE SCHEME FOR AN ILL-POSED CAUCHY PROBLEM

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Abstract. In this article, we obtain criteria for stability of two-layer difference schemes for an abstract ill-posed Cauchy problem. Method of proof is based on obtaining a priori difference weighted Carleman type estimates. Stability conditions for solutions of two-layer difference schemes are used to prove the theorem of conditional stability of a solution of three-layer scheme that approximates an ill-posed Cauchy problem for an integral-differential equation associated with a coefficient inverse problem.

1. Introduction

In this article, stability problems of difference schemes for an ill-posed Cauchy problem and their application to investigation of coefficient inverse problems are considered. The applied research method is based on concept of stability of a difference scheme on functions with compact support, and in obtaining difference a priori weighted Carleman type estimates. This concept was introduced and developed by Bukhgeim [7, 8] in connection with construction of the theory of difference schemes for ill-posed Cauchy problems, encompassing equations with variable coefficients.

Application of a priori estimates with weight for proof of uniqueness of a solution of the Cauchy problem originates from the work of Carleman [10]. Later this method was extended to a wider class of partial differential equations by many authors [12, 13, 19].

To inverse problems on determining coefficients of partial differential equations, the method of weighted a priori estimates was first applied in the work of Bukhgeim and M.V.Klibanov [9]. They proved uniqueness theorems for solutions of multidimensional inverse problems in "whole".

Stability of difference schemes for an ill-posed Cauchy problem with constant coefficients was first investigated by Chudov [11], by using the Fourier transform method. The approach based on definition of ρ-stability, introduced by Samarskii [20], and SM (Spectral Mimetic) stability with respect to ill-posed and inverse

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problems was investigated in the works of Vabishchevich [22, 23]. Methods of quasi-reversibility have been discussed in [6, 17], iterative methods have been investigated in [2, 3].

With reduction of the method to practical numerical algorithms for solving ill-posed inverse problems, the Carleman estimates method was first proposed by Klibanov and Timonov [18]. Further development and application of the method were considered in (see [1, 4, 14, 16] and their references). Current state and application of Carleman estimates in the theory of multidimensional coefficient inverse problems is given in the review paper of Klibanov [15].

Conditions for conditional stability of the solution of a three-layer difference scheme for an ill-posed Cauchy problem for an integro-differential equation are obtained. This equation is associated with one-dimensional coefficient inverse problem for the nonstationary Schrödinger equation. The stability of the difference scheme is proved by factoring the problem into a sequence of two-layer schemes. In contrast to [8], in which necessary and sufficient conditions for abstract two and three-layer schemes are obtained, we apply these results to the coefficient inverse problem and explicitly construct the spectral decomposition of the difference version of the operator $-i\partial_t$, that occurs in the main part of the Schrödinger operator.

2. Finite stability and stability of two-layer difference schemes

Let $\mathrm{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $u : \mathrm{Z} \to H$ be a function of integer arguments $j \in \mathrm{Z}$ with values in complex Hilbert space H, with the norm $\|u\|$ and dot product $\langle u, v\rangle$, τ be an arbitrary positive number. We define the difference derivatives, and use the usual notation for difference schemes:

$$u_t = (u_{j+1} - u_j)/\tau, u_{\bar{t}} = (u_j - u_{j-1})/\tau, u_{t\bar{t}} = (u_{j+1} - 2u_j + u_{j-1})/\tau^2,$$
$$\hat{u} = u_{j+1}, \quad \overset{\vee}{u} = u_{j-1}.$$

Consider an abstract two-layer difference scheme with weight:

$$\begin{gathered}(Pu)_j \equiv (u_{j+1} - u_j)/\tau - A(\sigma u_{j+1} + (1-\sigma)u_j) = f_j,\\ u_0 = g, \quad j = 0, 1, \dots, N-1.\end{gathered} \tag{2.1}$$

Here A is a linear bounded operator, acting in the space H, and, possibly, depending on $j; \sigma$ is a real parameter; g, f_j are given elements in the space H, $\tau N = T - const$. Using the notation introduced above, we write the difference scheme (2.1) in the compact form:

$$Pu \equiv u_t - A(\sigma \hat{u} + (1-\sigma)u) = f. \tag{2.2}$$

Introduce the corresponding weighted norms (see [8, p.133]). Let $\mathrm{Z}_0^N = \{0, 1, \dots N\}$, $\varphi : \mathrm{Z}_0^N \to R$ be a real-valued monotonic decreasing weighted function, i.e. $-\varphi_t > 0$. Using the function φ and number, we construct the function $\Psi : \mathrm{Z}_0^{N-1} \to R$ so that:

$$\Psi_t = s \overset{\wedge}{\Psi} \varphi_t, \ \Psi_0 = 1.$$

The function Ψ is a discrete analogue of the weighted function $\exp(s\varphi(t))$. For the function $u : \mathrm{Z}_0^{N-1} \to H$ we put:

$$\|u\|_s^2 = \tau \sum_{j=0}^{N-1} \Psi_j^2(s) \|u_j\|^2. \tag{2.3}$$

The norm (2.3) is a discrete analogue:

$$\int_0^T \exp(2s\varphi(t))\|u(t)\|^2 dt, \tag{2.4}$$

moreover, as $\tau \to 0$ the expression (2.3) converges to (2.4).

If we denote by $l_2(k,N;H)$ the Hilbert space of grid functions $u : \mathrm{Z}_k^N \to H$, $\mathrm{Z}_k^N = \{k, k+1, \dots, N\}$ with the norm $\|u\|^2_{l_2(k,N;H)} = \tau\sum_{j=k}^N \|u_j\|^2$, then due to the definition of the norm $\|u\|_s : \|u\|_0 = \|u\|_{l_2(0,N-1;H)}$.

We denote by $C_0(Z_0^N)$ the space of functions $u : Z_0^N \to H$ such that: $u_0 = u_N = 0$. The linear space $C_0(Z_0^N)$ is a discrete analogue of the space $C_0(0,T)$ of continuous functions $u(t) : u(0) = u(T) = 0$ with compact support on the interval $[0,T]$.

Definition. Difference scheme P of the form (2.2) is called stable on functions with compact support, if there exist independent from $\tau, \|A\|$ numbers $s_0 > 0$, $M > 0$, such that for all $s \geq s_0$, $u \in C_0(Z_0^N)$ the following estimate holds:

$$s\|u\|_s^2 \leq M\,\|Pu\|_s^2. \tag{2.5}$$

To obtain a stability estimate on the whole grid Z_1^N, it is necessary to take into account contribution of outside integral terms that arise when we use the formula of summation by parts, and, therefore, it is necessary to work not with functions with compact support $C_0(\mathrm{Z}_0^N)$, but with arbitrary $u : \mathrm{Z}_0^N \to H$. For brevity, we introduce the notation

$$\|u\|^2_{s(k,N)} = \tau\sum_{j=k}^N \Psi_j^2(s)\|u_j\|^2, \quad k \geq 0.$$

Consider the difference scheme

$$\begin{gathered} Pu \equiv u_t - (A+iB)u = f, \;\; i^2 = -1, \\ u_0 = g, \end{gathered} \tag{2.6}$$

where A, B are independent on j, selfadjoint, commuting, positive operators, i.e.

