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EFFECT OF KELVIN-VOIGT DAMPING ON SPECTRUM ANALYSIS OF A WAVE EQUATION

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Abstract. This article concerns a one-dimensional wave equation with a small amount of Kelvin-Voigt damping. We give a detailed spectrum analysis of the system operator, from which we show that the generalized eigenfunction forms a Riesz basis for the state Hilbert space. That is, the precise and explicit expression of the eigenvalues is deduced and the spectrum-determined growth condition is established. Hence the exponential stability of the system is obtained.

1. Introduction and statement of main results

In the past few decades, stimulated by a large quantity of applications of smart materials, there has been an increasing research on elastic system with viscoelastic damping. When the smart materials are added into the elastic structures, the exponential stability of the elastic system has attracted many research interests, especially for a linear vibration system governed by one-dimensional wave and Euler-Bernoulli beam equations. The results of exponential stability by bounded viscous damping can be found in [2, 3, 5, 11]. As a kind of unbounded viscoelastic damping and internal material damping, Kelvin-Voigt damping presents in all realistic materials and is specially important. The exponential stability of one-dimensional wave and Euler-Bernoulli beam equations with Kelvin-Voigt damping is discussed in [7, 10, 12, 13] and [1] for multi-dimensional case. The research of spectral analysis and Riesz basis can be found in [6, 8, 9, 15].

In infinite-dimensional control theory, Riesz basis property is one of the most wanted properties, particular for elastic vibrating systems for which the Riesz basis property is significant both theoretically and practically. Usually, the Riesz basis property will lead to the establishment such as the spectrum determined growth condition, and the exponential stability of the system.

In this work we consider the wave equation with a small amount of Keivin-Voigt damping,

$$\begin{gathered} w_{tt}(x,t) = (1 + d\partial_t)(w_{xx}(x,t) - cw(x,t)), \quad 0 < x < 1,\ t > 0, \\ w_x(0,t) = 0, \quad w(1,t) = 0, \quad t > 0, \end{gathered} \tag{1.1}$$

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where $c > 0$ is a system parameter and $d > 0$ is a small Kelvin-Voigt damping coefficient.

The energy function for(1.1) is

$$E(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t) + cw^2(x,t)]dx. \tag{1.2}$$

Formally, it is found that

$$\frac{d}{dt}E(t) = -d\int_0^1 w_{xt}^2(x,t)dx - cd\int_0^1 w_t^2(x,t)dx \le 0. \tag{1.3}$$

So $E(t)$ is non-increasing.

Let $H_R^1(0,1) = \{f(x) \in H^1(0,1)|f(1) = 0\}$. We consider system (1.1) in the energy state space $\mathcal{H} = H_R^1(0,1) \times L^2(0,1)$ with the inner product induced norm:

$$\|(f,g)\|^2 = \int_0^1 [|f'(x)|^2 + |g(x)|^2 + c|f(x)|^2]dx. \tag{1.4}$$

Then(1.1) can be written as an evolutionary equation in $\mathcal{H}$,

$$\frac{d}{dt}Y(t) = \mathcal{A}Y(t), \tag{1.5}$$

where $Y(t) = (w, w_t)$ and the operator $\mathcal{A}$ is given by

$$\begin{gathered}\mathcal{A}(f(x), g(x)) = (g(x), f''(x) - cf(x) + dg''(x) - cdg(x)),\\ \mathcal{D}(\mathcal{A}) = \{(f,g) \in [H^2(0,1) \cap H_R^1(0,1)]^2 : f'(0) = 0,\ g'(0) = 0\}.\end{gathered} \tag{1.6}$$

The following Lemma is straightforward.

Lemma 1.1. *Let $\mathcal{A}$ be given by* (1.6)*. Then its adjoint $\mathcal{A}^*$ has the form*

$$\begin{gathered}\mathcal{A}^*(f(x), g(x)) = (-g(x), -f''(x) + cf(x) + dg''(x) - cdg(x)),\\ \mathcal{D}(\mathcal{A}^*) = \{(f,g) \in [H^2(0,1) \cap H_R^1(0,1)]^2 : f'(0) = 0,\ g'(0) = 0\}.\end{gathered} \tag{1.7}$$

The following Definitions are brought from [4, Chapter 2].

Definition 1.2. A sequence of vectors $\{\phi_n,\ n \ge 1\}$ in a Hilbert space $\mathcal{H}$ forms a Riesz basis for $\mathcal{H}$ if the following two conditions hold:

(i) $\overline{\text{span}}_{n\ge1}\{\phi_n\} = \mathcal{H}$;
(ii) There exist positive constants m and M such that for arbitrary $N \in \mathbb{N}$ and arbitrary scalars α_n, $n = 1, \dots, N$, such that

$$m\sum_{n=1}^N |\alpha_n|^2 \le \|\sum_{n=1}^N \alpha_n\phi_n\|^2 \le M\sum_{n=1}^N |\alpha_n|^2.$$

Definition 1.3. Suppose that $\mathcal{A}$ is a linear closed operator on a Hilbert space, $\mathcal{H}$, with simple eigenvalues $\{\lambda_n,\ n \ge 1\}$ form a Riesz basis in $\mathcal{H}$. If the closure of $\{\lambda_n,\ n \ge 1\}$ is totally disconnected, then we call $\mathcal{A}$ a Riesz-spectral operator.

Remark 1.4. One can define a Riesz basis for $\mathcal{H}$ comprised of a sequence of vectors ϕ_n belonging to a countable subset of $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$.

The following two theorems will be proved in the next section.

Theorem 1.5. *Let $\mathcal{A}$ and $\mathcal{A}^*$ be given by*(1.6) *and*(1.7) *respectively. Then $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, and hence $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$. Moreover, $\mathcal{A}^{-1}$ exists.*

Theorem 1.6. *Let $\mathcal{A}$ be defined by*(1.6)*. Then there is a sequence of generalized eigenvectors of $\mathcal{A}$ which forms a Riesz basis for state space $\mathcal{H}$. Hence semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable if $d^2c < 4$.*

1.1. **Proof of main results.**

Proof of Theorem 1.5. Given any $(f,g) \in \mathcal{D}(\mathcal{A})$. We have

$$\begin{aligned}\langle \mathcal{A}(f,g),(f,g)\rangle &= \langle (g, f'' - cf + dg'' - cdg),(f,g)\rangle \\ &= \int_0^1 [f''(x)\overline{g(x)} - \overline{f''(x)}g(x)]dx + c\int_0^1 [\overline{f(x)}g(x) - f(x)\overline{g(x)}]dx \\ &\quad - d\int_0^1 [g'(x)]^2 dx - cd\int_0^1 [g(x)]^2 dx,\end{aligned}$$

and hence

$$\mathcal{R}e\langle \mathcal{A}(f,g),(f,g)\rangle = -d\int_0^1 [g'(x)]^2 dx - cd\int_0^1 [g(x)]^2 dx \le 0. \tag{1.8}$$

Similarly for any $(u,v) \in \mathcal{D}(\mathcal{A}^*)$,

$$\begin{aligned}&\langle \mathcal{A}^*(u,v),(u,v)\rangle \\ &= \langle (-v, -u'' + cu + dv'' - cdv),(u,v)\rangle \\ &= \int_0^1 [\overline{u''(x)}v(x) - u(x)\overline{v''(x)}]dx + c\int_0^1 [u(x)\overline{v(x)} - \overline{u(x)}v(x)]dx \\ &\quad - d\int_0^1 [v'(x)]^2 dx - cd\int_0^1 [v(x)]^2 dx,\end{aligned}$$

and hence

$$\mathcal{R}e\langle \mathcal{A}^*(u,v),(u,v)\rangle = -d\int_0^1 [v'(x)]^2 dx - cd\int_0^1 [v(x)]^2 dx \le 0. \tag{1.9}$$

Therefore, both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. By the Lumer-Phillips Theorem [14], $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$.

