Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 185, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SPREADING SOLUTIONS FOR A REACTION DIFFUSION EQUATION WITH FREE BOUNDARIES IN TIME-PERIODIC ENVIRONMENT

FANG LI, JUNFAN LU

Abstract. In this article, we consider a reaction diffusion equation with free boundaries in a time-periodic environment. Such models can be used to describe the spreading of a new or invasive species over a one-dimensional habitat, with the free boundaries representing the expanding fronts. We study an equation with a general time-periodic nonlinearity, and present some sufficient conditions for spreading phenomena. We also use time-periodic semi-waves to characterize the spreading solutions.

1. Introduction

In this article, we study the spreading phenomena of the time-periodic reaction diffusion equation with free boundaries,

$$\begin{gathered}
u_t = u_{xx} + f(t,u), \quad t>0,\ g(t)<x<h(t),\\
u(t,h(t)) = 0, h'(t) = -\mu u_x(t,h(t)), \quad t>0,\\
u(t,g(t)) = 0, g'(t) = -\mu u_x(t,g(t)), \quad t>0,\\
-g(0) = h(0) = h_0, u(0,x) = u_0(x), \quad -h_0 \le x \le h_0,
\end{gathered} \tag{1.1}$$

where μ and h_0 are given positive constants, u_0 is a nonnegative function with support in $[-h_0,h_0]$, $x=g(t)$ and $x=h(t)$ are the moving boundaries to be determined together with $u(t,x)$. Moreover, for some $T>0$, $\gamma\in(0,1)$ and some $\alpha(t)\in C(\mathbb{R})$ (T-periodic and $\alpha^0 := \max\alpha(t) > \alpha_0 := \min\alpha(t) > 0$), the function f is a general nonlinearity satisfying the assumption

(H1) $f(t,u)\in C^{\gamma/2,1}_{\rm loc}([0,T]\times\mathbb{R})$ is T-periodic in t, $f(t,0)=f(t,\alpha(t))\equiv 0$, $f_u(t,u)<0$ for any $t\in[0,T]$ and $u\in[\alpha_0,\alpha^0]$, $f(t,u)<0$ for $u>\alpha(t)$, and

$$\int_u^{\alpha_0} \min_{t\in[0,T]} f(t,s)ds > 0 \quad \text{for all } u\in[0,\alpha_0). \tag{1.2}$$

In the special case where $f(t,u)=u(a-bu)(a,b>0)$, the problem (1.1) was studied in [5]. Such a problem can be regarded as a model describing the spreading of a new or invasive species over a one-dimensional habitat, where $u(t,x)$ represents the density of the species at location x and time t, and its spreading fronts are represented by the free boundaries $x=g(t)$ and $x=h(t)$. The Stefan conditions

2010 *Mathematics Subject Classification.* 35B40, 35R35, 35K55, 92B05.
Key words and phrases. Spreading phenomenon; free boundary; time-periodic environment.

Submitted December 20, 2017. Published November 15, 2018.

$g'(t) = -\mu u_x(t, g(t))$ and $h'(t) = -\mu u_x(t, h(t))$ are interpreted as saying that the spreading fronts expand at a speed proportional to the population gradient at the front, a deduction of these conditions from ecological considerations can be found in [2]. Among others, Du and Lin [5] proved a spreading-vanishing dichotomy result for the asymptotic behavior of the solutions, namely, there is a barrier $R^* > 0$ such that

- (i) Spreading: the spreading fronts break the barrier at some finite time, and then the free boundaries go to infinity (i.e., $-g(t), h(t) \to \infty$ as $t \to \infty$), and the population successfully establishes itself in the new environment (i.e., $u(t, x) \to a/b$ as $t \to \infty$).
- (ii) Vanishing: the fronts never break the barrier (i.e., $h(t) - g(t) < R^*$ for all $t \geq 0$), and the population vanishes (i.e., $u(t, x) \to 0$ as $t \to \infty$).

Moreover, when spreading occurs, the asymptotic spreading speed can be determined (namely, $\lim_{t\to\infty} h(t)/t$ exists and is uniquely determined). The vanishing phenomena is a remarkable result since it shows that the presence of free boundaries makes spreading difficult and the *hair-trigger effect* in the Cauchy problem can be avoided for some small initial data. These results have subsequently been extended to more general situations in several directions. For example, Du and Lou [6] considered the monostable, bistable and combustion types nonlinearities and obtained a rather complete description on the asymptotic behavior of the solutions. For time dependent environments, Du, Guo and Peng [4] considered the time-periodic case and Li, Liang and Shen [12, 13] considered the time almost periodic case, both gave a spreading-vanishing dichotomy result, as in [5]. Especially, [4] specified the spreading solution by using the semi-wave. Other studies for time dependent problem includes [15] (for time-periodic reaction-advection-diffusion equations), [3] (for space-time periodic problem), etc.