$$A^* = A, \quad B^* = B, \quad [A,B] = 0, \quad A,\ B \geq 0\,.$$

To obtain the stable estimate we will estimate $\|Pu\|_s^2$ below. We have

$$\|Pu\|_s^2 = \tau\sum_{j=0}^{N-1} \|u_t - (A+iB)u\|^2 \Psi_j^2(s).$$

Put $\Psi u = v$. According to the formula of difference differentiation of product: $u_t = (\Psi^{-1}v)_t = (\Psi^{-1})_t \hat{v} + \Psi^{-1}v_t$. From (2.3) it follows that $(1/\Psi)_t = (-s\varphi_{tj})/\Psi_j$, thus,

$$u_t = (v_t - s\varphi_t \hat{v})/\Psi.$$

Then

$$
\begin{aligned}
&\|Pu\|_s^2\\
&= \tau \sum_{j=0}^{N-1} \|v_t - s\varphi_t \hat{v} - (A+iB)v\|^2\\
&= \tau \sum_{j=0}^{N-1} \Big\{ \|v_t - iBv\|^2 + + \|Av + s\varphi_t \hat{v}\|^2 - 2\operatorname{Re}\langle v_t - iBv, Av + s\varphi_t \hat{v}\rangle \Big\}\\
&= \tau \sum_{j=0}^{N-1} \Big\{ \|v_t - iBv\|^2 + \|Av + s\varphi_t \hat{v}\|^2 - 2\operatorname{Re}\langle v_t, Av\rangle - 2\operatorname{Re}\langle v_t, s\varphi_t \hat{v}\rangle\\
&\quad + 2\operatorname{Re}\langle iBv, Av\rangle + 2\operatorname{Re}\langle iBv, s\varphi_t \hat{v}\rangle \Big\} \equiv \sum_{k=1}^{6} I_k.
\end{aligned}
\tag{2.7}
$$

Here by I_k we denoted $\tau\sum_{j=0}^{N-1}$, corresponding to the k-term in curly brackets of the expression (2.7). For numerical functions of a discrete argument, we introduce the following notation:

$$
[x,y) \equiv \tau \sum_{j=0}^{N-1} x_j y_j, \quad (x,y) \equiv \tau \sum_{j=1}^{N-1} x_j y_j. \tag{2.8}
$$

Using this notation, from (2.7) we obtain:

$$
\begin{gathered}
I_1 = [1, \|v_t - iBv\|^2) \geq 0, \quad I_2 = [1, \|Av + s\varphi_t \hat{v}\|^2) \geq 0,\\
I_3 = -[1, 2\operatorname{Re}\langle v_t, Av\rangle), \quad I_4 = -[s\varphi_t, 2\operatorname{Re}\langle v_t, \hat{v}\rangle),\\
I_5 = [1, 2\operatorname{Re}\langle iBv, Av\rangle), \quad I_6 = [s\varphi_t, 2\operatorname{Re}\langle iBv, \hat{v}\rangle).
\end{gathered}
$$

For functions of discrete argument $v, w : Z \to H$ the formula of difference differentiation has the form:

$$
\partial\langle v, w\rangle = \langle v_t, \hat{w}\rangle + \langle v, w_t\rangle.
$$

In particular, when $w = v$, $\partial\|v\|^2 = \langle v_t, \hat{v}\rangle + \langle v - \hat{v}, v_t\rangle + \langle \hat{v}, v_t\rangle = -\tau\|v_t\|^2 + 2\operatorname{Re}\langle v_t, \hat{v}\rangle$, i.e. $2\operatorname{Re}\langle v_t, \hat{v}\rangle = \partial\|v\|^2 + \tau\|v_t\|^2$. $2\operatorname{Re}\langle v_t, \hat{v}\rangle = \partial\|v\|^2 + \tau\|v_t\|^2$. Then for I_4 we obtain: $I_4 = -[s\varphi_t, 2\operatorname{Re}\langle v_t, \hat{v}\rangle) = -[s\varphi_t, \partial\|v\|^2) - s\tau[\varphi_t, \|v_t\|^2)$. According to the formula $[x, \partial y) = -(\bar{\partial} x, y) + (x_{N-1}y_N - x_0 y_0)$ of summation by parts, we obtain: $I_4 = (s\varphi_{t\bar{t}}, \|v\|^2) - [s\tau\varphi_t, \|v_t\|^2) - s(\varphi_{t\,N-1}\|v_N\|^2 - \varphi_{t0}\|v_0\|^2)$. Suppose that $\tilde{\mu} \geq -\varphi_t \geq \mu > 0$. Taking this condition into account, we have

$$
I_4 \geq (s\varphi_{t\bar{t}}, \|v\|^2) - [s\tau\varphi_t, \|v_t\|^2) + s\mu\|v_N\|^2 - s\tilde{\mu}\|v_0\|^2. \tag{2.9}
$$

We now transform the expression $2\operatorname{Re}\langle iBv, \hat{v}\rangle$. $2\operatorname{Re}\langle iBv, \hat{v}\rangle = 2\operatorname{Re}\langle iBv, v + \tau v_t\rangle = 2\operatorname{Re}\langle iBv, v\rangle + \tau\cdot 2\operatorname{Re}\langle iBv, v_t\rangle$. Because of self-adjointness $B, \operatorname{Re}\langle iBv, v\rangle = 0$ for all $v \in H$, therefore, $2\operatorname{Re}\langle iBv, \hat{v}\rangle = \tau \cdot 2\operatorname{Re}\langle iBv, v_t\rangle$. From this equality we obtain

$$
|2\operatorname{Re}\langle iBv, \hat{v}\rangle| = \tau|2\operatorname{Re}\langle iBv, v_t\rangle| \leq \tau\big\{\alpha\|B\|^2\|v\|^2 + \alpha^{-1}\|v_t\|^2\big\}.
$$

Here we used α-inequality: $2ab \leq \alpha a^2 + \alpha^{-1}b^2, \alpha > 0$. Since $I_6 = s[\varphi_t, 2\operatorname{Re}\langle iBv, \hat{v}\rangle)$, we have $|I_6| \leq \tau s[|\varphi_t|, \alpha\|B\|^2\|v\|^2) + \tau s[|\varphi_t|, \alpha^{-1}\|v_t\|^2)$. Thus, writing $|\varphi_t|$ in the form $|\varphi_t| = -\varphi_t(-\varphi_t > 0)$, we obtain

$$\begin{aligned} I_6 &\geq \tau s\big\{\big[1, \varphi_t \alpha \|B\|^2 \|v\|^2\big) + \big[1, \varphi_t \alpha^{-1} \|v_t\|^2\big)\big\} \\ &= \tau^2 s \varphi_{t0} \alpha \|B\|^2 \|v_0\|^2 + \tau s\big(1, \varphi_t \alpha \|B\|^2 \|v\|^2\big) + \tau s\big[1, \varphi_t \alpha^{-1} \|v_t\|^2\big) \\ &\geq -\tau^2 s \tilde{\mu} \alpha \|B\|^2 \|v_0\|^2 + \tau s\big\{\big(1, \varphi_t \alpha \|B\|^2 \|v\|^2\big) + \big[1, \varphi_t \alpha^{-1} \|v_t\|^2\big)\big\}. \end{aligned} \tag{2.10}$$

From the equality $2\operatorname{Re}\langle v_t, Av\rangle = \partial\langle v, Av\rangle - \tau\langle v_t, Av_t\rangle$, and using the formula of summation by parts, we obtain

$$\begin{aligned} I_3 &= -[1,\ \partial\langle v, Av\rangle) + [1,\ \langle v_t, \tau Av_t\rangle) \\ &= [1,\ \langle v_t,\ \tau Av_t\rangle) - \langle v_N, Av_N\rangle + \langle v_0, Av_0\rangle \\ &\geq -\langle v_N, Av_N\rangle. \end{aligned} \tag{2.11}$$

Here we took into account the fact that $A^* = A$, $A \geq 0$.

