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LOGARITHMIC REGULARIZATION OF NON-AUTONOMOUS NON-LINEAR ILL-POSED PROBLEMS IN HILBERT SPACES

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Abstract. The regularization of non-autonomous non-linear ill-posed problems is established using a logarithmic approximation originally proposed by Boussetila and Rebbani, and later modified by Tuan and Trong. We first prove continuous dependence on modeling where the solution of the original ill-posed problem is estimated by the solution of an approximate well-posed problem. Finally, we illustrate the convergence via numerical experiments in L^2 spaces.

1. Introduction

In this paper, we study a class of non-linear non-autonomous ill-posed problems. In recent literature, the regularization of ill-posed problems is a topic of substantial investigation with applications to various natural phenomena, especially inverse processes such as backward diffusion (cf. [16]). Ill-posed problems such as the backward heat equation

$$\begin{gathered} u_t = -u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x) \end{gathered} \tag{1.1}$$

may lack existence and/or uniqueness of solutions corresponding to certain initial data, or may possess solutions that do not depend continuously on the initial data.

The regularization of ill-posed problems involves defining an "ϵ-close" well-posed problem whose solutions approximate solutions of the original ill-posed problem. Let us set $A = -\Delta$ and consider functions $t \mapsto u(t)$ having range in $L^2(\mathbb{R})$. Then (1.1) becomes the abstract Cauchy problem

$$\begin{gathered} \frac{du}{dt} = Au, \quad t > 0, \\ u(0) = \varphi. \end{gathered} \tag{1.2}$$

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Lattes and Lions [10] define the perturbation $f_\beta(A) = A - \beta A^2$, $\beta > 0$ yielding an approximate well-posed problem

$$\begin{aligned} \frac{dv}{dt} &= f_\beta(A)v, \quad t > 0, \\ v(0) &= \varphi. \end{aligned} \tag{1.3}$$

Moreover, if φ is replaced with φ_δ satisfying $\|\varphi - \varphi_\delta\|_2 \le \delta$, one may find $\beta = \beta(\delta)$ such that $\beta \to 0$ as $\delta \to 0$, and $\|v_\beta^\delta(t) - u(t)\|_2 \to 0$ as $\delta \to 0$ for each $t \ge 0$ (Here $v_\beta^\delta(t)$ is the solution of (1.3) corresponding to initial data φ_δ).

Many other authors including Miller [13], Showalter [15], and Mel'nikova [12] pioneered similar methods of regularization; for example, Showalter applies a bounded approximation $f_\beta(A) = A(I+\beta A)^{-1}$ in [15]. More recently, extensions to variations of (1.2) have been established by Ames and Hughes [1], Long and Dinh [11], Trong and Tuan [17, 18], Huang and Zheng [8, 9], Boussetila and Rebbani [2], and Fury [4, 5]. For instance, Trong and Tuan [18] consider the non-linear problem

$$\begin{aligned} \frac{du}{dt} &= Au + h(t, u(t)), \quad 0 < t < T, \\ u(0) &= \varphi \end{aligned} \tag{1.4}$$

with a Lipschitz condition on h. Applying Boussetila and Rebbani's logarithmic approximation

$$f_\beta(A) = -\frac{1}{pT} \ln(\beta + e^{-pTA}), \quad \beta > 0,\ p \ge 1, \tag{1.5}$$

which is of milder error order than $f_\beta(A) = A - \beta A^2$ or $f_\beta(A) = A(I + \beta A)^{-1}$, Trong and Tuan establish regularization for (1.4) where h satisfies a global Lipschitz condition. In a more recent paper [19], taking $p = T = 1$, Tuan and Trong modify (1.5) to

$$f_\beta(A) = -\ln(\beta A + e^{-A}), \quad 0 < \beta < 1 \tag{1.6}$$

in order to treat the case where h is locally Lipschitz.

In this paper, we apply a version of (1.6) to problems that are both non-linear and *non-autonomous*. We consider the problem with non-constant operators,

$$\begin{aligned} \frac{du}{dt} &= A(t, D)u(t) + h(t, u(t)) \quad 0 \le s < t < T \\ u(s) &= \varphi \end{aligned} \tag{1.7}$$

in a Hilbert space H where D is a positive, self-adjoint operator in H, $A(t, D) = \sum_{j=1}^k a_j(t) D^j$ with $a_j \in C([0, T] : \mathbb{R}^+) \cap C^1([0, T])$ for each $1 \le j \le k$, and $h : [s, T] \times H \to H$ satisfies (H1)–(H2) below (Section 2). Problem (1.7) is ill-posed since $\{A(t, D)\}_{t \in [0,T]}$ is not a stable family of generators; in fact since each $a_j(t) > 0$, none of the operators $A(t, D)$ generates a C_0 semigroup on H (cf. [14, Section 5.2], [7, Theorem 2.1.2]). Also, note that taking $D = -\Delta$, $k = 1$ and $a_k(t) = a_1(t) \equiv 1$, i.e. $A(t, D) = -\Delta$, problem (1.7) reduces to the non-linear backward heat equation (1.4) which is certainly ill-posed.