In this article, we extend the Fisher-KPP type nonlinearity to general ones (including monostable, bistable, combustion and other multi-stable nonlinearities as special cases). From the recent works [9, 10] (for Cauchy problems) one sees that, even for the homogeneous case (i.e. $f(t, u)$ is independent of t), when f is a multistable nonlinearity, the asymptotic behavior of the solutions can be very complicated, and it is characterized by terrace rather than traveling waves. Due to this reason, we mainly focus on the spreading phenomena of solutions to (1.1). We will provide some sufficient conditions for spreading, and then use the time-periodic semi-wave to characterize the spreading solutions.

To explain our results, we first list some special solutions of $(1.1)_1$ (which denotes the first equation in (1.1)), whose proofs are given in later sections.

(1) **Positive periodic solution** $P(t)$. It is easily to know that the ODE $u_t = f(t, u)$ has a unique maximal periodic solution $P(t)$ with $\alpha_0 \leq P(t) \leq \alpha^0$.

(2) **Compactly supported subsolutions**. Denote $\tilde{\rho}(u) := \min_{t\in[0,T]} f(t, u)$. By (H1) we have

$$\tilde{\rho}(\alpha_0) = 0,\ \tilde{\rho}(u) < 0 \text{ for } u > \alpha_0, \quad \int_u^{\alpha_0} \tilde{\rho}(s)ds > 0 \text{ for } 0 \leq u < \alpha_0.$$

We take a C^1 function $\rho(u)$ such that it is slightly smaller than $\tilde{\rho}$, $\rho'(\alpha_0) < 0$, and that, for given small $\varepsilon > 0$ and $\alpha_\varepsilon := \alpha_0 - \varepsilon$,

$$\rho(\alpha_\varepsilon) = 0,\ \rho(u) < 0 \text{ for } u > \alpha_\varepsilon, \quad \int_u^{\alpha_\varepsilon} \rho(s)ds > 0 \text{ for } 0 \leq u < \alpha_\varepsilon. \tag{1.3}$$

Denote

$$\theta := \max\{u < \alpha_\varepsilon : \rho(u) = 0\}, \quad \bar\theta := \inf\{u > \theta : \int_0^u \rho(s)ds > 0\}.$$

Then $\theta \in [0, \alpha_\varepsilon)$ and $\bar\theta \in [\theta, \alpha_\varepsilon)$. We will show in Lemma 2.1 that, for each $\beta \in (\bar\theta, \alpha_\varepsilon)$, the problem

$$v'' + \rho(v) = 0, \quad v(0) = \beta, \quad v'(0) = 0 \tag{1.4}$$

has a unique solution $V(x; \beta)$, positive in $(-L, L)$ for some $L > 0$ and $V(\pm L; \beta) = 0$. Clearly, each one of such functions is a subsolution of the original equation $(1.1)_1$.

(3) **Periodic rightward traveling semi-wave**. Consider the problem

$$\begin{gathered} U_t = U_{zz} - rU_z + f(t, U), \quad t \in [0, T], z > 0, \\ U(t, 0) = 0, U(t, \infty) = P(t), \quad t \in [0, T], \\ U(0, z) = U(T, z), U_z(t, z) > 0, \quad t \in [0, T], z > 0, \\ r(t) = \mu U_z(t, 0), \quad t \in [0, T]. \end{gathered} \tag{1.5}$$

We will show in Proposition 3.3 that this problem has a solution pair (r, U) with $r = r(t) \in \mathcal{P}_+$, where

$$\begin{gathered} \mathcal{P} := \{p \in C^{\gamma/2}([0, T]) : p(0) = p(T)\}, \\ \mathcal{P}_+ := \{p \in \mathcal{P} : p(t) > 0 \text{ for all } t \in [0, T]\}. \end{gathered}$$

With $R(t) := \int_0^t r(s)ds$, the function $u(t, x) = U(t, R(t) - x; -r)$ satisfies $(1.1)_1$, $u(t, R(t)) = 0$ and $R'(t) = -\mu u_x(t, R(t))$. We call $u = U(t, R(t) - x; -r)$ a *periodic rightward traveling semi-wave* since it is only defined in $x \le R(t)$ and $U(t, z; -r)$ is periodic in t.

