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BIFURCATION CURVES FOR SINGULAR AND NONSINGULAR PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We discuss a quadrature method for generating bifurcation curves of positive solutions to some autonomous boundary value problems with nonlinear boundary conditions. We consider various nonlinearities, including positone and semipositone problems in both singular and nonsingular cases. After analyzing the method in these cases, we provide an algorithm for the numerical generation of bifurcation curves and show its application to selected problems.

1. Introduction

We consider the two-point boundary value problem

$$\begin{aligned} -u''(t) &= \lambda f(u(t)), t \in (0,1), \\ u(0) &= 0, \\ u'(1) &= -c(u(1))u(1), \end{aligned} \tag{1.1}$$

where $f : (0,\infty) \to \mathbb{R}$ is a continuously differentiable function which is integrable on $(0,\epsilon)$ for some $\epsilon > 0$ and $c : [0,\infty) \to (0,\infty)$ is a continuous function. Positive solutions to equations of this form, but with linear boundary conditions, have been well-studied because of their applications in a number of fields, such as combustion theory, nonlinear heat generation, and population dynamics. See [2, 12, 21], respectively, for such examples. Further, problems with nonlinear boundary conditions have application in the study of thermal explosions and population dynamics with density dependent dispersal on the edges (see [19, 4], respectively for the derivation of such models), and have been the subject of recent mathematical study (see [3, 5, 8, 9, 11, 16, 20, 22]).

Here, we study positive solutions of (1.1) when the function f satisfies one of the additional hypotheses,

(H1) $f(s) > 0$ for all $s > 0$, or

(H2) there exist unique $\beta, \theta > 0$ so that $f(s) < 0$ for $s \in (0,\beta)$, $f(s) > 0$ for $s \in (\beta,\infty)$, and $F(\theta) = 0$ where $F(s) = \int_0^s f(r)\,dr$.

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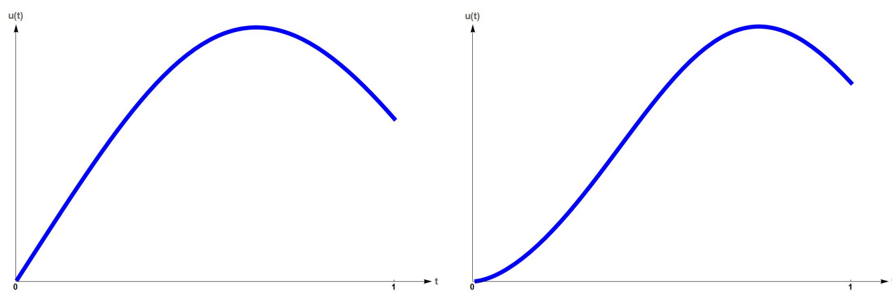
We note that any solution of (1.1) must be symmetric about any point $t_0 \in (0,1)$ where $u'(t_0) = 0$ (see proof of Lemma 2.2). To preserve the unique challenges posed by the presence of the nonlinear boundary condition, we consider only solutions where $u(1) > 0$, which implies that $u'(1) < 0$. When (H1) is satisfied, solutions to (1.1) are concave, while when (H2) is satisfied, solutions are convex near $t = 0$ (and possibly near $t = 1$) and are concave otherwise. See Figure 1 for examples.



Solution when f satisfies (H1). Solution when f satisfies (H2)

FIGURE 1. Shape of solution for positone and semipositone problems.

We further show in Section 2 that each positive solution of (1.1) has a unique interior maximum, and that if (H2) is satisfied, then $\|u\|_\infty \geq \theta$.

Of particular interest in this paper is the shape of bifurcation curves. Laetsch studied such problems in [14] with Dirichlet boundary conditions using a quadrature method (or time map analysis). The ideas of Laetsch have been been adapted to problems with a number of different boundary conditions, for example Neumann boundary conditions (see [18]), mixed boundary conditions (see [1]), and nonlinear boundary conditions (see [10]). In particular, in [10], the authors study a certain example of c arising in population dynamics involving density dependent dispersal on the boundary. The goal of this paper is to expand the ideas in [10] for general classes of c where f satisfies (H1) or (H2). In particular, we provide more detailed analysis of the quadrature method for such two-point boundary value problems involving nonlinear boundary conditions. Namely, we establish the following result.