Based on (1.7), consider the approximate well-posed problem

$$\begin{aligned} \frac{dv}{dt} &= f_\beta(t, D)v(t) + h(t, v(t)) \quad 0 \le s < t < T \\ v(s) &= \varphi \end{aligned} \tag{1.8}$$

where following Tuan and Trong [19], we define $f_\beta(t,D)$ by (2.1)–(2.2) below. We show that if $u(t)$ is a solution of (1.7) adhering to certain stabilizing conditions, then

$$\|u(t)-v_\beta(t)\| \le C'\beta^{\frac{T-t}{T-s}}\,[1-\ln\beta]^{\frac{s-t}{T-s}} \quad \text{for } 0\le s\le t\le T \tag{1.9}$$

where $v_\beta(t)$ is the unique solution of (1.8) and C' is a nonnegative constant independent of both β and t. Note that by letting $t=T$ in (1.9), we have $\|u(T)-v_\beta(T)\| \le C'(1-\ln\beta)^{-1}\to 0$ as $\beta\to 0$. Thus, the estimate (1.9) is a considerable improvement over other Hölder-continuous dependence results such as $\|u(\tau)-v_\beta(\tau)\|\le C\beta^{1-\frac{\tau}{T}}$, $0\le\tau<T$ which is inapplicable when $\tau=T$ (cf. [1, 5, 6, 11, 17, 18]).

In Section 4, we prove regularization for (1.7) which follows quickly from (1.9). In the last section of the paper, Section 5, we apply the theory to higher order partial differential equations with variable coefficients in L^2 spaces. We also provide some numerical experiments in order to demonstrate the convergence of the solutions $v_\beta^\delta(t)$ to $u(t)$ within concrete examples.

2. Approximate well-posed problem

Consider the generally ill-posed problem (1.7) where D is a positive, self-adjoint operator in a Hilbert space H and $A(t,D)=\sum_{j=1}^k a_j(t)D^j$ satisfies $a_j\in C([0,T]:\mathbb{R}^+)\cap C^1([0,T])$ for each $1\le j\le k$. Also let us assume the following conditions on $h:[s,T]\times H\to H$:

- (H1) h is uniformly Lipschitz in H, i.e. $\|h(t,\varphi_1)-h(t,\varphi_2)\|\le L\|\varphi_1-\varphi_2\|$ for some constant $L>0$ independent of $t\in[s,T]$ and every $\varphi_1,\varphi_2\in H$,
- (H2) for each $\varphi\in H$, $h(t,\varphi)$ is continuous from $[s,T]$ into H.

Set $\tau=T-s$. For $(t,\lambda)\in[0,T]\times[0,\infty)$, define the function

$$f_\beta(t,\lambda)=\max\{0,-\frac{1}{\tau}\ln(\beta\tau A(t,\lambda)+e^{-\tau A(t,\lambda)})\},\quad 0<\beta<1. \tag{2.1}$$

Then for each $0\le t\le T$, $f_\beta(t,D)$ is defined by means of the functional calculus for self-adjoint operators in the Hilbert space H. Particularly, since $f_\beta(t,\lambda)$ is a Borel function defined for $\lambda\in[0,\infty)$, the operator $f_\beta(t,D)$ is then defined by

$$\begin{gathered}\operatorname{Dom}(f_\beta(t,D))=\{\varphi\in H:\int_{\sigma(D)}|f_\beta(t,\lambda)|^2d(E(\lambda)\varphi,\varphi)<\infty\},\\ f_\beta(t,D)\varphi=\int_{\sigma(D)}f_\beta(t,\lambda)dE(\lambda)\varphi\quad\text{for }\varphi\in\operatorname{Dom}(f_\beta(t,D)),\end{gathered} \tag{2.2}$$

where $\{E(\cdot)\}$ denotes the resolution of the identity associated with the operator D and $\sigma(D)$ is its spectrum (cf. [3, Theorem XII.2.3, Theorem XII.2.6]). Note that since D is positive, self-adjoint, we have $\sigma(D)\subseteq[0,\infty)$.