Throughout this article we choose the initial data u_0 from the set

$$\begin{aligned} \mathcal{X}(h_0) = \big\{ & \phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0, \phi'(-h_0) > 0, \\ & \phi'(h_0) < 0, \phi(x) > 0 \text{ in } (-h_0, h_0) \big\}. \end{aligned} \tag{1.6}$$

By a similar argument as in [6], one can show that, for any $h_0 > 0$ and any initial data u_0, the problem (1.1) has a time-global solution $(u(t, x), g(t), h(t))$, with $u \in C^{1+\gamma/2, 2+\gamma}((0, \infty) \times [g(t), h(t)])$ and $g, h \in C^{1+\gamma/2}(0, \infty)$. Moreover, it follows from the maximum principal that, when $t > 0$, the solution u is positive in $(g(t), h(t))$, with $u_x(t, g(t)) > 0$ and $u_x(t, h(t)) < 0$. Thus $g'(t) < 0 < h'(t)$ for all $t > 0$. Denote

$$g_\infty := \lim_{t\to\infty} g(t), \quad h_\infty := \lim_{t\to\infty} h(t), \quad I_\infty := (g_\infty, h_\infty)$$

There are some possible situations on the asymptotic behavior of the solutions to (1.1). Spreading phenomenon is the most interesting one among them. Our first main result provides some sufficient conditions for spreading.

Theorem 1.1. *Assume* (H1). *If* $u_0 \in \mathcal{X}(h_0)$ *satisfies* $u_0 \ge V(x; \beta)$, *where* V *is the unique solution of the problem* (1.4) *for some* $\beta \in (\bar\theta, \alpha_\varepsilon)$, *then spreading happens in the sense that* $h_\infty = -g_\infty = \infty$, *and*

$$\lim_{t\to\infty} [u(t, \cdot) - P(t)] = 0 \quad \textit{locally uniformly in } \mathbb{R}. \tag{1.7}$$

Furthermore, when spreading happens, we will show that the right front of $u \approx U(t, R(t) - x)$ and the left front of $u \approx U(t, x + R(t))$. To construct precise sub- and supersolutions in our approach, we need the exponential stability of $P(t)$. For this purpose we have an additional condition:

(H2) the function f satisfies

$$\sigma(t) := f_u(t, P(t)) - \frac{f(t, P(t))}{P(t)} < 0 \quad \text{for } t \in [0, T]. \tag{1.8}$$

Theorem 1.2. *Assume* (H1), (H2). *When spreading happens, there exists $H_1, G_1 \in \mathbb{R}$ such that*

$$\lim_{t\to\infty}[h(t) - R(t)] = H_1, \quad \lim_{t\to\infty}[h'(t) - r(t)] = 0, \tag{1.9}$$

$$\lim_{t\to\infty}[g(t) + R(t)] = G_1, \quad \lim_{t\to\infty}[g'(t) + r(t)] = 0, \tag{1.10}$$

$$\lim_{t\to\infty} \|u(t,\cdot) - U(t, R(t) + H_1 - \cdot)\|_{L^\infty([0,h(t)])} = 0, \tag{1.11}$$

$$\lim_{t\to\infty} \|u(t,\cdot) - U(t, \cdot + R(t) - G_1)\|_{L^\infty([g(t),0])} = 0, \tag{1.12}$$

where $R(t) = \int_0^t r(s)ds$. Here we extend $U(t, z)$ to be zero for $z < 0$.

This article is organized as follows. In Section 2, we present the lower and upper estimates for the solution to (1.1) and prove Theorem 1.1. In Section 3, we construct a time-periodic traveling semi-wave and use it to characterize the profile of the spreading solutions, and prove Theorem 1.2.

2. Spreading happening

In this section, we give sufficient conditions to ensure the spreading phenomena happens. We first construct some subsolutions of $(1.1)_1$ which will be used for comparison, then we present the lower and upper estimates for u and prove Theorem 1.1. Throughout this section, we assume (H1) and use the notation $\rho, \alpha_\varepsilon, \theta, \bar{\theta}$ etc. as in Section 1.

2.1. **Subsolutions.** In this subsection, we construct the subsolutions to $(1.1)_1$, which are solutions to $v'' + \rho(v) = 0$ with compact supports.