Theorem 1.1. *For f satisfying either $(H1)$ or $(H2)$, there exists a positive solution $u \in C^2(0,1) \cap C^1[0,1]$ of* (1.1) *with $\|u\|_\infty = \rho$, $u(1) = q$, and $0 < q < \rho$ if and only if*

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c(q)q}{\sqrt{F(\rho) - F(q)}} = 0, \tag{1.2}$$

$$\sqrt{2\lambda} = \frac{c(q)q}{\sqrt{F(\rho) - F(q)}} \tag{1.3}$$

hold. Further, for a (λ, ρ, q) satisfying (1.2) *and* (1.3), (1.1) *has a positive solution u given by*

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}, \quad t \in [0, t_0),$$

$$(1-t)\sqrt{2\lambda} = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}, \quad t \in (t_0, 1],$$

$u(t_0)=\rho$ and $u(1)=q$, where t_0 satisfies
$$t_0=\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}\Big/\Big(\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}+\int_q^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}\Big).$$

Theorem 1.2. *If f satisfies (H1), then for every $\rho>0$, there exists a $q>0$ so that* (1.2) *is satisfied. Similarly, if f satisfies (H2), then for every $\rho\geq\theta$, there exists a $q>0$ so that* (1.2) *is satisfied.*

To continue our analysis, we assume that f satisfies one of the following hypothesis:

(H3) : (H1) and $f(0)>0$,
(H4) : (H1) and $\lim_{s\to 0+} f(s)=\infty$,
(H5) : (H2) and $f(0)<0$, or
(H6) : (H2) and $\lim_{s\to 0+} f(s)=-\infty$.

In cases (H3) and (H5) problems are referred in the literature as positone and semipositone, respectively, where we drop the requirement that f be nondecreasing. In [17], the author gives an overview of results for positone problems, while also addressing some difficulties encountered in dealing with semipositone problems. Semipositone problems were first treated in [6], and continue to be of great interest to mathematicians due to the difficulty in establishing positivity of solutions, and to scientists involved in management of natural resources. See [3] and [8] for recent work on semipositone problems with nonlinear boundary conditions of the form studied here.

In cases (H4) and (H6) problems are referred in the literature as infinite positone and infinite semipositone, respectively. For an overview of results for infinite positone and infinite semipositone problems, see [7] and [15]. For infinite positone and infinite semipositone problems with nonlinear boundary conditions, see [13] and [16]. In these cases, we establish the following theorem.

Theorem 1.3. *If f satisfies either* (H3) *or* (H4) *and $s+c(s)s$ is continuously differentiable and nondecreasing for all $s>0$, then for each fixed $\rho>0$, there exists a unique $q>0$ so that* (1.2) *is satisfied.*

Theorem 1.4. *If f satisfies either* (H5) *or* (H6)*, $c(s)s$ is continuously differentiable, and either*

(H7) *$\frac{s+c(s)s}{\sqrt{-F(s)}}$ is nondecreasing for $s\in(0,\beta)$ and $s+c(s)s$ is nondecreasing for all $s>0$, or*
(H8) *$(f(s)c(s)s)'>2f(s)$ for $s\in(0,\beta)$ and $c(s)s$ is nondecreasing for all $s>0$,*

is satisfied, then for each fixed $\rho\geq\theta$, there exists a unique $q>0$ so that (1.2) *is satisfied.*

In Section 2, we prove Theorems 1.1-1.4. In Section 3, we provide plots of the bifurcation curves for some specific problems generated by Mathematica. In Section 4, we present an interesting example and its bifurcation curve where the hypotheses of Theorem 1.4 are violated and for fixed ρ in a certain range, there exist multiple values of q satisfying (1.2).

2. Proofs of Theorems 1.1-1.4

Proof of Theorem 1.1. First we establish the following two lemmas needed to prove our results.