Let us find the maximum and minimum values of $f_\beta(t,\lambda)$ on $[0,T]\times[0,\infty)$. Note, the function $F(x)=-\frac{1}{\tau}\ln(\beta\tau x+e^{-\tau x})$, $x\ge 0$ has $F'(x)=\frac{e^{-\tau x}-\beta}{\beta\tau x+e^{-\tau x}}$. Hence, $F(x)$ attains an absolute maximum at $x_M=-\frac{1}{\tau}\ln\beta$ so that $F(x)\le F(x_M)=-\frac{1}{\tau}\ln[\beta(1-\ln\beta)]$ for $x\ge 0$. Furthermore, since $F(x_M)>0$ and $\lim_{x\to\infty}F(x)=-\infty$, we obtain a unique $x_\beta>x_M$ such that $F(x)\ge 0$ on $[0,x_\beta]$ and $F(x)<0$ on (x_β,∞). By (2.1), it follows that

$$0\le f_\beta(t,\lambda)\le -\frac{1}{\tau}\ln[\beta(1-\ln\beta)]\quad\text{for }(t,\lambda)\in[0,T]\times[0,\infty) \tag{2.3}$$

and so for each $t \in [0,T]$, $f_\beta(t,D)$ is a bounded operator on H satisfying

$$\|f_\beta(t,D)\| \le -\frac{1}{\tau}\ln\left[\beta(1-\ln\beta)\right] \quad \text{for all } 0 \le t \le T. \tag{2.4}$$

Proposition 2.1. *Let H be a Hilbert space and for $0<\beta<1$, let the operators $f_\beta(t,D), 0 \le t \le T$ be defined by (2.1)–(2.2). Assume the function $h:[s,T]\times H \to H$ satisfies conditions (H1) and (H2). Then (1.8) is well-posed, with unique classical solution $v_\beta(t)$ for every $\varphi \in H$ where $v_\beta(t)$ satisfies the integral equation*

$$v_\beta(t) = e^{\int_s^t f_\beta(q,D)dq}\varphi + \int_s^t e^{\int_r^t f_\beta(q,D)dq} h(r, v_\beta(r))dr. \tag{2.5}$$

Proof. See [5, Proposition 2.1]. In particular, $e^{\int_s^t f_\beta(q,D)dq}$ is an evolution system on H which by (2.4), satisfies

$$\|e^{\int_s^t f_\beta(q,D)dq}\| \le [\beta(1-\ln\beta)]^{\frac{s-t}{T-s}} \quad \text{for all } 0 \le s \le t \le T. \tag{2.6}$$

Well-posedness follows immediately from (2.6). □

The following lemma will aid in establishing continuous dependence on modeling and is motivated by the approximation condition, Condition A, of Ames and Hughes (cf. [1, Definition 1], and also [18, Definition p. 4]).

Lemma 2.2. *Let H be a Hilbert space and for $0<\beta<1$, let the operators $f_\beta(t,D), 0 \le t \le T$ be defined by (2.1)–(2.2). Define $B(\lambda) = \sum_{j=1}^k B_j\lambda^j$ where $B_j = \max_{t\in[0,T]} a_j(t)$ for each $1 \le j \le k$. Then for each $t \in [0,T]$,*

$$\operatorname{Dom}(B(D)e^{\tau B(D)}) \subseteq \operatorname{Dom}(A(t,D)) \cap \operatorname{Dom}(f_\beta(t,D)),$$
$$\|(-A(t,D)+f_\beta(t,D))\varphi\| \le \sqrt{2}\beta\|B(D)e^{\tau B(D)}\varphi\|$$

for all $\varphi \in \operatorname{Dom}(B(D)e^{\tau B(D)})$.

Proof. Let $t \in [0,T]$. For $\lambda \ge 0$, we have $0 \le A(t,\lambda) \le B(\lambda) \le B(\lambda)e^{\tau B(\lambda)}$ which by (2.2) shows that $\operatorname{Dom}(A(t,D)) \supseteq \operatorname{Dom}(B(D)e^{\tau B(D)})$. Certainly, $\operatorname{Dom}(f_\beta(t,D)) = H \supseteq \operatorname{Dom}(B(D)e^{\tau B(D)})$ as well since $f_\beta(t,D)$ is a bounded operator. Now let $\varphi \in \operatorname{Dom}(B(D)e^{\tau B(D)})$ and let x_β be as in the paragraph preceding inequality (2.3). Set $e_\beta = \{\lambda \ge 0 : B(\lambda) \le x_\beta\}$ and let e'_β be the complement of e_β in $[0,\infty)$. We have

$$\begin{aligned}
&\int_{e_\beta} |-A(t,\lambda)+f_\beta(t,\lambda)|^2 d(E(\lambda)\varphi,\varphi)\\
&= \int_{e_\beta} |A(t,\lambda) + \frac{1}{\tau}\ln(\beta\tau A(t,\lambda)+e^{-\tau A(t,\lambda)})|^2 d(E(\lambda)\varphi,\varphi)\\
&= \int_{e_\beta} |\frac{1}{\tau}\ln(e^{\tau A(t,\lambda)}) + \frac{1}{\tau}\ln(\beta\tau A(t,\lambda)+e^{-\tau A(t,\lambda)})|^2 d(E(\lambda)\varphi,\varphi)\\
&= \int_{e_\beta} |\frac{1}{\tau}\ln(\beta\tau A(t,\lambda)e^{\tau A(t,\lambda)}+1)|^2 d(E(\lambda)\varphi,\varphi).
\end{aligned}$$