Lemma 2.1. *For any $\beta \in (\bar{\theta}, \alpha_\varepsilon)$, the unique solution $V(x;\beta)$ of* (1.4) *exists in the interval $[-L, L]$ for some $L = L(\beta) > 0$, and*

$$V(\pm L;\beta) = 0, \quad V(x;\beta) = V(-x;\beta), \quad V'(x;\beta) < 0 \quad \text{for } 0 < x \le L. \tag{2.1}$$

Proof. We use the phase plane to consider the initial value problem (1.4) in a suitable interval $J \subset \mathbb{R}$. The equation in (1.4) is equivalent to the system

$$v'(x) = w, \quad w' = -\rho(v). \tag{2.2}$$

A solution $(v(x), w(x))$ of this system traces out a trajectory in the v-w phase plane. Such a trajectory has slope

$$\frac{dw}{dv} = -\frac{\rho(v)}{w} \tag{2.3}$$

at any point where $w \neq 0$. It is easily seen that $(\alpha_\varepsilon, 0)$ is one singular point on the phase plane. $w = \sqrt{2\int_v^{\alpha_\varepsilon} \rho(s)ds}$ is the unique strictly increasing solution of $v'' + \rho(v) = 0$ in $[0, \infty)$ connecting the regular point $(0, \sqrt{2\int_0^{\alpha_\varepsilon} \rho(s)ds})$ and the singular point $(\alpha_\varepsilon, 0)$. Since the solution depends on initial value β continuously, for any $\beta \in (\bar{\theta}, \alpha_\varepsilon)$, there exists a unique $L(\beta) > 0$ such that the problem (1.4) has a solution $V(x;\beta) \in C^2([-L(\beta), L(\beta)])$ with $V(\pm L(\beta);\beta) = 0$. Obviously, $V = V(x;\beta)$ satisfies the properties (2.1). $\square$

Collecting the solutions of (1.4) for different β we obtain a set

$$\mathcal{S} = \{v : v = V(x;\beta) \text{ for some } \beta \in (\bar{\theta}, \alpha_\varepsilon)\}.$$

The previous lemma indicates that this set is not empty. Moreover, from the above phase plane analysis, it is easily seen that $L(\beta)$ is continuous in $\beta \in (\bar{\theta}, \alpha_\varepsilon)$ and, as $\beta \to \alpha_\varepsilon$, $L(\beta) \to \infty$ and $V(x;\beta) \to \alpha_\varepsilon$ in $L^\infty_{\rm loc}(\mathbb{R})$ topology.

2.2. **Lower estimate.** Since $\alpha_0 > \alpha_\varepsilon > \bar{\theta}$, we have $\delta := (\alpha_0 - \bar{\theta})/3 > 0$. When we take $\varepsilon > 0$ small in α_ε we have $\alpha_0 - \delta < \alpha_\varepsilon < \alpha_0$.

Now we show that spreading happens for the solution (u, g, h) of (1.1) in a weak sense.

Lemma 2.2. *Let (u, g, h) be the solution triple of problem* (1.1) *with initial data u_0 as in Theorem 1.1. Then $h_\infty = -g_\infty = \infty$, and for any integer n, there exists $\tau(n) > 0$ such that*

$$u(t,x) \geq \alpha_0 - 2\delta, \quad x \in [-n, n],\ t \geq \tau(n). \tag{2.4}$$

Proof. Let u_0 be the initial data in Theorem 1.1. Consider the auxiliary problem

$$\begin{gathered} w_t = w_{xx} + \rho(w), \quad \tilde{g}(t) < x < \tilde{h}(t),\ t > 0, \\ w(t, \tilde{g}(t)) = 0, \quad \tilde{g}'(t) = -\mu w(t, \tilde{g}(t)),\ t > 0, \\ w(t, \tilde{h}(t)) = 0, \quad \tilde{h}'(t) = -\mu w(t, \tilde{h}(t)),\ t > 0, \\ -\tilde{g}(0) = \tilde{h}(0) = h_0, \quad w(0,x) = u_0(x), \quad -h_0 \leq x \leq h_0. \end{gathered} \tag{2.5}$$

By [6, Theorem 1.1], either $\tilde{h}(t) - \tilde{g}(t)$ remains bounded and $w(t,\cdot) \to 0$ as $t \to \infty$, or $\tilde{h}(t), -\tilde{g}(t) \to \infty$ and $w(t,\cdot)$ converges to a stationary solution $w_\infty(x)$ as $t \to \infty$. In particular, if $u_0(x) \geq V(x;\beta)$ as in Theorem 1.1, we have $w_\infty(x) \geq V(x;\beta)$ by the comparison principle, and so $\tilde{h}(t), -\tilde{g}(t) \to \infty$. Therefore, $w_\infty(x)$ is a solution of $v'' + \rho(v) = 0$, positive in $\mathbb{R}$ and larger than $V(x;\beta)$, which is nothing but α_ε. This implies that, for any integer n, there exists $\tau(n) > 0$ such that

$$w(t,x) \geq \alpha_\varepsilon - \delta, \quad x \in [-n, n],\ t \geq \tau(n).$$