Lemma 2.1. *If f satisfies* (H2) *and $\rho < \theta$, then a positive solution, u, to* (1.1) *with $\|u\|_\infty = \rho$ does not exist for any $\lambda > 0$.*

Proof: Assume to the contrary that u is a positive solution to (1.1) for some $\lambda > 0$ such that $\|u\|_\infty = \rho < \theta$. Note that $u'(1) < 0$, since we are only interested in the case where $u(1) > 0$. Hence, there exists $t_0 \in (0,1)$ such that $u'(t_0) = 0$ and $u(t_0) = \rho$. Now, multiplying the differential equation by u', we obtain

$$-\Big[\frac{(u'(t))^2}{2}\Big]' = \lambda\big(F(u(t))\big)'.$$

Further, integrating we obtain

$$(u'(t))^2 = 2\lambda\left[F(\rho) - F(u(t))\right], \quad t \in (0, t_0). \tag{2.1}$$

But this implies that $(u'(0))^2 = 2\lambda F(\rho) < 0$, a contradiction. Hence, no such solution can exist.

Lemma 2.2. *Any positive solution u of* (1.1) *has a unique interior maximum at some $t_0 \in (0,1)$, is strictly increasing on $(0, t_0)$, is strictly decreasing on $(t_0, 1)$, and is symmetric about t_0.*

Proof: Let $t_0 \in (0,1)$ be such that $\|u\|_\infty = u(t_0) = \rho$. Suppose there exists another local maximum. Then there must be a local minimum at some $t_1 \in (0,1)$, at which $u''(t_1) \geq 0$, which implies that $u(t_1) \leq \beta$. Let $E(t) = \lambda F(u(t)) + \frac{1}{2}(u'(t))^2$ for $t \in (0,1)$. A simple calculation will show that $E'(t) = 0$, and hence $E(t)$ is constant on $[0,1]$. But $E(t_0) = \lambda F(\rho) \geq 0$ while $E(t_1) = \lambda F(u(t_1)) < 0$, and hence we have a contradiction. Therefore, t_0 is the unique critical point and from (2.1), we easily see that

$$u'(t) = \begin{cases} \sqrt{2\lambda[F(\rho) - F(u(t))]} > 0, & t \in (0, t_0), \\ -\sqrt{2\lambda[F(\rho) - F(u(t))]} < 0, & t \in (t_0, 1). \end{cases} \tag{2.2}$$

Further, note that both $w_1(t) = u(t_0 + t)$ and $w_2(t) = u(t_0 - t)$ satisfy

$$\begin{gathered} -w''(t) = \lambda f(w(t)), t \in (0,1), \\ w(0) = \rho, \\ w'(0) = 0. \end{gathered}$$

Hence, by Picard's Theorem, we have $w_1(t) = w_2(t)$ which implies that u is symmetric about t_0.

We now begin the proof of Theorem 1.1 by showing first that if $u \in C^2(0,1) \cap C^1[0,1]$ is a positive solution to (1.1) with $\|u\|_\infty = u(t_0) = \rho$ and $u(1) = q$, then λ, ρ, and q must satisfy (1.2) and (1.3). We note here that the improper integral in (1.2) is convergent since $f(\rho) > 0$.

Integrating (2.2), we obtain

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \; t \in (0, t_0), \tag{2.3}$$

$$(1-t)\sqrt{2\lambda} = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \; t \in (t_0, 1). \tag{2.4}$$

Setting $t = t_0$, we obtain

$$t_0\sqrt{2\lambda} = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}, \tag{2.5}$$

$$(1-t_0)\sqrt{2\lambda} = \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{2.6}$$

Adding (2.5) and (2.6), we obtain

$$\sqrt{2\lambda} = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

and hence from (2.5) we obtain

$$t_0 = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \Big/ \Big(\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \Big). \tag{2.7}$$

Further, using the boundary conditions and (2.2), we obtain

$$-u'(1) = c(q)q = \sqrt{2\lambda\,[F(\rho) - F(q)]}.$$

Hence (1.2) and (1.3) are satisfied.