Applying the fact that $\ln(x+1) \le x$ for $x \ge 0$, we get

$$\int_{e_\beta} |-A(t,\lambda)+f_\beta(t,\lambda)|^2 d(E(\lambda)\varphi,\varphi) \le \int_{e_\beta} |\beta A(t,\lambda)e^{\tau A(t,\lambda)}|^2 d(E(\lambda)\varphi,\varphi)$$

$$\leq \int_0^\infty |\beta B(\lambda)e^{\tau B(\lambda)}|^2 d(E(\lambda)\varphi,\varphi)$$
$$= \beta^2 \|B(D)e^{\tau B(D)}\varphi\|^2.$$

Also, since $x_\beta > -\frac{1}{\tau}\ln\beta$, we have

$$\begin{aligned}\int_{e'_\beta} |-A(t,\lambda)+f_\beta(t,\lambda)|^2 d(E(\lambda)\varphi,\varphi) &= \int_{e'_\beta} |A(t,\lambda)|^2 d(E(\lambda)\varphi,\varphi)\\ &\leq \int_{e'_\beta} |e^{-\tau B(\lambda)}e^{\tau B(\lambda)}B(\lambda)|^2 d(E(\lambda)\varphi,\varphi)\\ &< \int_{e'_\beta} |\beta B(\lambda)e^{\tau B(\lambda)}|^2 d(E(\lambda)\varphi,\varphi)\\ &\leq \int_0^\infty |\beta B(\lambda)e^{\tau B(\lambda)}|^2 d(E(\lambda)\varphi,\varphi)\\ &= \beta^2\|B(D)e^{\tau B(D)}\varphi\|^2.\end{aligned}$$

Combining yields $\|(-A(t,D)+f_\beta(t,D))\varphi\|^2 \leq 2\beta^2\|B(D)e^{\tau B(D)}\varphi\|^2$, which proves the desired result. □

Following Lemma 2.2, let us define for $(t,\lambda)\in[0,T]\times[0,\infty)$,

$$g_\beta(t,\lambda) = -A(t,\lambda)+f_\beta(t,\lambda). \tag{2.7}$$

Note, $\ln(\beta\tau A(t,\lambda)+e^{-\tau A(t,\lambda)}) \geq \ln(e^{-\tau A(t,\lambda)}) = -\tau A(t,\lambda)$ which, after dividing through by $-\tau$, yields $f_\beta(t,\lambda)\leq A(t,\lambda)$ and hence

$$g_\beta(t,\lambda)\leq 0 \quad \text{for } (t,\lambda)\in[0,T]\times[0,\infty). \tag{2.8}$$

For each natural number n, set

$$e_n = \{\lambda\geq 0 : B(\lambda)\leq n\}. \tag{2.9}$$

Then by (2.3) and (2.7), we have $|g_\beta(t,\lambda)|\leq n-\frac{1}{\tau}\ln[\beta(1-\ln\beta)]$ for all $(t,\lambda)\in[0,T]\times e_n$. Thus, if we set $E_n = E(e_n)$, then each of $A(t,D)E_n$, $f_\beta(t,D)E_n$, and $g_\beta(t,D)E_n$ is a bounded operator on H for all $t\in[0,T]$. Following [5, Lemma 2.3, Corollary 2.4], we obtain evolution systems $U_n(t,s)$, $V_{\beta,n}(t,s)$, and $W_{\beta,n}(t,s)$ satisfying the following for all $\varphi_n\in E_nH$ and all $0\leq s\leq t\leq T$:

- (S1) $U_n(t,s)\varphi_n = e^{\int_s^t A(q,D)dq}\varphi_n$, $V_{\beta,n}(t,s)\varphi_n = e^{\int_s^t f_\beta(q,D)dq}\varphi_n$, and $W_n(t,s)\varphi_n = e^{\int_s^t g_\beta(q,D)dq}\varphi_n$
- (S2) $U_n(t,s)W_{\beta,n}(t,s)\varphi_n = V_{\beta,n}(t,s)\varphi_n = W_{\beta,n}(t,s)U_n(t,s)\varphi_n$.

3. Continuous dependence on modeling

In this section, we use the results from Section 2 to prove continuous dependence on modeling for the ill-posed problem (1.7) (Theorem 3.2 below).