Compute the term I_6. From commutativity of the operators A and B, we obtain

$$2\operatorname{Re}\langle iBv, Av\rangle = \langle iBv, Av\rangle + \langle Av, iBv\rangle = \langle iABv, v\rangle - \langle iBAv, v\rangle = \langle i[A,B]v, v\rangle = 0,$$

then

$$I_5 = [1, 2\operatorname{Re}\langle iBv, Av\rangle) = 0.$$

The obtained estimates yield the following result.

Lemma 2.1. *Let $Pu \equiv u_t - (A + iB)u, A^* = A \geq 0, B^* = B \geq 0, [A,B] = 0$. Then for all $u : Z_0^N \to H$:*

$$\|Pu\|_s^2 = \sum_{k=1}^{6} I_k, \quad I_1 = [1, \|v_t - iBv\|^2) \geq 0,$$

$$I_2 = [1, \|Av + s\varphi_t \hat{v}\|^2) \geq 0, \quad I_3 \geq -\langle v_N, Av_N\rangle,\ I_5 = 0,$$

$$I_4 \geq (s\varphi_{t\bar{t}}, \|v\|^2) - [s\tau\varphi_t, \|v_t\|^2) + s\mu\|v_N\|^2 - s\tilde{\mu}\|v_0\|^2,$$

$$I_6 \geq \tau s\big\{(1, \varphi_t \alpha \|B\|^2 \|v\|^2) + [1, \varphi_t \alpha^{-1} \|v_t\|^2)\big\} - \tau^2 s \tilde{\mu} \alpha \|B\|^2 \|v\|^2.$$

Here $v = \Psi u, -\varphi_t > 0$.

This lemma implies the following theorem.

Theorem 2.2. *Let in the conditions of Lemma 2.1 for all $s \geq s_0$ and some $\delta > 0$ the following conditions hold:*

$$M_1 \equiv (s\varphi_{t\bar{t}} + \tau s \varphi_t \alpha \|B\|^2)E \geq s\delta E, \tag{2.12}$$

$$M_0 \equiv -s\tau\varphi_t(1 - \alpha^{-1})E \geq 0, \alpha > 0. \tag{2.13}$$

Then for all $u : Z_0^N \to H, s \geq s_0$ for the difference scheme (2.6) *the following stability estimate holds*

$$s\|u\|_{s(1,N)}^2 \leq \mu_2^{-1}\big\{\|Pu\|_s^2 + s\mu_0\|u_0\|^2 + \Psi_N^2 \langle u_N, Au_N\rangle\big\}. \tag{2.14}$$

Here μ_0, μ_2 are certain positive constants.

Proof. Discarding the quantities $I_1, I_2 \geq 0$, and collecting separately the terms containing v and v_t, by Lemma 2.1 we obtain:

$$\begin{aligned} \|Pu\|_s^2 &\geq (1, \langle M_1 v, v\rangle) + [1, \langle M_0 v_t, v_t\rangle) \\ &\quad - s\tilde{\mu}(1 + \tau^2 \alpha \|B\|^2)\|v_0\|^2 - \langle v_N, Av_N\rangle + s\mu\|v_N\|^2. \end{aligned}$$

Thus, from (2.12) and (2.13), and taking into account that $v = \Psi u$, $\Psi_0 = 1$, we have

$$\begin{aligned} &\delta s\|u\|^2_{s(1,N-1)} + s\mu\Psi_N^2\|u_N\|^2 \\ &\le \|Pu\|^2_s + s\tilde{\mu}(1+\tau^2\alpha\|B\|^2)\|u_0\|^2 + \Psi_N^2\langle u_N, Au_N\rangle. \end{aligned} \tag{2.15}$$

When $0 < \tau \le \tau_0$ and $\mu_2 = \min(\delta, \frac{\mu}{\tau_0})$, we obtain

$$s\delta\|u\|^2_{s(1,N-1)} + s\mu\Psi_N^2\|u_N\|^2 \ge s\mu_2\|u\|^2_{s(1,N)}.$$

Taking this estimate into account, and assuming $\mu_0 = \tilde{\mu}(1+\tau_0^2\alpha\|B\|^2)$, after dividing the inequality (2.15) by μ_2, we obtain (2.14). The proof is complete. □

Furthermore, we assume that the following conditions are satisfied:

$$\varphi_{t\bar{t}} \ge 1,\ -\varphi_t \ge 1.$$

Theorem 2.3. *Let*

$$\tau\|B\|^2 \le c, \quad c > 0. \tag{2.16}$$

Then there exists a number $c_1 = c(\alpha, c)$such that when

$$\varphi_{t\bar{t}} + c\varphi_t \ge 1 \tag{2.17}$$

for the difference scheme (2.6) *the estimate* (2.14) *holds.*

Proof. By choosing sufficiently large numbers s_0 and α, we obtain non-negativity of the operatorM_0. Similarly, due to (2.16) and (2.17), we have:

$$\begin{aligned} \langle M_1 v, v\rangle &= s\varphi_{t\bar{t}}\|v\|^2 + \tau s\varphi_t\alpha\|B\|^2\|v\|^2 \\ &\ge s\varphi_{t\bar{t}}\|v\|^2 + s\varphi_t\alpha\cdot c\|v\|^2 \\ &= s(\varphi_{t\bar{t}} + c_1\varphi_t)\|v\|^2 \ge s\cdot\varepsilon\|v\|^2 \end{aligned}$$

at a large enough number and small ε. Reference to Theorem 2.2 completes the proof of the theorem. □

Consider now the difference scheme:

$$\begin{gathered} Pu \equiv u_t + (A+iB)u = f,\ i^2 = -1, \\ u_0 = g. \end{gathered} \tag{2.18}$$

Here operators A and B satisfy the same conditions as in (2.6). Similarly to Lemma 2.1, we establish the following result.

Lemma 2.4. *Let $Pu \equiv u_t + (A+iB)u$, where $A^* = A \ge 0$, $B^* = B \ge 0$, $[A,B] = 0$. Then for all $u : Z_0^N \to H$ we have:*

$$\|Pu\|^2_s = \sum_{k=1}^{6}\tilde{I}_k,$$

$$\begin{gathered} \tilde{I}_1 = [1, \|v_t + iBv\|^2) \ge 0, \quad \tilde{I}_2 = [1, \|Av - s\varphi_t\hat{v}\|^2) \ge 0, \\ \tilde{I}_3 \ge -[1, \langle v_t, \tau Av_t\rangle) - \langle v_0, \tau Av_0\rangle, \\ \tilde{I}_4 \ge \big(s\varphi_{t\bar{t}}, \|v\|^2\big) - \big[s\tau\varphi_t,\ \|v_t\|^2\big) - s\tilde{\mu}\|v_0\|^2 + s\mu\|v_N\|^2, \quad \tilde{I}_5 = 0, \\ \tilde{I}_6 \ge \tau s\big\{(1, \varphi_t\alpha\|B\|^2\|v\|^2) + [1, \varphi_t\alpha^{-1}\|v_t\|^2)\big\} - \tau^2 s\tilde{\mu}\alpha\|B\|^2\|v\|^2. \end{gathered}$$

This lemma yields the following two theorems.