Next, if λ, ρ, and q satisfy (1.2) and (1.3), let t_0 be defined by (2.7), and define $u : [0,1] \to [0,\rho]$ via (2.3) and (2.4) for $t \in (0,t_0) \cup (t_0,1)$ with $u(0) = 0$, $u(t_0) = \rho$, $u(1) = q$. Note that u is well defined on $(0,t_0)$ since both

$$\int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

and $t\sqrt{2\lambda}$ increase from 0 to

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

as u increases from 0 to ρ and t increases from 0 to t_0, respectively. Also, u is well defined on $(t_0,1)$ since both

$$\int_q^u \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

and $(1-t)\sqrt{2\lambda}$ decrease from

$$\int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

to 0 as u decreases from ρ to q and t increases from t_0 to 1, respectively. Now, define $H : (0,t_0) \times (0,\rho) \to \mathbb{R}$ by

$$H(\ell, v) = \int_0^v \frac{ds}{\sqrt{F(\rho) - F(s)}} - \ell\sqrt{2\lambda}.$$

Clearly H is C^1, $H(t,u(t)) = 0$; $t \in (0,t_0)$ and

$$H_v\,|_{(t,u(t))} = \frac{1}{\sqrt{F(\rho) - F(u(t))}} \neq 0.$$

Hence, by the Implicit Function Theorem, u is C^1 on $(0,t_0)$. Similarly, u is C^1 on $(t_0,1)$, and from (2.3)-(2.4), we get

$$u'(t)=\begin{cases}\sqrt{2\lambda[F(\rho)-F(u(t))]}, & t\in(0,t_0),\\ -\sqrt{2\lambda[F(\rho)-F(u(t))]}, & t\in(t_0,1).\end{cases} \tag{2.8}$$

Differentiating (2.8) again, we get

$$-u''(t)=\lambda f(u(t)),\quad t\in(0,t_0)\cup(t_0,1).$$

But $u(t_0)=\rho$ and f is continuous, and hence $u\in C^2(0,1)\cap C^1[0,1]$. Further, (2.8) implies that $-u'(1)=\sqrt{2\lambda[F(\rho)-F(q)]}$, and hence by (1.3) we have $u'(1)+c(u(1))u(1)=0$. Thus u is a solution of (1.1).

Proof of Theorem 1.2. Define

$$J(\rho,q):=\int_0^\rho\frac{ds}{\sqrt{F(\rho)-F(s)}}+\int_q^\rho\frac{ds}{\sqrt{F(\rho)-F(s)}}-\frac{c(q)q}{\sqrt{F(\rho)-F(q)}},$$

and note that if (H1) is satisfied, then for every fixed $\rho>0$, there exists a $q>0$ so that $J(\rho,q)=0$ since

$$J(\rho,0)=2\int_0^\rho\frac{ds}{\sqrt{F(\rho)-F(s)}}>0 \text{ and } \lim_{q\to\rho}J(\rho,q)=-\infty.$$

Hence, ρ,q satisfy (1.2). Similarly, if (H2) is satisfied, then the claim holds for all $\rho>\theta$. For $\rho=\theta$, we again have

$$\lim_{q\to\theta}J(\theta,q)=-\infty,$$

and observe that

$$\begin{aligned}\lim_{q\to0}J(\theta,q)&=2\int_0^\theta\frac{ds}{\sqrt{-F(s)}}-\lim_{q\to0^+}\frac{c(q)q}{\sqrt{-F(q)}}\\ &=2\int_0^\theta\frac{ds}{\sqrt{-F(s)}}-\lim_{q\to0^+}\frac{c(q)q}{\sqrt{-qf(z)}}\\ &=2\int_0^\theta\frac{ds}{\sqrt{-F(s)}}>0\end{aligned}$$

for some $z\in(0,q)$. Hence, there exists $q>0$ satisfying (1.2) for all $\rho\geq\theta$.

Proof of Theorem 1.3. Let $\rho>0$ be fixed. The existence of $q>0$ satisfying (1.2) follows from Theorem 1.2. As for the uniqueness of q, a straightforward calculation will show

$$J_q(\rho,q)=-\frac{2[1+(c(q)q)'](F(\rho)-F(q))+f(q)c(q)q}{2\,(F(\rho)-F(q))^{\frac{3}{2}}} \tag{2.9}$$

Since $f(q)>0$ and $1+(c(s)s)'=(s+c(s)s)'>0$ by assumption, $J_q(\rho,q)<0$ for all $q>0$, and hence there cannot be two values of q such that $J(\rho,q)=0$.