Lemma 3.1. *Let $u(t)$ and $v_\beta(t)$ be classical solutions of* (1.7) *and* (1.8) *respectively where the operators $f_\beta(t,D), 0\leq t\leq T$ are defined by* (2.1)–(2.2) *and $h:[s,T]\times H\to H$ satisfies the hypotheses of Proposition 2.1. Also, set $\varphi_n = E_n\varphi$ and $h_n(t,\varphi) = E_nh(t,\varphi)$ for all $(t,\varphi)\in[s,T]\times H$. Then*

$$E_nu(t) = U_n(t,s)\varphi_n + \int_s^t U_n(t,r)h_n(r,u(r))dr,$$

$$E_n v_\beta(t) = V_{\beta,n}(t,s)\varphi_n + \int_s^t V_{\beta,n}(t,r)h_n(r,v_\beta(r))dr$$

for all $t \in [s,T]$.

Proof. The first identity follows from uniqueness of solutions since both sides of the equation are classical solutions of the linear inhomogeneous problem

$$\begin{gathered} \frac{dw}{dt} = A(t,D)E_n w(t) + h_n(t,u(t)) \quad 0 \le s \le t < T \\ w(s) = \varphi_n. \end{gathered} \tag{3.1}$$

The second identity holds by a similar argument with $A(t,D)E_n$ replaced by $f_\beta(t,D)E_n$ in (3.1). □

As in Lemma 2.2, set $B(\lambda) = \sum_{j=1}^k B_j\lambda^j$ where $B_j = \max_{t\in[0,T]} a_j(t)$ for each $1 \le j \le k$. We have

Theorem 3.2. *Let $u(t)$ and $v_\beta(t)$ be classical solutions of (1.7) and (1.8) respectively where the operators $f_\beta(t,D), 0 \le t \le T$ are defined by (2.1)–(2.2) and $h : [s,T] \times H \to H$ satisfies the hypotheses of Proposition 2.1. Then if there exist constants $M', M'' \ge 0$ such that $\|B(D)e^{(T-s)B(D)}e^{\int_s^t A(q,D)dq}\varphi\| \le M'$ and $\|B(D)e^{(T-s)B(D)}e^{\int_s^t A(q,D)dq}h(t,u(t))\| \le M''$ for all $t \in [s,T]$, then there exist constants C and L independent of β such that*

$$\|u(t) - v_\beta(t)\| \le \beta^{\frac{T-t}{T-s}}(1-\ln\beta)^{\frac{s-t}{T-s}}Ce^{L(T-s)} \quad \text{for } 0 \le s \le t \le T. \tag{3.2}$$

Proof. Set $\varphi_n = E_n\varphi$ and $h_n(t,\varphi) = E_n h(t,\varphi)$ for all $(t,\varphi) \in [s,T] \times H$. From Lemma 3.1, for $0 \le s \le t \le T$,

$$\begin{aligned} &\|E_n u(t) - E_n v_\beta(t)\| \\ &\le \|U_n(t,s)\varphi_n - V_{\beta,n}(t,s)\varphi_n\| \\ &\quad + \int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,v_\beta(r))\|dr \\ &\le \|U_n(t,s)\varphi_n - V_{\beta,n}(t,s)\varphi_n\| && (3.3) \\ &\quad + \int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,u(r))\|dr && (3.4) \\ &\quad + \int_s^t \|V_{\beta,n}(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,v_\beta(r))\|dr. && (3.5) \end{aligned}$$

For the first expression, by (S2) and [14, Theorem 5.1.2], we have

$$\begin{aligned} (3.3) &= \|(I - W_{\beta,n}(t,s))U_n(t,s)\varphi_n\| \\ &= \|(W_{\beta,n}(t,t) - W_{\beta,n}(t,s))U_n(t,s)\varphi_n\| \\ &= \Big\|\int_s^t \frac{\partial}{\partial p}W_{\beta,n}(t,p)U_n(t,s)\varphi_n dp\Big\| \\ &= \Big\|\int_s^t (-W_{\beta,n}(t,p)g_\beta(p,D)E_n)U_n(t,s)\varphi_n dp\Big\| \\ &\le \int_s^t \|W_{\beta,n}(t,p)g_\beta(p,D)U_n(t,s)\varphi_n\|dp. \end{aligned}$$

Next from (2.9) and (2.2), note that $U_n(t,s)\varphi_n \in \mathrm{Dom}(B(D)e^{(T-s)B(D)})$. Therefore, by (S1), (2.8), and Lemma 2.2, we have

$$\begin{aligned}(3.3) &\le \int_s^t \|g_\beta(p,D)U_n(t,s)\varphi_n\|dp.\\ &\le \sqrt{2}\beta(t-s)\|B(D)e^{(T-s)B(D)}U_n(t,s)\varphi_n\|.\end{aligned}$$