Theorem 2.5. *Let in the conditions of Lemma 2.4 for all $s \geq s_0$ and for some $\delta > 0$ the following inequality hold:*

$$\tilde{M}_1 \equiv \big(s\varphi_{t\bar{t}} + \tau s\varphi_t \alpha \|B\|^2\big)E \geq s\delta E,$$

$$\tilde{M}_0 \equiv -s\tau\varphi_t(1-\alpha^{-1})E - \tau A \geq 0.$$

Then for all $s \geq s_0$, $u : Z_0^N \to H$ to solve the difference scheme (2.18) *we have the stability estimate:*

$$s\|u\|^2_{s(1,N)} \leq \mu_2^{-1}\big\{\, \|Pu\|_s^2 + s\mu_0\|u_0\| + \langle u_0, Au_0\rangle\big\}. \tag{2.19}$$

Theorem 2.6. *Let for some $m, c > 0$ the following inequality hold:*

$$\tau A \leq mE, \quad \tau\|B\|^2 \leq c. \tag{2.20}$$

Then there exists a number $c_1 = c_1(\alpha, m, c)$ such that when

$$\varphi_{t\bar{t}} + c_1\varphi_t \geq 1$$

to solve the difference scheme (2.18) *we have the estimate* (2.19).

Theorems 2.5 and 2.6 are proved simlarly to Theorems 2.2 and 2.3.

Remark 2.7. To obtain Lemmas 2.1, 2.4 we used the method of proofs of [8, Lemma 2.1 p. 142], where it is used to obtain sufficient conditions for stability with compact support of two-layer difference schemes.

Remark 2.8. Theorem 2.3 was proved by Bukhgeim the case of functions with compact support without assumption of positivity of the operators [8].

3. An ill-posed Cauchy problem for integral-differential equation

In this section we give an ill-posed Cauchy problem related to the one-dimensional coefficient inverse problem for the Schrödinger equation. Such an ill-posed Cauchy problem arises instudy of questions of uniqueness and stability of solutions of coefficient inverse problems in non-stationary formulation [5, 8]. Scheme of proof of these theorems consists in reducing the inverse problem to an ill-posed Cauchy problem for integral-differential equations, and the subsequent application of a priori weighted Carleman type estimates. Therefore, justifying the difference methods for solving these inverse problems, it becomes necessary to obtain stability estimates for solutions of difference schemes that approximate an ill-posed Cauchy problem for the corresponding integral-differential equations or inequalities.

Let $\Omega = \{x, t : x > 0, t^2 + (x - r) < 0\}$ (see Figure 1).

Consider on the domain Ω the following ill-posed Cauchy problem:

$$\begin{aligned} iv_t + v_{xx} = a_1(x)v + (b_1\partial + b_0)(u_2(x,t)f(x)/f_2(x) \\ + \int_0^t K(x,t,\tau)v(x,\tau)d\tau), \end{aligned} \tag{3.1}$$

$$v(0,t) = g'(t) - {g'}_2(t)g(t)/g_2(t), \quad v_x(0,t) = 0, \tag{3.2}$$

where $K(x,t,\tau) = u_2(x,t)/u_2(x,\tau)$, $\partial = \partial/\partial x, i^2 = -1$. Concerning to the coefficients in (3.1), we assume that they are smooth enough functions of their variables (conclusion of the integro-differential equation (3.1) is given in [8, p.40]). Writing the equation (3.1) in more detail, and assuming

$$\bar{b}(x,t) \equiv b_0(x,t)u_2(x,t)/f_2(x) + b_1(x,t)(u_2(x,t)/f_2(x))_x,$$

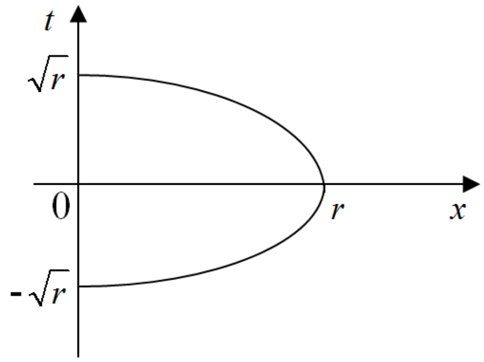

FIGURE 1. Source area.

$$\begin{aligned}\overline{b}_1(x,t) &\equiv b_1(x,t)u_2(x,t)/f_2(x),\\ \overline{K}(x,t,\tau) &\equiv b_0(x,t)K(x,t,\tau)+b_1(x,t)K_x(x,t,\tau),\\ \overline{K_1}(x,t,\tau) &\equiv b_1(x,t)K(x,t,\tau),\end{aligned}$$

it can be rewritten in the form

$$\begin{aligned} iv_t + v_{xx} &= a_1(x)v + \overline{b}(x,t)f(x) + \overline{b_1}(x,t)f'(x)\\ &\quad + \int_0^t (\overline{K}(x,t,\tau)v(x,\tau) + \overline{K_1}(x,t,\tau)v_x(x,\tau))d\tau. \end{aligned} \tag{3.3}$$

We make the change of variables in the equation (3.1). Put $x = \xi - t^2$, $t = t$. Then

$$v(x,t) = w(\xi,t) = w(x+t^2,t), \quad v_t = 2t \cdot w_\xi + w_t, \quad v_x = w_\xi, v_{xx} = w_{\xi\xi}.$$

After simple transformations and using the formula $\int \delta(p(\tau))\, u(\tau)d\tau = \frac{u(\tau_0)}{|p'(\tau_0)|}$, where τ_0 is a unique root of the equation $p(\tau) = 0$ (in our case $p(\tau) = \tau^2 - t^2 + \xi - \eta$), we obtain:

$$\begin{aligned} &2itw_\xi + iw_t + w_{\xi\xi}\\ &= \tilde{a}_1(\xi,t)w(\xi,t) + \tilde{b}(\xi,t)\tilde{f}(\xi,t) + \tilde{b}_1(\xi,t)\tilde{f}_\xi(\xi,t)\\ &\quad \pm \int_{\xi-t^2}^{\xi} \frac{\tilde{K}(\eta,t,\pm\sqrt{\eta-\xi+t^2})w(\eta,\pm\sqrt{\eta-\xi+t^2})}{2\sqrt{\eta-\xi+t^2}}d\eta\\ &\quad \pm \int_{\xi-t^2}^{\xi} \frac{\tilde{K}_1(\eta,t,\pm\sqrt{\eta-\xi+t^2})w_\eta(\eta,\pm\sqrt{\eta\xi+t^2})}{2\sqrt{\eta-\xi+t^2}}d\eta. \end{aligned} \tag{3.4}$$