Now we show that $\mathcal{A}^{-1}$ exists. For any given $(f_1, g_1) \in \mathcal{H}$, solving

$$\mathcal{A}(f,g) = (g, f'' - cf + dg'' - cdg) = (f_1, g_1)$$

gives $g(x) = f_1(x)$ with f satisfying

$$\begin{gathered} f''(x) - cf(x) = g_1(x) + cdf_1(x) - df_1''(x), \\ f(1) = 0, \quad f'(0) = 0. \end{gathered}$$

The solution of the above ODE has the form

$$f(x) = \int_0^1 [g_1(s) + cdf_1(s) - df_1''(s)]k(x,s)ds,$$

where

$$k(x,s) = \frac{1}{\sqrt{c}(1 + e^{-2\sqrt{c}})}\begin{cases} (e^{\sqrt{c}(s-2)} - e^{-\sqrt{c}s})\cosh(\sqrt{c}x), & x < s, \\ \cosh(\sqrt{c}s)(e^{\sqrt{c}(x-2)} - e^{-\sqrt{c}x}), & x > s. \end{cases} \tag{1.10}$$

From this and $g(x) = f_1(x)$, we get the unique $(f,g) \in \mathcal{D}(\mathcal{A})$. Hence, $\mathcal{A}^{-1}$ exists. This completes the proof. $\square$

Let us now consider the eigenvalue problem of $\mathcal{A}$. It is seen that $\mathcal{A}(f,g) = \lambda(f,g)$, where $(f,g) \in \mathcal{D}(\mathcal{A}),\ \lambda \in \mathbb{C}$, if and only if $g(x) = \lambda f(x)$, and $f(x) \neq 0$ satisfies the eigenvalue problem

$$\begin{gathered} (1+d\lambda)f''(x) - (\lambda^2 + cd\lambda + c)f(x) = 0, \\ f(1) = 0, \quad f'(0) = 0. \end{gathered} \tag{1.11}$$

To prove Theorem 1.6 we first prove the following lemma, which gives the precise and explicit expression of the eigenvalues of the system operator $\mathcal{A}$.

Lemma 1.7. *Let $\mathcal{A}$ be defined by(1.6). Then $\mathcal{R}e\lambda < 0$ for every $\lambda \in \sigma_p(\mathcal{A})$. Furthermore,*
(i) the point spectrum $\sigma_p(\mathcal{A})$ has three families:

$$\begin{gathered} \sigma_p(\mathcal{A}) = \{\lambda_{1n},\ n \geq N+1,\ n \in \mathbb{Z}\} \cup \{\lambda_{2n},\ n \geq N+1,\ n \in \mathbb{Z}\} \\ \cup \{\lambda_{3n},\ \bar{\lambda}_{3n},\ 0 \leq n \leq N-1,\ n \in \mathbb{Z}\} \cup \{\lambda_N\}, \end{gathered}$$

where

$$\begin{gathered} \lambda_{1n} = \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{\Delta_n}], \quad n \in \mathbb{Z},\ n \geq N+1, \\ \lambda_{2n} = \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{\Delta_n}], \quad n \in \mathbb{Z},\ n \geq N+1, \\ \lambda_{3n} = \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{-\Delta_n} i], \quad n \in \mathbb{Z},\ 0 \leq n \leq N-1, \\ \lambda_N = -\frac{2}{d}, \end{gathered} \tag{1.12}$$

where the algebraic multiplicity of λ_N is two, the algebraic multiplicity of other eigenvalues is one. The quantities σ_n, Δ_n, N will be given later.

Moreover there are two sets of the eigenvalues of the system operator $\mathcal{A}$: For lower n the eigenvalues reside on the circle

$$[\mathcal{R}e(\lambda_n) + \frac{1}{d}]^2 + [\mathcal{I}m(\lambda_n)]^2 = \frac{1}{d^2}, \tag{1.13}$$

and for higher n the eigenvalues are real, with one branch accumulating towards $-\frac{1}{d}$ as $n \to \infty$ and the other branch converging to $-\infty$.
(ii) For any $\lambda_n \in \sigma_p(\mathcal{A})$, $n \neq N$, there is only one associated (linearly independent) eigenfunction which takes the form $(\frac{1}{\lambda_n} f_n, f_n)$, where f_n is found to be

$$f_n(x) = \cos[(n + \frac{1}{2})\pi x], \quad n = 0, 1, 2, \dots. \tag{1.14}$$

Remark 1.8. It should be noted that a wave equation with Kelvin-Voigt damping cannot be categorized as a hyperbolic PDE. With at most a finite number of conjugate-complex eigenvalues in its spectrum, such a PDE should be categorized as a parabolic/hyperbolic hybrid.

Proof of Lemma 1.7. Denote $\sigma^2 = \frac{\lambda^2 + cd\lambda + c}{1 + d\lambda}$. Then, by the equivalence, the ODE

$$\begin{gathered} f''(x) - \sigma^2 f(x) = 0,, \\ f(1) = f'(0) = 0, \end{gathered} \tag{1.15}$$

has a non-trivial solution if and only if

$$\sigma_n = (n + \frac{1}{2})\pi i, \quad n = 0, \pm 1, \pm 2, \dots.$$

The corresponding eigenfunctions are

$$f_n(x) = \cos[(n+\frac{1}{2})\pi x], \quad n = 0, \pm 1, \pm 2, \dots .$$

Hence the eigenvalues of $\mathcal{A}$ satisfy the characteristic equation

$$\lambda_n^2 + d(c-\sigma_n^2)\lambda_n + c - \sigma_n^2 = 0. \tag{1.16}$$

Setting $\Delta_n = d^2(c-\sigma_n^2)^2 - 4(c-\sigma_n^2)$, we have

$$\Delta_n = d^2(c-\sigma_n^2)[(n+\frac{1}{2})^2\pi^2 - (\frac{4}{d^2} - c)].$$

Considering $d > 0$ as a small Kelvin-Voigt damping coefficient such that $\frac{4}{d^2} > c$, there exists a positive integer N such that

$$(N+\frac{1}{2})^2\pi^2 = \frac{4}{d^2} - c < (N+\frac{3}{2})^2\pi^2. \tag{1.17}$$

Then for $n \in \mathbb{Z},\ n \geq N+1$ or $n \leq -N-2$, we have $\Delta_n > 0$, and in this case, the eigenvalues of $\mathcal{A}$ are

$$\begin{aligned} \lambda_{1n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{\Delta_n}], \\ \lambda_{2n} &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{\Delta_n}], \end{aligned} \tag{1.18}$$

where

$$\begin{aligned} \sigma_n &= (n+\frac{1}{2})\pi i, \\ \Delta_n &= d^2(c-\sigma_n^2)^2 - 4(c-\sigma_n^2). \end{aligned} \tag{1.19}$$