Similarly, for the second expression,

$$\begin{aligned}(3.4) &= \int_s^t \|(I-W_{\beta,n}(t,r))U_n(t,r)h_n(r,u(r))\|dr\\ &\le \int_s^t \sqrt{2}\beta(t-r)\|B(D)e^{(T-s)B(D)}U_n(t,r)h_n(r,u(r))\|dr.\end{aligned}$$

Combining the above we have

$$\begin{aligned}&\|U_n(t,s)\varphi_n - V_{\beta,n}(t,s)\varphi_n\|\\ &+\int_s^t \|U_n(t,r)h_n(r,u(r)) - V_{\beta,n}(t,r)h_n(r,u(r))\|dr \le \beta C\end{aligned} \tag{3.6}$$

where C is a constant independent of β and also independent of n and t by our stabilizing constants M' and M''. Finally, by (S1), (2.6), and (H1), the third expression satisfies

$$\begin{aligned}(3.5) &= \int_s^t \|V_{\beta,n}(t,r)(h_n(r,u(r)) - h_n(r,v_\beta(r)))\|dr\\ &\le \int_s^t [\beta(1-\ln\beta)]^{\frac{r-t}{T-s}}\|h_n(r,u(r)) - h_n(r,v_\beta(r))\|dr\\ &\le L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have shown that

$$\|E_n u(t) - E_n v_\beta(t)\| \le \beta C + L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr,$$

and since all constants on the right are independent of n, we may let $n\to\infty$ to obtain

$$\|u(t)-v_\beta(t)\| \le \beta C + L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-t}{T-s}}\|u(r)-v_\beta(r)\|dr. \tag{3.8}$$

Note that $0<\beta<1$ implies

$$0 < [\beta(1-\ln\beta)]^{\frac{t-s}{T-s}} < 1 \quad \text{for all } t\in[s,T]. \tag{3.9}$$

Hence multiplying (3.8) through by $[\beta(1-\ln\beta)]^{\frac{t-s}{T-s}}$ and applying (3.9), we obtain

$$[\beta(1-\ln\beta)]^{\frac{t-s}{T-s}}\|u(t)-v_\beta(t)\| \le \beta C + L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-s}{T-s}}\|u(r)-v_\beta(r)\|dr.$$

Gronwall's inequality (cf. [14, Theorem 6.1.2]) then yields the estimate

$$[\beta(1-\ln\beta)]^{\frac{t-s}{T-s}}\|u(t)-v_\beta(t)\| \le \beta C e^{L(T-s)}$$

which is equivalent to (3.2). □

4. Regularization for problem (1.7)

Below, Theorem 4.1 establishes the main result of the paper, that is regularization for (1.7). Its proof uses our estimate from Theorem 3.2.

Theorem 4.1. *Let $u(t)$ be a classical solution of (1.7) and assume the hypotheses of Theorem 3.2. Then given $\delta > 0$, there exists $\beta = \beta(\delta) > 0$ such that*

(i) $\beta \to 0$ *as* $\delta \to 0$,
(ii) $\|u(t) - v_\beta^\delta(t)\| \to 0$ *as* $\delta \to 0$ *for* $s \le t \le T$ *whenever* $\|\varphi - \varphi_\delta\| \le \delta$

where $v_\beta^\delta(t)$ is the solution of (1.8) with initial data φ_δ.

Proof. Let $\delta > 0$ be given and let $\|\varphi - \varphi_\delta\| \le \delta$. Also, let $v_\beta(t)$ be the solution of (1.8) as in Theorem 3.2. For $s \le t \le T$, by Theorem 3.2, then

$$\begin{aligned} \|u(t) - v_\beta^\delta(t)\| &\le \|u(t) - v_\beta(t)\| + \|v_\beta(t) - v_\beta^\delta(t)\| \\ &\le \beta^{\frac{T-t}{T-s}}(1-\ln\beta)^{\frac{s-t}{T-s}} Ce^{L(T-s)} + \|v_\beta(t) - v_\beta^\delta(t)\|. \end{aligned} \tag{4.1}$$

Consider the second quantity in (4.1). By (2.6) and (H1), we have

$$\begin{aligned} &\|v_\beta(t) - v_\beta^\delta(t)\| \\ &\le \|e^{\int_s^t f_\beta(q,D)dq}(\varphi - \varphi_\delta)\| + \int_s^t \|e^{\int_r^t f_\beta(q,D)dq}(h(r, v_\beta(r)) - h(r, v_\beta^\delta(r)))\|dr \\ &\le \delta\,[\beta(1-\ln\beta)]^{\frac{s-t}{T-s}} + L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-t}{T-s}} \|v_\beta(r) - v_\beta^\delta(r)\|dr. \end{aligned}$$