Here the sign (+) corresponds to the case $t > 0$, and the sign (-) to the case $t < 0$. For convenience we rename the variables ξ, t : $\xi := t$, $t := x$. Then the equation (3.4) has the form

$$\begin{aligned} &2ixw_t + iw_x + w_{tt} \\ &= \tilde{a}_1(t,x)w(t,x) + \tilde{b}(t,x)\tilde{f}(t,x) + \tilde{b}_1(t,x)\tilde{f}_t(t,x) \\ &\quad \pm \int_{t-x^2}^{t} \frac{\tilde{K}(\eta, x, \pm\sqrt{\eta - t + x^2})w(\eta, \pm\sqrt{\eta - t + x^2})}{2\sqrt{\eta - t + x^2}} d\eta \\ &\quad \pm \int_{t-x^2}^{t} \frac{\tilde{K}_1(\eta, x, \pm\sqrt{\eta - t + x^2})w_\eta(\eta, \pm\sqrt{\eta - t + x^2})}{2\sqrt{\eta - t + x^2}} d\eta, \end{aligned} \tag{3.5}$$

$$w\big|_\gamma = g_x(t,x) - g_{2x}(t,x)g(t,x)/g_2(t,x), w_t\big|_\gamma = 0 \quad , \tag{3.6}$$

where $\gamma : t = x^2$. Note that the original domain Ω after the change of variables, and renaming, goes into the domain bounded between the parabola $t = x^2$ and line $t = r$, which we also denote by Ω (see Figure 2).

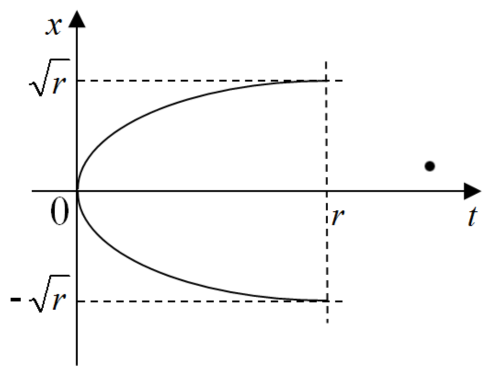

FIGURE 2. Area after the change of variables and renaming.

Assuming $w_0 = w\big|_\gamma$. and introducing a new function $\tilde{w} = w - w_0$, the problem (3.5) - (3.6) can be reduced to a problem with homogeneous boundary conditions on γ:

$$2ix\tilde{w}_t + i\tilde{w}_x + \tilde{w}_{tt} = F, \tag{3.7}$$

$$\tilde{w}\big|_\gamma = 0, \quad \tilde{w}_t\big|_\gamma = 0\,. \tag{3.8}$$

where F is the right side of the equation (3.5). We extend the function $\tilde{w}$ in (3.7) by continuity by zero to a rectangular domain, and rename $\tilde{w} := w$. The condition (3.8), unbounded generality, is replaced by $w(0,x) = g(x), w_t(0,x) = 0$. Then in the domain we obtain the following problem:

$$w_{tt} + iw_x = \tilde{a}_1(t,x)w - 2ixw_t + Kw + f(t,x), \tag{3.9}$$

$$w(0,x) = g(x), w_t(0,x) = 0. \tag{3.10}$$

Here

$$\begin{aligned} &Kw = K_0 w + K_1 w, \;\; K_0 w(t,x) \\ &= \pm \int_{t-x^2}^{t} \frac{\tilde{K}(\eta, x, \pm\sqrt{\eta - t + x^2})\, w\,(\eta, \pm\sqrt{\eta\; - t + x})}{2\sqrt{\eta - t + x^2}} d\eta, \end{aligned}$$

$$K_1w(t,x) = \pm \int_{t-x^2}^{t} \frac{\tilde{K}_1(\eta, x, \pm\sqrt{\eta - t + x^2})\, w_\eta(\eta, \pm\sqrt{\eta - t + x^2})}{2\sqrt{\eta - t + x^2}} d\eta,$$
$$f(t,x) = \tilde{b}(t,x)\tilde{f}(t,x) + \tilde{b}_1(t,x)\tilde{f}_t(t,x).$$

Our goal is to obtain an estimate of conditional stability of a solution of difference scheme that approximates the ill-posed Cauchy problem (3.9)-(3.10). Construction and proof of the conditional stability of difference scheme solution for this problem are carried out in the next section. Unconditional stability of three-layer difference scheme for the problem (3.9)-(3.10), depending on two parameters, was obtained in [21].

4. Stability of difference scheme for an ill-posed Cauchy problem

We associate with the problem (3.9)-(3.10) the following three-layer difference scheme:

$$\begin{aligned} &\frac{u^k_{j+1} - 2u^k_j + u^k_{j-1}}{\tau^2} - Au^k_{j-1} \\ &= \tilde{a}^k_{1\,j-1} u^k_{j-1} - 2ikh(u^k_j - u^k_{j-1})/\tau + K_{\tau,\,h} u^k_{j-1} + f^k_{j-1}, \end{aligned} \tag{4.1}$$

$$\begin{gathered} u_0 = g^k, \quad u_1 = u_0, \\ j = 1, 2, \dots, N-1, \quad \tau N = r, \\ k = 0, \pm 1, \dots, \pm(N_1 - 1), \quad \pm hN_1 = \pm\sqrt{r} \equiv \pm T. \end{gathered} \tag{4.2}$$

Here $Au^k_{j-1} = -i\frac{u^{k+1}_{j-1} - u^{k-1}_{j-1}}{2h}$, $K_{\tau,h}$ is an approximation of the operator $K = K_0 + K_1$ such that

$$\|K_{\tau,h}u\| \le c(\|u\| + \|u_t\|). \tag{4.3}$$

Naturalness of this assumption follows from boundedness and Volterra property of the integral operator K_0 by t. We will consider the operator as an operator acting in the complex Hilbert space. As H we take the space of grid functions $u(x)$ defined on the grid $\tilde{\omega}_h = \{x_k = kh,\, k = 0, \pm 1, \dots, \pm N_1, \pm hN_1 = \pm T\}$ and vanishing for $k = -N_1,\ k = N_1$. The dot product and norm in space $H$are introduced in the usual form:

$$\langle u, v\rangle = h \sum_{k=-(N_1-1)}^{N_1-1} u^k \bar{v}^k \quad \|u\|^2 = h \sum_{k=-(N_1-1)}^{N_1-1} |u^k|^2$$

Obviously, A will be a self-adjoint operator in H. Furthermore, we shall omit the index k in all the functions. We denote the right-hand side of (4.1) by F_j, and write the equation (4.1) with a shift by a step to the right:

$$\frac{u_{j+2} - 2u_{j+1} + u_j}{\tau^2} - Au_j = F_j. \tag{4.4}$$

Put:

$$u_{j+1} - Ru_j = \tau v_j, \tag{4.5}$$
$$v_{j+1} - Sv_j = \tau F_j, \tag{4.6}$$

and we will try to pick up the operators R and S so that after exclusion v the system (4.5) - (4.6) goes into equation (4.4). Eliminating v_j from (4.6) with the help of (4.5), we obtain

$$u_{j+2} - (R+S)u_{j+1} + SRu_j = \tau^2 F_{j+1};$$

thus

$$R + S = 2E, \tag{4.7}$$

$$SR = E - \tau^2 A. \tag{4.8}$$

We represent the operator A in the form of difference of two nonnegative commuting operators $A_\pm$: $A = A_+ - A_-, (A_\pm)^* = A_\pm$, $A_+ = QA$, $A_- = (Q - E)A$, where Q is an orthogonal projection, projecting the space onto the subspace of eigenfunctions corresponding to the nonnegative part of spectrum of the operator$A$. Since $A_\pm \geq 0$, the nonnegative operators $\sqrt{A_\pm}$ are uniquely determined. By definition,