Noticing that $\sigma_n^2 = \sigma_{-n-1}^2$, and $\lambda_n = \lambda_{(-n-1)}$, for any $n \in \mathbb{Z}$, $n \geq N+1$, we have that

$$\begin{aligned} \lambda_{1n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{\Delta_n}], \\ \lambda_{2n} &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{\Delta_n}]. \end{aligned} \tag{1.20}$$

It is easy to check that $\mathcal{R}e\lambda_{1n} < 0$, and $\mathcal{R}e\lambda_{2n} < 0$. Furthermore, for any $n \in \mathbb{Z}$ with $n \geq N+1$,

$$\begin{aligned} \lambda_{1n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{\Delta_n}] \\ &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{d^2(c-\sigma_n^2)^2 - 4(c-\sigma_n^2)}] \\ &= \frac{d}{2}(\sigma_n^2 - c)[1 - \sqrt{1 - \frac{4}{d^2(c-\sigma_n^2)}}], \\ \lambda_{2n} &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{\Delta_n}] \\ &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{d^2(c-\sigma_n^2)^2 - 4(c-\sigma_n^2)}], \end{aligned}$$

where $c - \sigma_n^2 = c + (n+\frac{1}{2})^2\pi^2$ converges to $+\infty$ as $n \to \infty$. Moreover

$$d(\sigma_n^2 - c) < \lambda_{2n} < \frac{1}{2}d(\sigma_n^2 - c).$$

Then λ_{1n} accumulates towards $-1/d$ and λ_{2n} converges to $-\infty$ as $n \to \infty$.

On the other hand, in the case $n \in \mathbb{Z}$ with $0 \le n \le N-1$, we have $\Delta_n < 0$, and the eigenvalues of $\mathcal{A}$ are

$$\begin{aligned} \lambda_{3n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{-\Delta_n} i], \\ \bar{\lambda}_{3n} &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{-\Delta_n} i]. \end{aligned} \tag{1.21}$$

For $n \in \mathbb{Z}$ with $0 \le n \le N-1$, we can easily obtain

$$[\mathcal{R}e(\lambda_n) + \frac{1}{d}]^2 + [\mathcal{I}m(\lambda_n)]^2 = \frac{1}{d^2}.$$

Moreover, $\Delta_N = 0$, and in this case, $\lambda_N = -2/d$. (the algebraic multiplicity of λ_N is two.) This completes the proof. □

Remark 1.9. If there exists a positive integer N such that

$$(N + \frac{1}{2})^2\pi^2 < \frac{4}{d^2} - c < (N + \frac{3}{2})^2\pi^2,$$

in this case, the spectrum $\sigma_p(\mathcal{A})$ has three families:

$$\begin{aligned} \sigma_p(\mathcal{A}) = &\{\lambda_{1n},\ n \ge N+1,\ n \in \mathbb{Z}\} \cup \{\lambda_{2n},\ n \ge N+1,\ n \in \mathbb{Z}\} \\ &\cup \{\lambda_{3n},\ \bar{\lambda}_{3n},\ 0 \le n \le N,\ n \in \mathbb{Z}\}, \end{aligned}$$

where

$$\begin{aligned} \lambda_{1n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{\Delta_n}], \quad n \in \mathbb{Z},\ n \ge N+1, \\ \lambda_{2n} &= \frac{1}{2}[d(\sigma_n^2 - c) - \sqrt{\Delta_n}], \quad n \in \mathbb{Z},\ n \ge N+1, \\ \lambda_{3n} &= \frac{1}{2}[d(\sigma_n^2 - c) + \sqrt{-\Delta_n} i], \quad n \in \mathbb{Z},\ 0 \le n \le N. \end{aligned} \tag{1.22}$$

where σ_n, Δ_n were given by (1.19).

Lemma 1.10. *Let $\mathcal{A}$ be defined by(1.6). Then the residual spectrum, $\sigma_r(\mathcal{A}) = \emptyset$.*

Proof. It is sufficient to show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. Let us now consider the eigenvalue problem of $\mathcal{A}^*$. It is seen that $\mathcal{A}^*(f,g) = \lambda(f,g)$, where $(f,g) \in \mathcal{D}(\mathcal{A}^*)$, $\lambda \in \mathbb{C}$, if and only if $g(x) = -\lambda f(x)$, and $f(x) \neq 0$ satisfies the eigenvalue problem

$$\begin{gathered} (1 + d\lambda)f''(x) - (\lambda^2 + cd\lambda + c)f(x) = 0, \\ f(1) = 0, \quad f'(0) = 0. \end{gathered} \tag{1.23}$$

It is seen that (1.23) is the same with (1.11). Hence, $\lambda \in \sigma_p(\mathcal{A}^*)$ if and only if $\lambda \in \sigma_p(\mathcal{A})$. Since the eigenvalues of $\mathcal{A}^*$ are symmetric with real axis, we have $\sigma_r(\mathcal{A}) = \emptyset$. □

Now we are in a position to prove the system operator $\mathcal{A}$ has the Riesz basis property which will lead to the establishment of the spectrum determined growth condition, and the exponential stability of the system.

Proof of Theorem 1.6. We first prove that $\{\Phi_n,\ n = 0, 1, 2, \dots,\}$ is maximal in $\mathcal{H}$, where Φ_n satisfies

$$\Phi_{1n} = \sqrt{\frac{2\lambda_{1n}}{d(\sigma_n^2 - c)}}\Big(\frac{1}{\lambda_{1n}}\cos[(n + \frac{1}{2})\pi x], \cos[(n + \frac{1}{2})\pi x]\Big), \quad n \in \mathbb{Z},\ n \ge N+1,$$

$$
\begin{gathered}
\Phi_{2n}=\sqrt{\frac{2\lambda_{2n}}{d(\sigma_n^2-c)}}(\frac{1}{\lambda_{2n}}\cos[(n+\frac{1}{2})\pi x],\cos[(n+\frac{1}{2})\pi x]),\quad n\in\mathbb{Z},\ n\geq N+1,\\
\Phi_{3n}=(\frac{1}{\lambda_{3n}}\cos[(n+\frac{1}{2})\pi x],\cos[(n+\frac{1}{2})\pi x]),\quad n\in\mathbb{Z},\ 0\leq n\leq N-1,\\
\bar{\Phi}_{3n}=(\frac{1}{\bar{\lambda}_{3n}}\cos[(n+\frac{1}{2})\pi x],\cos[(n+\frac{1}{2})\pi x]),\quad n\in\mathbb{Z},\ 0\leq n\leq N-1,\\
\Phi_N=(\frac{1}{\lambda_N}\cos[(N+\frac{1}{2})\pi x],\cos[(N+\frac{1}{2})\pi x]),\\
\tilde{\Phi}_N=\frac{1}{\sqrt{\frac{4}{d^2}+\frac{d^2}{8}-1}}(\cos[(N+\frac{1}{2})\pi x],(\frac{d}{2}-\frac{2}{d})\cos[(N+\frac{1}{2})\pi x]),
\end{gathered}
$$

and $\tilde{\Phi}_N$ is the generalized eigenvector of λ_N.