Proof of Theorem 1.4. Let $\rho \geq \theta$ be fixed. The existence of $q > 0$ satisfying (1.2) again follows from Theorem 1.2.

If (H7) holds, then for $s \in (0, \beta)$,

$$\Big(\ln\Big(\frac{s + c(s)s}{\sqrt{-F(s)}}\Big)\Big)' \geq 0.$$

A straightforward calculation will show that this implies that

$$\frac{1 + (c(s)s)'}{s + c(s)s} \geq \frac{-f(s)}{2(-F(s))}, \tag{2.10}$$

and we observe from (2.10) that for $s \in (0, \beta)$,

$$\frac{1 + (c(s)s)'}{c(s)s} \geq \frac{1 + (c(s)s)'}{s + c(s)s} \geq \frac{-f(s)}{2(-F(s))} \geq \frac{-f(s)}{2(F(\rho) - F(s))}. \tag{2.11}$$

Hence, using (2.11), we conclude that

$$2[1 + (c(s)s)'](F(\rho) - F(s)) + f(s)c(s)s > 0, \tag{2.12}$$

for $s \in (0, \beta)$. Since $f(s) \geq 0$ for all $s \in [\beta, \infty)$, it is easy to see that the inequality (2.12) also holds for $s \in [\beta, \rho)$. Therefore, by (2.9), we have $J_q(\rho, q) < 0$ for all $q > 0$, and the result follows.

If (H8) holds, then let

$$g(s) = 2(F(\rho) - F(s)) + f(s)c(s)s,$$

and observe that g is continuous on $[0, \rho]$, $g(0) = 2F(\rho) \geq 0$, and $g'(s) > 0$ for $s \in (0, \beta)$ by (H8). Hence, $g(s) > 0$ on $(0, \beta]$. Now, $(c(s)s)' \geq 0$ implies $1 + (c(s)s)' \geq 1$, and therefore, $J_q(\rho, q) < 0$ for $q \in (0, \beta]$. For $q \in (\beta, \rho)$, since $f(s) > 0$ for all $s \in (\beta, \rho)$, it easily follows that $J_q(\rho, q) < 0$ for all $q > 0$ from (2.9), and the result follows.

3. Application of the method to some examples

Below, we provide several examples of bifurcation diagrams which are numerically generated in Mathematica. The general procedure is outlined below.

```
begin
  N = 1000;
  pts = {};
  ρ_step = (ρ_max − ρ_min)/N;
  for i := 0 to N
      ρ = ρ_min + i ∗ ρ_step;
      q = FindRoot[J(ρ, s), s];
      λ = (c(q) ∗ q)^2/(2[F(ρ) − F(q)]);
      pts = AppendTo[pts, {λ, ρ}]
  end
  ListPlot[pts]
end
```

We apply this algorithm to (1.1) with the following nonlinearities,

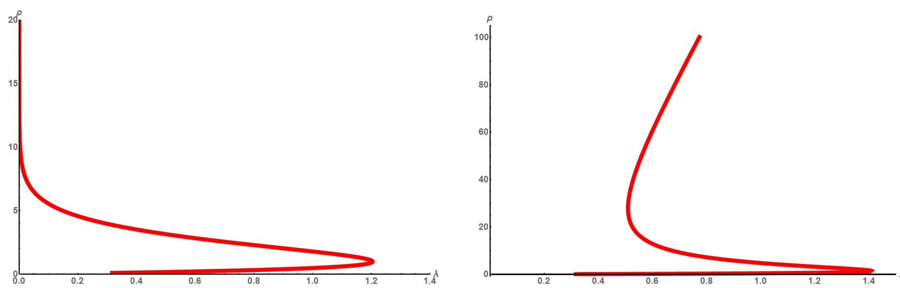
$$f(u) = e^u, \tag{3.1}$$

$$f(u) = e^{\frac{6u}{6+u}}, \tag{3.2}$$

$$f(u) = \frac{u-1}{\sqrt{u}}, \tag{3.3}$$

$$f(u) = u^3 - 10u^2 + 40u - 10, \tag{3.4}$$

with the nonlinearity in the boundary condition fixed as $c(s) = \frac{1}{s+1}$ for each problem. Note that the nonlinearities (3.1) and (3.2) are both positone and that $s+c(s)s$ is nondecreasing. Hence, the result of Theorem 1.3 holds. Bifurcation diagrams for these problems are shown in Figure 2.