$$\sqrt{A} = \sqrt{A_+} + i\sqrt{A_-}.$$

Since by construction $A_\pm A_\mp = 0$, then $E - \tau^2 A = (E - \tau\sqrt{A})(E + \tau\sqrt{A})$. Assuming now $S = (E - \tau\sqrt{A}), R = (E + \tau\sqrt{A})$, we find the solution of the system of equations (4.7)-(4.8):

$$u_{j+1} - (E + \tau\sqrt{A})u_j = \tau v_j, \quad u_0 = g,$$

$$v_{j+1} - (E - \tau\sqrt{A})v_j = \tau F_{j+1}, \quad v_0 = -\sqrt{A}u_0,$$

or

$$u_t - \sqrt{A}u = v, \quad u_0 = g, \tag{4.9}$$

$$v_t + \sqrt{A}v = \tilde{F}, \quad v_0 = -\sqrt{A}u_0, (\tilde{F} = F_{j+1}). \tag{4.10}$$

Applying Theorem 2.3 from Section 1 to the difference scheme (4.9), and Theorem 2.6 to the scheme (4.10) with $N := N - 1$, and taking into account that in our case $A = \sqrt{A_+},\ B = \sqrt{A_-}$, and the conditions (2.16), (2.20) have the form

$$\tau\sqrt{A_+} \leq mE, \quad \tau\|\sqrt{A_-}\|^2 \leq c, \tag{4.11}$$

we obtain

$$s\|u\|^2_{s(1,N)} \leq \mu_2^{-1}\Big\{\tau_0\|v_0\|^2 + \|v\|^2_{s(1,N-1)} + s\mu_0\|u_0\|^2 + \Psi_N^2\langle u_N, \sqrt{A_+}u_N\rangle\Big\}$$

(here we noted that $\Psi_0 = 1,\ \tau \leq \tau_0$),

$$s\|v\|^2_{s(1,N-1)} \leq \mu_2^{-1}\big\{\|\tilde{F}\|^2_{s(0,N-2)} + s\mu_0\|v_0\|^2 + \langle v_0, \sqrt{A_+}v_0\rangle\big\}. \tag{4.12}$$

Combining these estimates, we have

$$\begin{aligned} s^2\|u\|^2_{s(1,N)} &\leq s\mu_2^{-1}\big\{s\mu_0\|u_0\|^2 + \Psi_N^2\langle u_N, \sqrt{A_+}u_N\rangle\big\} + s\tilde{\mu}_0\|v_0\|^2 \\ &\quad + \mu_2^{-2}\big\{\|\tilde{F}\|^2_{s(0,N-2)} + \langle v_0, \sqrt{A_+}v_0\rangle\big\}, \end{aligned}$$

where $\tilde{\mu}_0 = \tau_0\mu_2^{-1} + \mu_2^{-2}\mu_0$. Combining this inequality with (4.12), and taking into account the fact that $v = u_t - \sqrt{A}u$, we obtain

$$\begin{aligned} &s^2\|u\|^2_{s(1,N)} + s\|u_t - \sqrt{A}u\|^2_{s(1,N-1)} \\ &\leq s\mu_2^{-1}\big\{s\mu_0\|u_0\|^2 + \Psi_N^2\langle u_N, \sqrt{A_+}u_N\rangle\big\} \\ &\quad + sc_1\|v_0\|^2 + c_2\big\{\|\tilde{F}\|^2_{s(0,N-2)} + \langle v_0, \sqrt{A_+}v_0\rangle\big\}. \end{aligned} \tag{4.13}$$

Here $c_1 = \tilde{\mu}_0 + \mu_2^{-1}\mu_0$, $c_2 = \mu_2^{-2} + \mu_2^{-1}$. Applying Theorem 2.6 to the difference problem

$$u_t + \sqrt{A}u = v, \quad u_0 = g, \quad (R = E - \tau\sqrt{A}),$$

and Theorem 2.3 to the difference problem

$$v_t - \sqrt{A}v = \tilde{F}, \quad v_0 = \sqrt{A}u_0, \quad (S = E + \tau\sqrt{A}),$$

we obtain

$$\begin{gathered} s\|u\|^2_{s(1,N)} \le \mu_2^{-1}\big\{\tau_0\|v_0\|^2 + \|v\|^2_{s(1,N-1)} + s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle\big\}, \\ s\|v\|^2_{s(1,N-1)} \le \\ \le \mu_2^{-1}\big\{\|\tilde{F}\|^2_{s(0,N-2)} + s\mu_0\|v_0\|^2 + \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-}\rangle\big\}. \end{gathered} \tag{4.14}$$

As above, combining these estimates, we have

$$\begin{aligned} s^2\|u\|^2_{s(1,N)} \le{}& s\mu_2^{-1}\big\{s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle\big\} + s\tilde{\mu}_0\|v_0\|^2 \\ &+ \mu_2^{-2}\big\{\|\tilde{F}\|^2_{s(0,N-2)} + \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-1}\rangle\big\}. \end{aligned}$$

Combining this inequality with (4.14) and taking into account that $v = u_t + \sqrt{A}u$, we have

$$\begin{aligned} &s^2\|u\|^2_{s(1,N)} + s\|u_t + \sqrt{A}u\|^2_{s(1,N-1)} \\ &\le s\mu_2^{-1}\big\{s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle\big\} + sc_1\|v_0\|^2 \\ &\quad + c_2\big\{\|\tilde{F}\|^2_{s(0,N-2)} + \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-1}\rangle\big\}. \end{aligned} \tag{4.15}$$

Adding now the estimate (4.15) with the estimate (4.13), and taking into account the identity:

$$\|u_t - \sqrt{A}u\|^2 + \|u_t + \sqrt{A}u\|^2 = 2\|u_t\|^2 + 2\|\sqrt{A}u\|^2,$$

we obtain

$$\begin{aligned} &2s^2\|u\|^2_{s(1,N)} + 2s(\|u_t\|^2_{s(1,N-1)} + \|\sqrt{A}u\|^2_{s(1,N-1)}) \\ &\le s\mu_2^{-1}\big\{2s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle + \Psi^2_N\langle u_N, \sqrt{A_+}u_N\rangle\big\} + 2sc_1\|v_0\|^2 \\ &\quad + c_2\big\{2\|\tilde{F}\|^2_{s(0,N-2)} + \langle v_0, \sqrt{A_+}v_0\rangle + \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-1}\rangle\big\}. \end{aligned} \tag{4.16}$$