Suppose that $z=(z_1,z_2)\in\mathcal{H}$ is orthogonal to every Φ_n. Then

$$
\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}\frac{1}{\lambda_n}\cos[(n+\frac{1}{2})\pi x)]\\ \cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle=0,\quad\forall n\in\mathbb{Z},\ n\geq 0,\ n\neq N,
$$

i.e.,

$$
\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(n+\frac{1}{2})\pi x)]\\ \lambda_n\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle=0,\quad\forall n\in\mathbb{Z},\ n\geq 0,\ ;n\neq N. \tag{1.24}
$$

And

$$
\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}\frac{1}{\lambda_N}\cos[(N+\frac{1}{2})\pi x)]\\ \cos[(N+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle=0, \tag{1.25}
$$

$$
\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(N+\frac{1}{2})\pi x]\\ (\frac{d}{2}-\frac{2}{d})\cos[(N+\frac{1}{2})\pi x]\end{pmatrix}\Big\rangle=0, \tag{1.26}
$$

where $\lambda_N=-2/d$. Next we distinguish three cases.

Case 1. For $n\in\mathbb{Z},\ n\geq N+1$, noticing that $\lambda_{1n}<0$, $\lambda_{2n}<0$,

$$
\begin{aligned}
&\int_0^1(\lambda_{1n}-\lambda_{2n})z_2(x)\cos[(n+\frac{1}{2})\pi x]dx\\
&=\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}0\\ (\lambda_{1n}-\lambda_{2n})\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle\\
&=\sqrt{\frac{d(\sigma_n^2-c)\lambda_{2n}}{2}}\langle z,\Phi_{2n}\rangle-\sqrt{\frac{d(\sigma_n^2-c)\lambda_{1n}}{2}}\langle z,\Phi_{1n}\rangle=0,
\end{aligned}
$$

where $\lambda_{1n}-\lambda_{2n}=\sqrt{\Delta_n}>0$. Then for any $n\in\mathbb{Z}$ with $n\geq N+1$,

$$
\int_0^1 z_2(x)\cos[(n+\frac{1}{2})\pi x]dx=0. \tag{1.27}
$$

From(1.24), (1.27) and for any $n\in\mathbb{Z}$ with $n\geq N+1$,

$$
\begin{aligned}
0&=\Big\langle\begin{pmatrix}z_1(x)\\ z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(n+\frac{1}{2})\pi x)]\\ \lambda_{1n}\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle\\
&=\int_0^1-(n+\frac{1}{2})\pi z_1'(x)\sin[(n+\frac{1}{2})\pi x)]dx
\end{aligned}
$$

$$\begin{aligned}&+c\int_0^1 z_1(x)\cos[(n+\frac{1}{2})\pi x)]dx\\&+\int_0^1 \lambda_{1n}z_2(x)\cos[(n+\frac{1}{2})\pi x)]dx\\&=(c-\sigma_n^2)\int_0^1 z_1(x)\cos[(n+\frac{1}{2})\pi x)]dx,\end{aligned}$$

i.e.,

$$\int_0^1 z_1(x)\cos[(n+\frac{1}{2})\pi x)]dx=0, \tag{1.28}$$

for any $n\in\mathbb{Z}$, $n\geq N+1$.

Case 2. For $0\leq n\leq N-1$, $n\in\mathbb{Z}$,

$$\begin{aligned}&(\lambda_{3n}-\bar\lambda_{3n})\int_0^1 z_2(x)\cos[(n+\frac{1}{2})\pi x)]dx\\&=\Big\langle\begin{pmatrix}z_1(x)\\z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(n+\frac{1}{2})\pi x)]\\\lambda_{3n}\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle-\Big\langle\begin{pmatrix}z_1(x)\\z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(n+\frac{1}{2})\pi x)]\\\bar\lambda_{3n}\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle\\&=\lambda_{3n}\langle z,\Phi_{3n}\rangle-\bar\lambda_{3n}\langle z,\bar\Phi_{3n}\rangle=0,\end{aligned}$$

where $\lambda_{3n}-\bar\lambda_{3n}=\sqrt{-\Delta_n}i$. Then for any $0\leq n\leq N-1$ with $n\in\mathbb{Z}$,

$$\int_0^1 z_2(x)\cos[(n+\frac{1}{2})\pi x)]dx=0. \tag{1.29}$$

Similarly,

$$\begin{aligned}0&=\Big\langle\begin{pmatrix}z_1(x)\\z_2(x)\end{pmatrix},\begin{pmatrix}\cos[(n+\frac{1}{2})\pi x)]\\\lambda_{3n}\cos[(n+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle\\&=(c-\sigma_n^2)\int_0^1 z_1(x)\cos[(n+\frac{1}{2})\pi x)]dx\\&\quad+\bar\lambda_{3n}\int_0^1 z_2(x)\cos[(n+\frac{1}{2})\pi x)]dx.\end{aligned} \tag{1.30}$$

Combining (1.29) and(1.30), for $0\leq n\leq N-1,\ n\in\mathbb{Z}$, we have

$$\int_0^1 z_1(x)\cos[(n+\frac{1}{2})\pi x)]dx=0. \tag{1.31}$$

Case 3. Multiplying (1.26) by $d/2$ and adding it to (1.25), we obtain

$$0=\Big\langle\begin{pmatrix}z_1(x)\\z_2(x)\end{pmatrix},\begin{pmatrix}0\\\frac{4}{d^2}\cos[(N+\frac{1}{2})\pi x)]\end{pmatrix}\Big\rangle=\frac{d^2}{4}\int_0^1 z_2(x)\cos[(N+\frac{1}{2})\pi x)]dx,$$

i.e.,

$$\int_0^1 z_2(x)\cos[(N+\frac{1}{2})\pi x)]dx=0. \tag{1.32}$$

Combining (1.26) and(1.32), we can easily obtain

$$\int_0^1 z_1(x)\cos[(N+\frac{1}{2})\pi x)]dx=0, \tag{1.33}$$

Since $\{\cos[(n+\frac{1}{2})\pi x)],\ n \in \mathbb{Z},\ n \geq 0\}$ is maximal in $L^2(0,1)$, and considering (1.27)-(1.29) and (1.31)-(1.33), we have that $z = (z_1, z_2) = 0$. Thus $\{\Phi_n,\ n \in \mathbb{Z},\ n \geq 0\}$ is maximal in $\mathcal{H}$.

For any $n, k \geq N+1$, $n \in \mathbb{Z}$,

$$
\begin{aligned}
&\Big\langle \begin{pmatrix} \frac{1}{\lambda_{1n}} \cos[(n+\frac{1}{2})\pi x)] \\ \cos[(n+\frac{1}{2})\pi x)] \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_{1k}} \cos[(k+\frac{1}{2})\pi x)] \\ \cos[(k+\frac{1}{2})\pi x)] \end{pmatrix} \Big\rangle \\
&= \frac{(n+\frac{1}{2})(k+\frac{1}{2})\pi^2}{\lambda_{1n}\lambda_{1k}} \int_0^1 \sin[(n+\frac{1}{2})\pi x)] \sin[(k+\frac{1}{2})\pi x)]dx \\
&\quad + \frac{c}{\lambda_{1n}\lambda_{1k}} \int_0^1 \cos[(n+\frac{1}{2})\pi x)] \cos[(k+\frac{1}{2})\pi x)]dx \\
&\quad + \int_0^1 \cos[(n+\frac{1}{2})\pi x)] \cos[(k+\frac{1}{2})\pi x)]dx \\
&= \frac{(n+\frac{1}{2})(k+\frac{1}{2})\pi^2}{2\lambda_{1n}\lambda_{1k}} \int_0^1 \cos[(n-k)\pi x] - \cos[(n+k+1)\pi x]dx \\
&\quad + \frac{1}{2}(\frac{c}{\lambda_{1n}\lambda_{1k}} + 1) \int_0^1 \cos[(n-k)\pi x] + \cos[(n+k+1)\pi x]dx \\
&= \begin{cases} 0, & n \neq k, \\ \frac{c-\sigma_n^2+\lambda_{1n}^2}{2\lambda_{1n}^2}, & n = k. \end{cases} \\
&= \begin{cases} 0, & n \neq k, \\ \frac{d(\sigma_n^2-c)}{2\lambda_{1n}}, & n = k. \end{cases}
\end{aligned}
$$