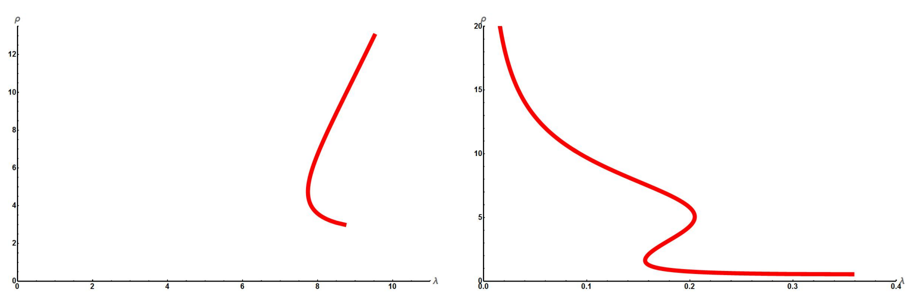


Bifurcation Curve for (3.1) Bifurcation Curve for (3.2)

FIGURE 2. Bifurcation diagrams for some positone problems.

The nonlinearities (3.3) and (3.4) are infinite semipositone and semipositone, respectively, and satisfy (H8). Hence, the results of Theorem 1.4 apply. Bifurcation diagrams for these problems are shown in Figure 3.

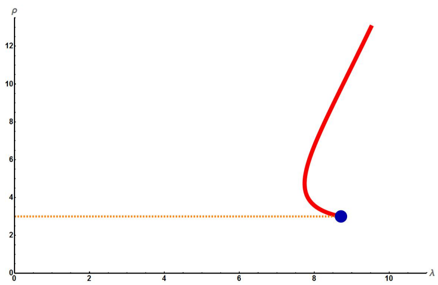
It is well known that the shape of bifurcation curves depends on characteristics of the nonlinearity f (see [17]). The nonlinearities (3.2) and (3.3) are both sublinear at infinity, while the nonlinearities (3.1) and (3.4) are both superlinear at infinity. Furthermore, the nonlinearities in (3.2) and (3.4) give rise to what are referred to in the literature as S-shaped and reverse S-shaped bifurcation curves. See [2] and [6] for early work on S-shaped and reverse S-shaped bifurcation curves, respectively.



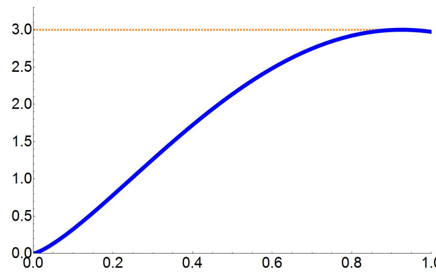
Bifurcation Curve for (3.3) Bifurcation Curve for (3.4)

FIGURE 3. Bifurcation diagrams for some semipositone problems.

Of particular interest in the semipositone problems (3.3) and (3.4) is the shape of the solution when $\rho = \theta$. As we exhibit in Figures 4 and 5, our computations illustrate that solutions to (3.3) or (3.4) with $\|u\|_\infty = \theta$ also satisfy $u'(0) = 0$.

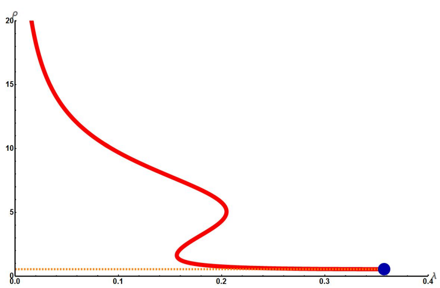


Bifurcation curve ends when $(\lambda, \rho) \approx (8.71082, 3)$

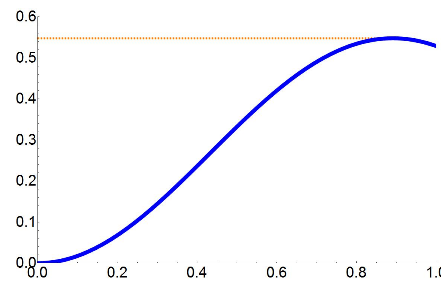


Solution plot with $(\lambda, \rho) \approx (8.71082, 3)$. in the case $u'(0) \approx 6 \times 10{-}2$

Figure 4. Behavior of solutions at endpoint of bifurcation curve for a sublinear infinite semipositone problem.