Hence,

$$[\beta(1-\ln\beta)]^{\frac{t-s}{T-s}} \|v_\beta(t) - v_\beta^\delta(t)\| \le \delta + L\int_s^t [\beta(1-\ln\beta)]^{\frac{r-s}{T-s}} \|v_\beta(r) - v_\beta^\delta(r)\|dr$$

which by Gronwall's Inequality gives us

$$[\beta(1-\ln\beta)]^{\frac{t-s}{T-s}} \|v_\beta(t) - v_\beta^\delta(t)\| \le \delta e^{L(T-s)}.$$

Therefore, $\|v_\beta(t) - v_\beta^\delta(t)\| \le \delta\,[\beta(1-\ln\beta)]^{\frac{s-t}{T-s}} e^{L(T-s)}$ and choosing $\beta = \delta$ yields

$$\|v_\beta(t) - v_\beta^\delta(t)\| \le \beta^{\frac{T-t}{T-s}}(1-\ln\beta)^{\frac{s-t}{T-s}} e^{L(T-s)}. \tag{4.2}$$

Thus $\beta \to 0$ as $\delta \to 0$, and combining (4.1) with (4.2), we obtain

$$\|u(t) - v_\beta^\delta(t)\| \le \beta^{\frac{T-t}{T-s}}(1-\ln\beta)^{\frac{s-t}{T-s}}(C+1)e^{L(T-s)} \to 0 \quad \text{as } \delta \to 0.$$

□

5. Examples

The theory of this paper may be applied to a wide class of ill-posed partial differential equations in L^2 spaces including the backward heat equation with a time-dependent diffusion coefficient. Let us examine a concrete example of higher order with $H = L^2(0,\pi)$ where for $\varphi \in L^2(0,\pi)$, $\|\varphi\|_2 = \left(\int_0^\pi |\varphi(x)|^2 dx\right)^{1/2}$. Also define $D\varphi = -\varphi''$ for all twice-differentiable $\varphi \in L^2(0,\pi)$ whose first and second derivatives in the sense of distributions are also members of $L^2(0,\pi)$. Consider the fourth-order non-linear partial differential equation

$$\begin{gathered} u_t + u_{xx} - e^t u_{xxxx} = \psi(u) - e^{e^t}\sin x - e^{2e^t}\sin^2 x, \\ (x,t)\in(0,\pi)\times(0,1) \\ u(0,t) = u(\pi,t) = 0, \quad t\in[0,1] \\ u(x,0) = e\sin x, \quad x\in[0,\pi] \end{gathered} \tag{5.1}$$

where $\psi(u)$ is a compactly supported continuous function which coincides with u^2 on a sufficiently large interval centered at the origin. For example, following [18, Section 4], let us fix M large and positive, and define

$$\psi(u) = \begin{cases} u^2 & |u|\le M \\ Mu + 2M^2 & -2M \le u < -M \\ -Mu + 2M^2 & M < u \le 2M \\ 0 & |u| > 2M \end{cases}$$

(see Figure 1).

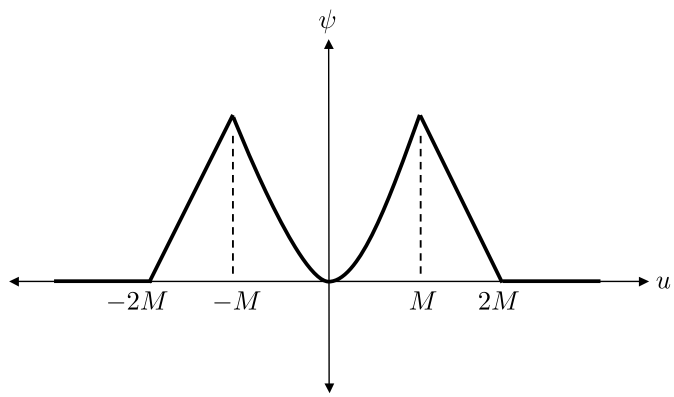

Figure 1. $\psi(u)$

Note, (5.1) is an example of (1.7) where $A(t,D) = D + e^t D^2$, $a_1(t)\equiv 1$, $a_k(t) = a_2(t) = e^t$, $h(x,t,u(x,t)) = \psi(u(x,t)) - e^{e^t}\sin x - e^{2e^t}\sin^2 x$, and $\varphi(x) = e\sin x$. It is straight-forward to check that the function h satisfies conditions (H1) and (H2), and that the exact solution of (5.1) is $u(x,t) = e^{e^t}\sin x$.