Then

$$
\begin{gathered}
\langle \Phi_{1n}, \Phi_{1k} \rangle = 0, \quad n,\ k \geq N+1,\ n \in \mathbb{Z},\ n \neq k, \\
\langle \Phi_{1n}, \Phi_{1n} \rangle = 1, \quad n \geq N+1,\ n \in \mathbb{Z}.
\end{gathered} \tag{1.34}
$$

Similarly for any $n,\ k \geq N+1$, $n \in \mathbb{Z}$, we have

$$
\begin{aligned}
&\Big\langle \begin{pmatrix} \frac{1}{\lambda_{1n}} \cos[(n+\frac{1}{2})\pi x)] \\ \cos[(n+\frac{1}{2})\pi x)] \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_{2k}} \cos[(k+\frac{1}{2})\pi x)] \\ \cos[(k+\frac{1}{2})\pi x)] \end{pmatrix} \Big\rangle \\
&= \frac{(n+\frac{1}{2})(k+\frac{1}{2})\pi^2}{\lambda_{1n}\lambda_{2k}} \int_0^1 \sin[(n+\frac{1}{2})\pi x)] \sin[(k+\frac{1}{2})\pi x)]dx \\
&\quad + \frac{c}{\lambda_{1n}\lambda_{2k}} \int_0^1 \cos[(n+\frac{1}{2})\pi x)] \cos[(k+\frac{1}{2})\pi x)]dx \\
&\quad + \int_0^1 \cos[(n+\frac{1}{2})\pi x)] \cos[(k+\frac{1}{2})\pi x)]dx \\
&= \frac{(n+\frac{1}{2})(k+\frac{1}{2})\pi^2}{2\lambda_{1n}\lambda_{2k}} \int_0^1 \cos[(n-k)\pi x] - \cos[(n+k+1)\pi x]dx \\
&\quad + \frac{1}{2}(\frac{c}{\lambda_{1n}\lambda_{2k}} + 1) \int_0^1 \cos[(n-k)\pi x] + \cos[(n+k+1)\pi x]dx.
\end{aligned}
$$

Then

$$
\langle \Phi_{1n}, \Phi_{2k} \rangle = 0, \quad n \neq k,\ n,\ ; k \geq N+1,\ n \in \mathbb{Z}, \tag{1.35}
$$

and by the fact that $\lambda_{1n}\lambda_{2n} = c - \sigma_n^2$, where $\sigma_n = (n+\frac{1}{2})\pi i$, we have

$$\langle \Phi_{1n}, \Phi_{2n}\rangle = \sqrt{\frac{2\lambda_{1n}}{d(\sigma_n^2 - c)}}\sqrt{\frac{2\lambda_{2n}}{d(\sigma_n^2 - c)}}\frac{c-\sigma_n^2+\lambda_{1n}\lambda_{2n}}{2\lambda_{1n}\lambda_{2n}} = \frac{2}{d\sqrt{c-\sigma_n^2}}. \tag{1.36}$$

For any $n \geq N+1$, $0 \leq m \leq N-1$, $n, m \in \mathbb{Z}$,

$$\begin{aligned}
&\Big\langle \begin{pmatrix} \frac{1}{\lambda_{1n}}\cos[(n+\frac{1}{2})\pi x)] \\ \cos[(n+\frac{1}{2})\pi x)] \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_{3m}}\cos[(m+\frac{1}{2})\pi x)] \\ \cos[(m+\frac{1}{2})\pi x)] \end{pmatrix} \Big\rangle \\
&= \frac{(n+\frac{1}{2})(m+\frac{1}{2})\pi^2}{\lambda_{1n}\bar{\lambda}_{3m}} \int_0^1 \sin[(n+\frac{1}{2})\pi x)]\sin[(m+\frac{1}{2})\pi x)]dx \\
&\quad + \frac{c}{\lambda_{1n}\bar{\lambda}_{3m}} \int_0^1 \cos[(n+\frac{1}{2})\pi x)]\cos[(m+\frac{1}{2})\pi x)]dx \\
&\quad + \int_0^1 \cos[(n+\frac{1}{2})\pi x)]\cos[(m+\frac{1}{2})\pi x)]dx \\
&= \frac{(n+\frac{1}{2})(m+\frac{1}{2})\pi^2}{2\lambda_{1n}\bar{\lambda}_{3m}} \int_0^1 \cos[(n-m)\pi x] - \cos[(n+m+1)\pi x]dx \\
&\quad + \frac{1}{2}(\frac{c}{\lambda_{1n}\bar{\lambda}_{3m}} + 1)\int_0^1 \cos[(n-m)\pi x] + \cos[(n+m+1)\pi x]dx = 0,
\end{aligned}$$

i.e.,

$$\langle \Phi_{1n}, \Phi_{3m}\rangle = 0. \tag{1.37}$$

Furthermore,

$$\begin{aligned}
\langle \Phi_{3m}, \bar{\Phi}_{3m}\rangle &= \Big\langle \begin{pmatrix} \frac{1}{\lambda_{3m}}\cos[(n+\frac{1}{2})\pi x)] \\ \cos[(n+\frac{1}{2})\pi x)] \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_{3m}}\cos[(m+\frac{1}{2})\pi x)] \\ \cos[(m+\frac{1}{2})\pi x)] \end{pmatrix} \Big\rangle \\
&= \frac{(m+\frac{1}{2})^2\pi^2}{(\lambda_{3m})^2}\int_0^1 \sin^2[(m+\frac{1}{2})\pi x)]dx \\
&\quad + (\frac{c}{(\lambda_{3m})^2} + 1)\int_0^1 \cos^2[(m+\frac{1}{2})\pi x)]dx \\
&= \frac{(m+\frac{1}{2})^2\pi^2}{2(\lambda_{3m})^2} + \frac{1}{2}(\frac{c}{(\lambda_{3m})^2} + 1) \\
&= \frac{(m+\frac{1}{2})^2\pi^2 + c + (\lambda_{3m})^2}{2(\lambda_{3m})^2},
\end{aligned} \tag{1.38}$$

where $\lambda_{3m} = \frac{1}{2}[d(\sigma_m^2 - c) + \sqrt{-\Delta_m} i]$, $0 \leq m \leq N-1$, $m \in \mathbb{Z}$, and

$$\sigma_m = i(m+\frac{1}{2})\pi,$$
$$\Delta_m = d^2(c-\sigma_m^2)^2 - 4(c-\sigma_m^2).$$