Estimate the term $\|\tilde{F}\|^2_{s(0,N-2)}$ in the right-hand side of (4.16). Noting that $\tilde{F} = F_{j+1}$ in the right-hand side of (4.1), due to the obvious inequality $\|u\|_{s(0,N-2)} \le \|u\|_s \le \|u\|_{s(0,N)}$ and the condition (4.3), we have

$$\|F_{j+1}\|^2_{s(0,N-2)} \le c(\|u\|^2_{s(1,N)} + \|u_t\|^2_{s(1,N-1)} + \|f\|^2_s + \|u_0\|^2) \tag{4.17}$$

with some constant c. Here we used the condition $u_1 = u_0$ and $\Psi_0 = 1$, $\tau \le \tau_0$. From (4.17) and (4.16) we obtain

$$\begin{aligned} &2s^2(1 - \frac{c_2c_0}{s^2})\|u\|^2_{s(1,N)} + 2s(1 - \frac{c_2c_0}{s})\|u_t\|^2_{s(1,N-1)} + 2s\|\sqrt{A}u\|^2_{s(1,N-1)} \\ &\le s\mu_2^{-1}\big\{2s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle + \Psi^2_N\langle u_N, \sqrt{A_+}u_N\rangle\big\} + 2c_2c_0\|u_0\|^2 \\ &\quad + 2sc_1\|v_0\|^2 + c_2\big\{\langle v_0, \sqrt{A_+}v_0\rangle + \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-1}\rangle\big\} + 2c_2c_0\|f\|^2_s. \end{aligned}$$

Choosing s_0 large enough, it is possible to achieve the condition $1 - \frac{c_2c_0}{s} \ge \frac{1}{2}$ when $s \ge s_0$ (especially $1 - \frac{c_2c_0}{s^2} \ge \frac{1}{2}$). Therefore,

$$\begin{aligned} s^2\|u\|^2_{s(1,N)} \le{}& s\mu_2^{-1}\big\{2s\mu_0\|u_0\|^2 + \langle u_0, \sqrt{A_+}u_0\rangle + \Psi^2_N\langle u_N, \sqrt{A_+}u_N\rangle\big\} \\ &+ 2c_2c_0\|u_0\|^2 + c_2\big\{\langle v_0, \sqrt{A_+}v_0\rangle \\ &+ \Psi^2_{N-1}\langle v_{N-1}, \sqrt{A_+}v_{N-1}\rangle a\big\} + 2c_2c_0\|f\|^2_s. \end{aligned} \tag{4.18}$$

Note that $v_{N-1}=u_{tN-1}-\sqrt{A}u_{N-1}, v_0=-\sqrt{A}u_0$, by (4.9) and (4.10). Taking into account that $\sqrt{A}=\sqrt{A_+}+i\sqrt{A_-}$, $(\sqrt{A_\pm})^*=\sqrt{A_\pm}\ge 0$, $\sqrt{A_\pm}\sqrt{A_\mp}=0$ it is easy to establish the following equalities:

$$
\begin{aligned}
\|v_0\|^2=\langle A_+u_0,u_0\rangle+\langle A_-u_0,u_0\rangle, \langle v_0,\sqrt{A_+}v_0\rangle=\langle\sqrt{A_+}\sqrt{A_+}u_0,\sqrt{A_+}u_0\rangle,\\
\langle v_{N-1},\sqrt{A_+}v_{N-1}\rangle=\langle\sqrt{A_+}u_{tN-1},u_{tN-1}\rangle-2\operatorname{Re}\langle A_+u_{tN-1},u_{N-1}\rangle\\
+\langle\sqrt{A_+}\sqrt{A_+}u_{N-1},\sqrt{A_+}u_{N-1}\rangle
\end{aligned}
$$

According to these equalities, and taking into account the obvious inequalities

$$
\begin{gathered}
\Psi_{j+1}<\Psi_j, \|u\|^2_{s(1,N)}\ge\|u\|^2_{s(1,N-1)}\ge\Psi^2_{N-1}\|u\|^2_{l_2(1,N-1;H)},\\
\|f\|^2_s\le\|f\|^2_{l_2(1,N-1;H)},
\end{gathered}
$$

from (4.18) we obtain the estimate

$$
\begin{aligned}
&s^2\Psi^2_{N-1}\|u\|^2_{l_2(1,N-1;H)}\\
&\le s\mu_2^{-1}\Big\{2s\mu_0\|u_0\|^2+\langle u_0,\sqrt{A_+}u_0\rangle+\Psi^2_{N-1}\langle u_N,\sqrt{A_+}u_N\rangle\Big\}+2c_2c_0\|u_0\|^2\\
&\quad+2sc_1(\langle A_+u_0,u_0\rangle+\langle A_-u_0,u_0\rangle)+c_2\Big\{\langle\sqrt{A_+}\sqrt{A_+}u_0,\sqrt{A_+}u_0\rangle\\
&\quad+\Psi^2_{N-1}\Big(\langle\sqrt{A_+}u_{tN-1},u_{tN-1}\rangle-2\operatorname{Re}\langle A_+u_{tN-1},u_{N-1}\rangle\\
&\quad+\langle\sqrt{A_+}\sqrt{A_+}u_{N-1},\sqrt{A_+}u_{N-1}\rangle\Big)\Big\}+2c_2c_0\|f\|^2_{l_2(0,N-1;H)}.
\end{aligned}
$$

Assuming $s\ge 1$ and

$$
\tilde{c}_1=\max\big(\frac{1}{\mu_2},c_2\big),\quad \tilde{c}_2=\frac{2\mu_0}{\mu_2}+2c_2c_0,\quad \tilde{c}_3=\max\big(\frac{1}{\mu_2},\ 2c_1,\ c_2,\ 2c_2c_0\big)
$$

from the above estimate we have

$$
\begin{aligned}
&s^2\Psi^2_{N-1}\|u\|^2_{l_2(1,N-1;H)}\\
&\le s\tilde{c}_1\Psi^2_{N-1}\Big\{\langle\sqrt{A_+}u_N,u_N\rangle+\langle\sqrt{A_+}u_{tN-1},u_{tN-1}\rangle\\
&\quad-2\operatorname{Re}\langle A_+u_{tN-1},u_{N-1}\rangle+\langle\sqrt{A_+}\sqrt{A_+}u_{N-1},\sqrt{A_+}u_{N-1}\rangle\Big\}\\
&\quad+s^2\tilde{c}_2\|u_0\|^2+s\tilde{c}_3\Big\{\langle\sqrt{A_+}u_0,u_0\rangle+\langle A_+u_0,u_0\rangle+\langle A_-u_0,u_0\rangle\\
&\quad+\langle\sqrt{A_+}\sqrt{A_+}u_0,\sqrt{A_+}u_0\rangle+\|f\|^2_{l_2(0,N-1;H)}\Big\}.
\end{aligned}
$$

Dividing both sides of this inequality by $s^2\Psi^2_{N-1}$, and supposing

$$
\varepsilon^2=\tilde{c}_1/s,\quad c^2(\varepsilon)=\max\big\{\tilde{c}_2\exp(2s\,\tilde{m}\,\tau),\ \tilde{c}_3s^{-1}\exp(2s\,\tilde{m}\,\tau)\big\},
$$

from inequality $1/\Psi_{N-1}<1/\Psi_N\le\exp(s\tilde{m}\tau)$ (see [8, Lemma 1.1, p.132]) we obtain

$$
\begin{aligned}
&\|u\|^2_{l_2(1,N-1;H)}\\
&\le\varepsilon^2\Big\{\langle\sqrt{A_+}u_N,u_N\rangle+\langle\sqrt{A_+}u_{tN-1},u_{tN-1}\rangle\\
&\quad-2\operatorname{Re}\langle A_+u_{tN-1},u_{N-1}\rangle+\langle\sqrt{A_+}\sqrt{A_+}u_{N-1},\sqrt{A_+}u_{N-1}\rangle\Big\}\\
&\quad+c^2(\varepsilon)\Big\{\|u_0\|^2+\langle\sqrt{A_+}u_0,u_0\rangle+\langle A_+u_0,u_0\rangle+\langle A_-u_0,u_0\rangle\\
&\quad+\langle\sqrt{A_+}\sqrt{A_+}u_0,\sqrt{A_+}u_0\rangle+\|f\|^2_{l_2(0,N-1;H)}\Big\}.
\end{aligned}
$$