Bifurcation curve ends when $(\lambda, \rho) \approx (0.357438, 0.547992)$



Solution plot with $(\lambda, \rho) \approx (0.357438, 0.547992)$. In this case, $u'(0) \approx 8 \times 10^{-8}$

Figure 5. Behavior of solutions at endpoint of bifurcation curve for a superlinear semipositone problem.

4. Multiplicity generated by $s + c(s)s$ oscillation

In the case that $(s^* + c(s^*)s^*)' < 0$ for some $s^* \in [0, \infty)$, Theorems 1.3 and 1.4 do not apply. In such cases, it is possible that for some fixed $\rho \geq \theta$, there are multiple values of $q > 0$ so that (1.2) is satisfied. Below, we provide such an example. Consider the problem

$$
\begin{gathered}
-u''(t) = \lambda\left((u(t))^2 - 3\right), \quad t \in (0,1), \\
u(0) = 0, \\
u'(1) = -\left(\frac{1}{2}(u(1) - 10)^2 + 1\right) u(1),
\end{gathered}
\tag{4.1}
$$

and note that though $\frac{s+c(s)s}{\sqrt{-F(s)}}$ is nondecreasing on $(0, \sqrt{3})$, $s + c(s)s$ is decreasing on the interval

$$
\Big(\frac{20 - 2\sqrt{22}}{3}, \frac{20 + 2\sqrt{22}}{3}\Big).
$$

Applying the method from the previous section, we now need to consider the possibility that for a fixed $\rho \geq \theta$, there may exist multiple q values so that (1.2) is satisfied.

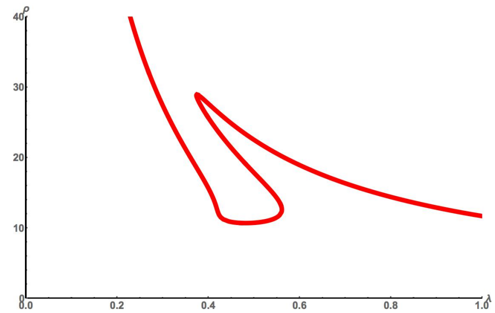
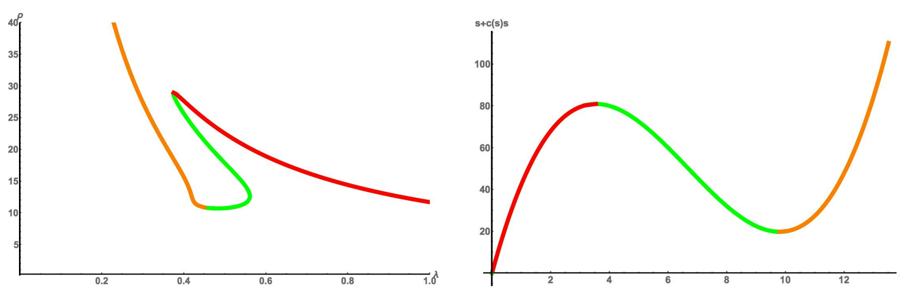


FIGURE 6. A bifurcation curve of (4.1).

In Figure 6, we provide the numerically generated bifurcation curve, and observe that the oscillation of $s + c(s)s$ has introduced multiplicity of solutions for some range of λ. In particular, if we track q values as we plot the bifurcation diagram, we observe numerical evidence of some correspondence to changes in the sign of $(s + c(s)s)'$.



Bifurcation Curve for (4.1) Graph of $s + c(s)s$

FIGURE 7. Correspondence between shape of the bifurcation diagram and shape of $s + c(s)s$.

Many problems related to the existence, uniqueness, and exact multiplicity of solutions to (1.1) remain open. Our aim in this paper has been to provide a quadrature method framework for addressing such problems, proofs of some results related to solutions of (1.2), and numerically generated bifurcation curves, which may motivate further inquiry.

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