Moreover, $|\lambda_{3m}|^2 = c - \sigma_m^2$. Then

$$\langle \Phi_N, \Phi_N\rangle = \Big\langle \begin{pmatrix} \frac{1}{\lambda_N}\cos[(N+\frac{1}{2})\pi x] \\ \cos[(N+\frac{1}{2})\pi x] \end{pmatrix}, \begin{pmatrix} \frac{1}{\lambda_N}\cos[(N+\frac{1}{2})\pi x] \\ \cos[(N+\frac{1}{2})\pi x] \end{pmatrix} \Big\rangle = 1. \tag{1.39}$$

$$
\begin{aligned}
\langle\tilde{\Phi}_N,\tilde{\Phi}_N\rangle &= \frac{1}{\frac{4}{d^2}+\frac{d^2}{8}-1}\Big\langle\begin{pmatrix}\cos[(N+\frac{1}{2})\pi x]\\ (\frac{d}{2}-\frac{2}{d})\cos[(N+\frac{1}{2})\pi x]\end{pmatrix},\\
&\quad \begin{pmatrix}\cos[(N+\frac{1}{2})\pi x]\\ (\frac{d}{2}-\frac{2}{d})\cos[(N+\frac{1}{2})\pi x]\end{pmatrix}\Big\rangle = 1.
\end{aligned} \tag{1.40}
$$

$$
\begin{aligned}
\langle\Phi_N,\tilde{\Phi}_N\rangle &= \frac{1}{\sqrt{\frac{4}{d^2}+\frac{d^2}{8}-1}}\Big\langle\begin{pmatrix}\frac{1}{\lambda_N}\cos[(N+\frac{1}{2})\pi x]\\ \cos[(N+\frac{1}{2})\pi x]\end{pmatrix},\begin{pmatrix}\cos[(N+\frac{1}{2})\pi x]\\ (\frac{d}{2}-\frac{2}{d})\cos[(N+\frac{1}{2})\pi x]\end{pmatrix}\Big\rangle\\
&= \frac{d^2-8}{\sqrt{2d^4-16d^2+64}}.
\end{aligned} \tag{1.41}
$$

Similarly, for any $n,\ k\ge N+1,\ 0\le m,\ l\le N-1,\ n,k,m,l\in\mathbb{Z}$, the following holds

$$
\begin{gathered}
\langle\Phi_{1n},\bar{\Phi}_{3m}\rangle = 0,\quad \langle\Phi_{1n},\Phi_N\rangle = 0,\quad \langle\Phi_{1n},\tilde{\Phi}_N\rangle = 0,\\
\langle\Phi_{2n},\Phi_{2k}\rangle = 0\ (n\ne k),\quad \langle\Phi_{2n},\Phi_{2n}\rangle = 1,\quad \langle\Phi_{2n},\Phi_{3m}\rangle = 0,\\
\langle\Phi_{2n},\bar{\Phi}_{3m}\rangle = 0,\quad \langle\Phi_{2n},\Phi_N\rangle = 0,\quad \langle\Phi_{2n},\tilde{\Phi}_N\rangle = 0,\\
\langle\Phi_{3m},\Phi_{3l}\rangle = 0\ (m\ne l),\quad \langle\Phi_{3m},\Phi_{3m}\rangle = \frac{c-\sigma_m^2+|\lambda_{3m}|^2}{2|\lambda_{3m}|^2} = 1,\\
\langle\Phi_{3m},\bar{\Phi}_{3m}\rangle = \frac{c-\sigma_m^2+\lambda_{3m}^2}{2\lambda_{3m}^2},\quad \langle\Phi_{3m},\bar{\Phi}_{3l}\rangle = 0\ (m\ne l),\\
\langle\Phi_{3m},\Phi_N\rangle = 0,\quad \langle\Phi_{3m},\tilde{\Phi}_N\rangle = 0,\quad \langle\bar{\Phi}_{3m},\bar{\Phi}_{3l}\rangle = 0\ (m\ne l),\\
\langle\bar{\Phi}_{3m},\bar{\Phi}_{3m}\rangle = \frac{c-\sigma_m^2+|\bar{\lambda}_{3m}|^2}{2|\bar{\lambda}_{3m}|^2} = 1,\quad \langle\bar{\Phi}_{3m},\Phi_N\rangle = 0,\quad \langle\bar{\Phi}_{3m},\tilde{\Phi}_N\rangle = 0.
\end{gathered}
$$

From the above equalities we deduce that

$$
\|\sum_{m=0}^{N-1}\alpha_m\Phi_{3m}\|^2 = \sum_{m=0}^{N-1}|\alpha_m|^2,\quad \|\sum_{m=0}^{N-1}\beta_m\bar{\Phi}_{3m}\|^2 = \sum_{m=0}^{N-1}|\beta_m|^2. \tag{1.42}
$$

and

$$
\begin{aligned}
&\|\sum_{m=0}^{N-1}\alpha_m\Phi_{3m}+\beta_m\bar{\Phi}_{3m}\|^2\\
&= \Big\langle\sum_{m=0}^{N-1}\alpha_m\Phi_{3m}+\beta_m\bar{\Phi}_{3m},\sum_{l=0}^{N-1}\alpha_l\Phi_{3l}+\beta_l\bar{\Phi}_{3l}\Big\rangle\\
&= \sum_{m=0}^{N-1}(|\alpha_m|^2\|\Phi_{3m}\|^2+|\beta_m|^2\|\bar{\Phi}_{3m}\|^2)\\
&\quad + \sum_{m=0}^{N-1}\alpha_m\bar{\beta}_m\langle\Phi_{3m},\bar{\Phi}_{3m}\rangle+\beta_m\bar{\alpha}_m\langle\bar{\Phi}_{3m},\Phi_{3m}\rangle\\
&= \sum_{m=0}^{N-1}(|\alpha_m|^2+|\beta_m|^2)+\sum_{m=0}^{N-1}2\mathcal{R}e(\alpha_m\bar{\beta}_m\langle\Phi_{3m},\bar{\Phi}_{3m}\rangle).
\end{aligned} \tag{1.43}
$$

A direct calculation from (1.38) indicates, for any $0 \le m \le N-1$ and $m \in \mathbb{Z}$, we have

$$|\langle \Phi_{3m}, \bar{\Phi}_{3m} \rangle| = \frac{d}{2}\sqrt{c-\sigma_m^2}, \tag{1.44}$$

and hence

$$|2\mathcal{R}e(\alpha_m \bar{\beta_m} \langle \Phi_{3m}, \bar{\Phi}_{3m} \rangle| \le (|\alpha_m|^2 + |\bar{\beta}_m|^2)\frac{d}{2}\sqrt{c-\sigma_m^2}. \tag{1.45}$$

Combining (1.43) and (1.45), we obtain

$$\|\sum_{m=0}^{N-1} \alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m}\|^2 \le (1 + \frac{d}{2}\sqrt{c-\sigma_{N-1}^2}) \sum_{m=0}^{N-1} (|\alpha_m|^2 + |\beta_m|^2), \tag{1.46}$$

and

$$\begin{aligned} \|\sum_{m=0}^{N-1} \alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m}\|^2 &\ge \sum_{m=0}^{N-1} (1 - \frac{d}{2}\sqrt{c-\sigma_m^2})(|\alpha_m|^2 + |\beta_m|^2) \\ &> (1 - \frac{d}{2}\sqrt{c-\sigma_{N-1}^2}) \sum_{m=0}^{N-1} (|\alpha_m|^2 + |\beta_m|^2), \end{aligned} \tag{1.47}$$

where $1 - \frac{d}{2}\sqrt{c-\sigma_{N-1}^2} > 1 - \frac{d}{2}\sqrt{c-\sigma_N^2} = 0$.