Denoting $\|u\|_D^2 = \langle Du, u\rangle$, where the operator $D \ge 0$, we rewrite the last inequality in the form

$$\begin{aligned} \|u\|^2_{l_2(1,N-1;H)} \le \varepsilon^2 \Big\{ & \|u_N\|^2_{\sqrt{A_+}} + \|u_{tN-1}\|^2_{\sqrt{A_+}} \\ & - 2\,\mathrm{Re}\langle A_+ u_{tN-1}, u_{N-1}\rangle + \|\sqrt{A_+} u_{N-1}\|^2_{\sqrt{A_+}} \Big\} \\ & + c^2(\varepsilon)\Big\{ \|u_0\|^2 + \|u_0\|^2_{\sqrt{A_+}} + \|A_+^{1/2} u_0\| + \\ & + \|A_-^{1/2} u_0\| + \|\sqrt{A_+} u_0\|^2_{\sqrt{A_+}} + \|f\|^2_{l_2(0,N-1;H)} \Big\}. \end{aligned} \tag{4.19}$$

We now turn to the construction of the operators A_+, A_-, $\sqrt{A_+}$, $\sqrt{A_-}$. Consider for the operator

$$(Au)^k = -i\frac{u^{k+1} - u^{k-1}}{2h}, \quad i^2 = -1,$$

and the eigenvalue problem

$$\begin{gathered} Au^k = \lambda u^k, \quad k = 0, \pm 1, \dots, \pm(N_1 - 1), \quad \pm hN_1 = \pm T, \\ u^{N_1} = u^{-N_1} = 0. \end{gathered} \tag{4.20}$$

By direct calculations it is not difficult to show that eigenvalues of the operator A and the corresponding eigenfunctions are determined by the formulas

$$\lambda_m = \frac{1}{h}\sin\frac{\pi m}{2N_1}, \quad u_m^k = e^{ik\frac{\pi m}{2N_1}} - (-1)^{k-N_1} e^{i(2N_1-k)\frac{\pi m}{2N_1}},$$

for $k, m = 0, \pm 1, \dots, \pm(N_1 - 1)$, and norm of the eigenfunctions u_m^k in the sense of the above dot product is $\|u_m^k\|^2 = 4T$.

Since the eigenfunctions u_m^k are orthogonal, and consequently, linearly independent, then the functions $\mu_m^k = \frac{1}{2\sqrt{T}} u_m^k$ for orthonormal basis in the space H, consisting of eigenfunctions of the operator A, corresponding to $\{\lambda_{\rm m}\}$.

Since $A^* = A$, we have spectral decomposition of the operator A:

$$A = \sum_{m=-(N_1-1)}^{N_1-1} \lambda_m P_m = \sum_{m=0}^{N_1-1} \lambda_m P_m - \sum_{m=-(N_1-1)}^{-1} (-\lambda_m) P_m,$$

where P_m projector, defined by the relation $P_m u = \langle u, \mu_m^k\rangle \mu_m^k$, $u \in H$. Hence we see that the operators $A_\pm$ have the form:

$$A_+ = \sum_{m=0}^{N_1-1} \lambda_m P_m\,, \quad A_- = \sum_{m=1-N_1}^{-1} (-\lambda_m) P_m.$$

We define the operators $\sqrt{A_\pm}$ by

$$\sqrt{A_+} = \sum_{m=0}^{N_1-1} \lambda_m^{1/2} P_m\,, \quad \sqrt{A_-} = \sum_{m=1-N_1}^{-1} (-\lambda_m)^{1/2} P_m.$$

It is obvious that

$$\begin{gathered} \|A\| = \|A_+\| = \|A_-\| = \frac{1}{h}\sin\frac{\pi(N_1-1)}{2N_1} \le \frac{1}{h}, \\ \|\sqrt{A_+}\| = \|\sqrt{A_-}\| = \sqrt{\frac{1}{h}\sin\frac{\pi(N_1-1)}{2N_1}} \le \frac{1}{\sqrt{h}}\,. \end{gathered} \tag{4.21}$$

We transform the conditions

$$\tau\|\sqrt{A_+}\|^2 \le c, \tau\sqrt{A_+} \le mE, c, m > 0, \tag{4.22}$$

under which we obtained stability of the two-layer difference schemes (4.9), (4.10). Note that the condition $\tau\sqrt{A_+} \le mE$ is satisfied if $\tau\|\sqrt{A_+}\| \le m$. Indeed,

$$\begin{aligned}\langle (mE - \tau\sqrt{A_+})u, u\rangle &= m\|u\|^2 - \tau\langle\sqrt{A_+}u, u\rangle \\ &\ge m\|u\|^2 - \tau\|\sqrt{A_+}\|\|u\|^2 \\ &= (m - \tau\|\sqrt{A_+}\|)\|u\|^2 \ge 0,\end{aligned}$$

if $\tau\|\sqrt{A_+}\| \le m$. By (4.21) and this remark, instead of the condition (4.22) we obtain the conditions

$$\begin{gathered}\frac{\tau}{h}\sin\frac{\pi(N_1-1)}{2N_1} \le c, \quad \tau \le \frac{ch}{sin\frac{\pi(N_1-1)}{2N_1}}, \\ \frac{\tau^2}{h}\sin\frac{\pi(N_1-1)}{2N_1} \le m^2, \quad \tau^2 \le \frac{m^2h}{sin\frac{\pi(N_1-1)}{2N_1}}.\end{gathered} \tag{4.23}$$

Since $\sin\frac{\pi(N_1-1)}{2N_1} = \sin\frac{\pi}{2}(\frac{T-h}{T}) = O(1)$ at small h, from conditions (4.23) we obtain the condition:

$$\tau^2 \le c \cdot h,$$

where $c = \min(c, ; m)$, c and m are constants from (4.22). Therefore, the following theorem of conditional stability of solution of the difference scheme (4.1).

Theorem 4.1. *Let $\tau^2 \le ch$ and $c > 0$. Then for all $\tau \in (0, \tau_0]$, ($\tau_0 = \sqrt{ch}$), $\varepsilon > 0$, $u : Z_0^N \to H$, to solve the difference scheme* (4.1) *we have the stability estimate* (4.19).

Conclusions. On the basis of notion of stability of a difference scheme on functions with compact support, stability criteria for two-layer difference schemes, that approximate an ill-posed abstract Cauchy problem, are obtained. Stability of difference schemes is based on obtaining a priori difference weighted Carleman type estimates. Obtained stability criteria are used to prove conditional stability of the solution of a three-layer difference scheme for an ill-posed Cauchy problem. In connection with cumbersomeness and technical complexity of obtaining a difference analogue of weighted stability estimates for three-layer schemes, preliminary factorization of the problem into a sequence of two-layer schemes was carried out.

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