For the corresponding well-posed problem, following work in [11] and [18], let us assume an approximate solution of the form $v_N(x,t) = \sum_{n=1}^N v_n(t)\sin(nx)$. Set $\varphi_\delta(x) = (e + \delta\sqrt{\frac{2}{\pi}})\sin x$ so that $\|\varphi - \varphi_\delta\|_2 = \delta$. Then solving (1.8) is equivalent to solving the system of N differential equations

$$\begin{aligned} &v_m'(t) + \ln(\beta(m^2 + e^t m^4) + e^{-(m^2+e^t m^4)})v_m(t) \\ &= \frac{2}{\pi}\int_0^\pi h(x,t,v(x,t))\sin(mx)\,dx, \quad t\in(0,1),\ 1\le m\le N, \\ &v_1(0) = e + \delta\sqrt{\frac{2}{\pi}}, \quad v_2(0) = v_3(0) = \dots = v_N(0) = 0 \end{aligned} \tag{5.2}$$

where $h(x,t,v(x,t)) = \psi(v(x,t)) - e^{e^t}\sin x - e^{2e^t}\sin^2 x$.

We apply a finite difference method in order to estimate the solution $v_N(x,t)$ of (5.2). Let

$$\Delta t = \frac{1}{100}, \quad t_i = i\Delta t, \quad 0 \le i \le 100.$$

For each $i = 0, 1, 2, \dots$, we solve the N difference equations

$$\begin{aligned} &\frac{v_m(t_{i+1}) - v_m(t_i)}{\Delta t} + \ln\big(\beta(m^2 + e^{t_i}m^4) + e^{-(m^2+e^{t_i}m^4)}\big)\big(\frac{v_m(t_{i+1}) + v_m(t_i)}{2}\big) \\ &= \frac{2}{\pi}\int_0^\pi \big([\sum_{n=1}^N v_n(t_i)\sin(nx)]^2 - e^{e^{t_i}}\sin x - e^{2e^{t_i}}\sin^2 x\big)\sin(mx)\,dx, \quad 1 \le m \le N \end{aligned}$$

for the unknown $v_m(t_{i+1})$. Tables 1 and 2 illustrate our calculations with $N = 5$, $i = 0, 1, 2, 3, 4$, and the indicated values for δ. Note as in the proof of Theorem 4.1, β is chosen to be the same value as δ in each table. As expected, we find a smaller L^2-difference between $u(x,t)$ and $v_N(x,t)$ for each t as δ is taken closer to zero.

Table 1. $\beta = \delta = 10^{-3}$

t	$u(x,t)$	$v_N(x,t)$	$\|u - v_N\|_2$
0	$e\sin x$	$2.719079713\sin x$	0.001
0.01	$2.74574\sin x$	$2.74619\sin x - 0.0000074548\sin(3x)$ $-0.00000105443\sin(5x)$	0.00056407
0.02	$2.77375\sin x$	$2.77382\sin x - 0.0000121535\sin(3x)$ $-0.00000161556\sin(5x)$	0.0000890675
0.03	$2.80234\sin x$	$2.80199\sin x - 0.0000135129\sin(3x)$ $-0.00000163352\sin(5x)$	0.000438992
0.04	$2.83151\sin x$	$2.8307\sin x - 0.0000109582\sin(3x)$ $-0.00000107001\sin(5x)$	0.00101528
0.05	$2.86129\sin x$	$2.85997\sin x - 0.00000372291\sin(3x)$ $+0.000000129345\sin(5x)$	0.00165438

Table 2. $\beta = \delta = 10^{-6}$

t	$u(x,t)$	$v_N(x,t)$	$\|u - v_N\|_2$
0	$e\sin x$	$2.718282626344\sin x$	0.000001
0.01	$2.74574\sin x$	$2.74574\sin x - 0.00000000772376\sin(3x)$ $-0.00000000109207\sin(5x)$	0.00000000977658
0.02	$2.77375\sin x$	$2.77375\sin x - 0.0000000209072\sin(3x)$ $-0.00000000284423\sin(5x)$	0.0000000264447
0.03	$2.80234\sin x$	$2.80234\sin x + 0.00000000765303\sin(3x)$ $+0.00000000151195\sin(5x)$	0.00000000977704
0.04	$2.83151\sin x$	$2.83151\sin x + 0.0000000017265\sin(3x)$ $+0.000000000610782\sin(5x)$	0.00000000229526
0.05	$2.86129\sin x$	$2.86128\sin x + 0.0000000201077\sin(3x)$ $+0.00000000322633\sin(5x)$	0.0000125332

For a future research, it is worthwhile to examine similar partial differential equations of higher order where the function h satisfies a local Lipschitz condition rather than global. The numerical experiments presented in this paper may also be strengthened by directly solving the system of differential equations (5.2).

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