Without loss of generality, let $K_1, K_2 \in \mathbb{Z}$, $K_1 \ge K_2 > 0$,

$$\begin{aligned} &\|\sum_{n=N+1}^{N+K_1} \alpha_n \Phi_{1n} + \sum_{n=N+1}^{N+K_2} \beta_n \Phi_{2n}\|^2 \\ &= \sum_{n=N+1}^{N+K_1} |\alpha_n|^2 + \sum_{n=N+1}^{N+K_2} |\beta_n|^2 \\ &\quad + \sum_{n=N+1}^{N+K_2} \langle \alpha_n \Phi_{1n}, \beta_n \Phi_{2n} \rangle + \langle \beta_n \Phi_{2n}, \alpha_n \Phi_{1n} \rangle. \end{aligned} \tag{1.48}$$

While

$$\begin{aligned} \langle \alpha_n \Phi_{1n}, \beta_n \Phi_{2n} \rangle + \langle \beta_n \Phi_{2n}, \alpha_n \Phi_{1n} \rangle &= 2\mathcal{R}e(\alpha_n \beta_n \langle \Phi_{1n}, \Phi_{2n} \rangle \\ &\le 2|\alpha_n||\beta_n||\langle \Phi_{1n}, \Phi_{2n} \rangle| \\ &\le \frac{2}{d\sqrt{c-\sigma_n^2}} (|\alpha_n|^2 + |\beta_n|^2) \\ &\le \frac{2}{d\sqrt{c-\sigma_{N+1}^2}} (|\alpha_n|^2 + |\beta_n|^2). \end{aligned} \tag{1.49}$$

Inserting (1.49) into (1.48), we obtain

$$
\begin{aligned}
&\|\sum_{n=N+1}^{N+K_1}\alpha_n\Phi_{1n}+\sum_{n=N+1}^{N+K_2}\beta_n\Phi_{2n}\|^2\\
&\le \sum_{n=N+1}^{N+K_1}|\alpha_n|^2+\sum_{n=N+1}^{N+K_2}|\beta_n|^2+\sum_{n=N+1}^{N+K_2}\frac{2}{d\sqrt{c-\sigma_n^2}}(|\alpha_n|^2+|\beta_n|^2)\\
&< \sum_{n=N+1}^{N+K_1}(1+\frac{2}{d\sqrt{c-\sigma_{N+1}^2}})|\alpha_n|^2+\sum_{n=N+1}^{N+K_2}(1+\frac{2}{d\sqrt{c-\sigma_{N+1}^2}})|\beta_n|^2\\
&= (1+\frac{2}{d\sqrt{c-\sigma_{N+1}^2}})[\sum_{n=N+1}^{N+K_1}|\alpha_n|^2+\sum_{n=N+1}^{N+K_2}|\beta_n|^2],
\end{aligned}
\tag{1.50}
$$

and

$$
\begin{aligned}
&\|\sum_{n=N+1}^{N+K_1}\alpha_n\Phi_{1n}+\sum_{n=N+1}^{N+K_2}\beta_n\Phi_{2n}\|^2\\
&\ge \sum_{n=N+1}^{N+K_1}|\alpha_n|^2+\sum_{n=N+1}^{N+K_2}|\beta_n|^2-\sum_{n=N+1}^{N+K_2}\frac{2}{d\sqrt{c-\sigma_n^2}}(|\alpha_n|^2+|\beta_n|^2)\\
&> (1-\frac{2}{d\sqrt{c-\sigma_{N+1}^2}})[\sum_{n=N+1}^{N+K_1}|\alpha_n|^2+\sum_{n=N+1}^{N+K_2}|\beta_n|^2],
\end{aligned}
\tag{1.51}
$$

where $1-\frac{2}{d\sqrt{c-\sigma_{N+1}^2}}>1-\frac{2}{d\sqrt{c-\sigma_N^2}}=0$. Similarly,

$$
\begin{aligned}
&\|\alpha_N\Phi_N+\tilde\alpha_N\tilde\Phi_N\|^2\\
&=|\alpha_N|^2\|\Phi_N\|^2+|\tilde\alpha_N|^2\|\tilde\Phi_N\|^2+2\mathcal{R}e(\alpha_N\tilde\alpha_N\langle\Phi_N,\tilde\Phi_N\rangle),
\end{aligned}
\tag{1.52}
$$

and

$$
\begin{aligned}
|2\mathcal{R}e(\alpha_N\tilde\alpha_N\langle\Phi_N,\tilde\Phi_N\rangle)| &\le (|\alpha_N|^2+|\tilde\alpha_N|^2)|\langle\Phi_N,\tilde\Phi_N\rangle|\\
&=\frac{|d^2-8|}{\sqrt{2d^4-16d^2+64}}(|\alpha_N|^2+|\tilde\alpha_N|^2|).
\end{aligned}
\tag{1.53}
$$

Combining (1.52) and (1.53), we obtain

$$
\|\alpha_N\Phi_N+\tilde\alpha_N\tilde\Phi_N\|^2\le(1+\frac{|d^2-8|}{\sqrt{2d^4-16d^2+64}})(|\alpha_N|^2+|\tilde\alpha_N|^2|),\tag{1.54}
$$

$$
\|\alpha_N\Phi_N+\tilde\alpha_N\tilde\Phi_N\|^2\ge(1-\frac{|d^2-8|}{\sqrt{2d^4-16d^2+64}})(|\alpha_N|^2+|\tilde\alpha_N|^2|),\tag{1.55}
$$

where

$$
1-\frac{|d^2-8|}{\sqrt{2d^4-16d^2+64}}>0.
$$

Considering(1.46), (1.47), (1.50), (1.51) (1.54), and (1.55), by Definition 1.2 we conclude that

$$
\{\Phi_{1n},\Phi_{2n},\Phi_{3m},\bar\Phi_{3m},\Phi_N,\tilde\Phi_N,\ n\ge N+1,\,0\le m\le N-1,\,n,m\in\mathbb{Z}\}
$$

form a Riesz basis. Furthermore the spectrum-determined growth condition holds for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. This completes the proof of Theorem 1.6. Hence semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable for $d^2c < 4$. $\square$

Remark 1.11. If there exists a positive integer N such that

$$(N+\frac{1}{2})^2\pi^2 < \frac{4}{d^2} - c < (N+\frac{3}{2})^2\pi^2, \tag{1.56}$$

the operator $\mathcal{A}$ has the eigenvectors

$$\big\{\Phi_{1n}, \Phi_{2n}, \Phi_{3m}, \bar{\Phi}_{3m},\ n \ge N+1,\, 0 \le m \le N,\, n, m \in \mathbb{Z}\big\},$$

where

$$\begin{gathered}
\Phi_{1n} = \sqrt{\frac{2\lambda_{1n}}{d(\sigma_n^2 - c)}}(\frac{1}{\lambda_{1n}}\cos[(n+\frac{1}{2})\pi x], \cos[(n+\frac{1}{2})\pi x]), \quad n \in \mathbb{Z},\ n \ge N+1, \\
\Phi_{2n} = \sqrt{\frac{2\lambda_{2n}}{d(\sigma_n^2 - c)}}(\frac{1}{\lambda_{2n}}\cos[(n+\frac{1}{2})\pi x], \cos[(n+\frac{1}{2})\pi x]), \quad n \in \mathbb{Z},\ n \ge N+1, \\
\Phi_{3m} = (\frac{1}{\lambda_{3m}}\cos[(m+\frac{1}{2})\pi x], \cos[(m+\frac{1}{2})\pi x]), \quad m \in \mathbb{Z},\ 0 \le m \le N, \\
\bar{\Phi}_{3m} = (\frac{1}{\bar{\lambda}_{3m}}\cos[(m+\frac{1}{2})\pi x], \cos[(m+\frac{1}{2})\pi x]), \quad n \in \mathbb{Z},\ 0 \le n \le N,
\end{gathered} \tag{1.57}$$

and λ_n were given by(1.22).

First we can easily check that

$$\overline{\operatorname{span}}\{\Phi_{1n}, \Phi_{2n}, \Phi_{3m}, \bar{\Phi}_{3m},\ n \ge N+1, 0 \le m \le N,\ n,\ m \in \mathbb{Z}\} = \mathcal{H}.$$

Furthermore, for any $n, k \ge N+1$, $0 \le m,\ l \le N,\ n, k, m, l \in \mathbb{Z}$, the following holds:

$$\begin{gathered}
\langle \Phi_{1n}, \Phi_{1k}\rangle = 0\ (n \ne k), \quad \langle \Phi_{1n}, \Phi_{1n}\rangle = 1, \quad \langle \Phi_{1n}, \Phi_{2n}\rangle = \frac{2}{d\sqrt{c-\sigma_n^2}}, \\
\langle \Phi_{1n}, \Phi_{2k}\rangle = 0\ (n \ne k), \quad \langle \Phi_{2n}, \Phi_{2n}\rangle = 1, \quad \langle \Phi_{2n}, \Phi_{2k}\rangle = 0\ (n \ne k), \\
\langle \Phi_{1n}, \Phi_{3m}\rangle = 0, \quad \langle \Phi_{1n}, \bar{\Phi}_{3m}\rangle = 0, \quad \langle \Phi_{2n}, \Phi_{3m}\rangle = 0, \\
\langle \Phi_{2n}, \bar{\Phi}_{3m}\rangle = 0, \quad \langle \Phi_{3m}, \Phi_{3m}\rangle = 1, \quad \langle \Phi_{3m}, \Phi_{3l}\rangle = 0\ (m \ne l), \\
\langle \bar{\Phi}_{3m}, \bar{\Phi}_{3m}\rangle = 1, \quad \langle \bar{\Phi}_{3m}, \bar{\Phi}_{3l}\rangle = 0\ (m \ne l), \quad \langle \Phi_{3m}, \bar{\Phi}_{3l}\rangle = 0\ (m \ne l), \\
\langle \Phi_{3m}, \bar{\Phi}_{3m}\rangle = \frac{c - \sigma_m^2 + \lambda_{3m}^2}{2\lambda_{3m}^2}.
\end{gathered}$$

Let $K_1, K_2 \in \mathbb{Z}$, $K_1 \geq K_2 > 0$. Then

$$\begin{aligned}
&\| \sum_{n=N+1}^{N+K_1} \alpha_n \Phi_{1n} + \sum_{n=N+1}^{N+K_2} \beta_n \Phi_{2n} \|^2 \\
&\leq \sum_{n=N+1}^{N+K_1} |\alpha_n|^2 + \sum_{n=N+1}^{N+K_2} |\beta_n|^2 + \sum_{n=N+1}^{N+K_2} \frac{2}{d\sqrt{c-\sigma_n^2}} (|\alpha_n|^2 + |\beta_n|^2) \\
&< \sum_{n=N+1}^{N+K_1} (1 + \frac{2}{d\sqrt{c-\sigma_{N+1}^2}}) |\alpha_n|^2 + \sum_{n=N+1}^{N+K_2} (1 + \frac{2}{d\sqrt{c-\sigma_{N+1}^2}}) |\beta_n|^2 \\
&= (1 + \frac{2}{d\sqrt{c-\sigma_{N+1}^2}}) [\sum_{n=N+1}^{N+K_1} |\alpha_n|^2 + \sum_{n=N+1}^{N+K_2} |\beta_n|^2].
\end{aligned} \tag{1.58}$$

Similarly, we can deduce that

$$\begin{aligned}
&\| \sum_{n=N+1}^{N+K_1} \alpha_n \Phi_{1n} + \sum_{n=N+1}^{N+K_2} \beta_n \Phi_{2n} \|^2 \\
&\geq (1 - \frac{2}{d\sqrt{c-\sigma_{N+1}^2}}) [\sum_{n=N+1}^{N+K_1} |\alpha_n|^2 + \sum_{n=N+1}^{N+K_2} |\beta_n|^2],
\end{aligned} \tag{1.59}$$

where $1 - \frac{2}{d\sqrt{c-\sigma_{N+1}^2}} > 0$. Moreover,

$$\begin{aligned}
\| \sum_{m=0}^{N} \alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m} \|^2 &= \langle \sum_{m=0}^{N} \alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m}, \sum_{l=0}^{N} \alpha_l \Phi_{3l} + \beta_l \bar{\Phi}_{3l} \rangle \\
&= \sum_{m=0}^{N} (|\alpha_m|^2 \|\Phi_{3m}\|^2 + |\beta_m|^2 \|\bar{\Phi}_{3m}\|^2) \\
&\quad + \sum_{m=0}^{N} \alpha_m \bar{\beta}_m \langle \Phi_{3m}, \bar{\Phi}_{3m} \rangle + \beta_m \bar{\alpha}_m \langle \bar{\Phi}_{3m}, \Phi_{3m} \rangle \\
&= \sum_{m=0}^{N} (|\alpha_m|^2 + |\beta_m|^2) + \sum_{m=0}^{N} 2\mathcal{R}e(\alpha_m \bar{\beta}_m \langle \Phi_{3m}, \bar{\Phi}_{3m} \rangle).
\end{aligned}$$

By estimating the term $2\mathcal{R}e(\alpha_m \bar{\beta}_m \langle \Phi_{3m}, \bar{\Phi}_{3m} \rangle)$, we can obtain

$$\begin{aligned}
\| \sum_{m=0}^{N} \alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m} \|^2 &\leq (1 + \frac{d}{2}\sqrt{c-\sigma_N^2}) \sum_{m=0}^{N} (|\alpha_m|^2 + |\beta_m|^2), \\
\| \sum_{m=0}^{N} (\alpha_m \Phi_{3m} + \beta_m \bar{\Phi}_{3m} \|^2) &\geq (1 - \frac{d}{2}\sqrt{c-\sigma_N^2}) \sum_{m=0}^{N} |\alpha_m|^2 + |\beta_m|^2,
\end{aligned}$$

where $1 - \frac{d}{2}\sqrt{c-\sigma_N^2} > 0$. Combining the above inequalities with (1.58), and (1.59), we conclude that, in the case of (1.57),

$$\{\Phi_{1n}, \Phi_{2n}, \Phi_{3m}, \bar{\Phi}_{3m},\ n \geq N+1,\ 0 \leq m \leq N,\ n, m \in \mathbb{Z}\},$$

also forms a Riesz basis of the state space $\mathcal{H}$.

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