Since $\rho(u) \leq f(t,u)$ by the definition of ρ, we see that $(w, \tilde{g}, \tilde{h})$ is a subsolution of (1.1), and so

$$\begin{gathered} h(t) \geq \tilde{h}(t) \to \infty, \quad g(t) \leq \tilde{g}(t) \to -\infty \quad \text{as } t \to \infty, \\ u(t,x) \geq w(t,x) \geq \alpha_\varepsilon - \delta > \alpha_0 - 2\delta, \quad x \in [-n, n],\ t \geq \tau(n). \end{gathered}$$

This completes the proof. $\square$

In the rest of this subsection we will use (2.4) to give the lower estimate $P(t) - \epsilon$ for u. Define

$$\begin{gathered} k_1(t,\eta) = \frac{f(t,\alpha_0)}{2\delta}[\eta - (\alpha_0 - 2\delta)], \quad t \in [0,T],\ \eta \in \mathbb{R}, \\ k(t,\eta) = \min\{k_1(t,\eta),\ f(t,\eta)\}, \quad t \in [0,T],\ \eta \geq \alpha_0 - 2\delta. \end{gathered} \tag{2.6}$$

For any large integer $n > 0$, let $\tau(n)$ be the time in (2.4), then there exists a large integer k_n such that $k_nT > \tau(n)$. Consider the problem

$$\begin{gathered}\eta_t = \eta_{xx} + k(t,\eta), \quad -n < x < n,\ t > 0,\\ \eta(t,\pm n) = \alpha_0 - 2\delta, \quad t > 0,\\ \eta(0,x) = u(k_nT,x), \quad -n \le x \le n.\end{gathered} \tag{2.7}$$

By [1, Theorem 1], the solution $\eta(t,x;n)$ of (2.7) converges as $t \to \infty$ to a time-periodic solution $\eta^{\rm per}(t,x;n)$ of (2.7). Note that $k(t,\eta)$ is a Fisher-KPP type nonlinearity (above $\alpha_0 - 2\delta$), by [11, 14], $\eta^{\rm per}(t,x;n) \to P(t)$ as $n \to \infty$ in $L^\infty_{\rm loc}(\mathbb{R})$ topology.

Lemma 2.3. *Under the assumption of Theorem 1.1, for any $\epsilon > 0$ and any $M > 0$, there exists $\tau(M,\epsilon) > 0$ such that*

$$u(t,x) \ge P(t) - \epsilon, \quad x \in [-M,M],\ t \ge \tau(M,\epsilon). \tag{2.8}$$

Proof. By $\eta^{\rm per}(t,x;n) \to P(t)$ as $n \to \infty$, there exists an integer $n_0 > M$ depending on M and ϵ such that

$$\eta^{\rm per}(t,x;n_0) > P(t) - \frac{\epsilon}{2}, \quad x \in [-M,M],\ t \in [0,T]. \tag{2.9}$$

For this fixed n_0, we see that the solution $\eta(t,x;n_0)$ of (2.7) (with $n = n_0$) converges as $t \to \infty$ to $\eta^{\rm per}(t,x;n_0)$. Thus, there exists a integer n_1 such that for any integer $m \ge n_1$ we have

$$\eta(mT + t,x;n_0) \ge \eta^{\rm per}(t,x;n_0) - \frac{\epsilon}{2}, \quad x \in [-n_0,n_0],\ t \in [0,T]. \tag{2.10}$$

Finally, using (2.4) and the comparison principle to compare $u(k_{n_0}T + t,x)$ with the solution $\eta(t,x;n_0)$ of (2.7) we have

$$u(k_{n_0}T + t,x) \ge \eta(t,x;n_0), \quad x \in [-n_0,n_0],\ t > 0. \tag{2.11}$$

Combining the inequalities in (2.9), (2.10) and (2.11) we obtain

$$u(k_{n_0}T + mT + t,x) \ge P(t) - \epsilon, \quad x \in [-M,M],\ t \in [0,T],\ m \ge n_1.$$

Choosing $\tau(M,\epsilon) = k_{n_0}T + n_1T$ we obtain (2.8). $\square$

Proof of Theorem 1.1. Consider the initial value problem, of ODE,

$$\begin{gathered}\zeta_t = f(t,\zeta), \quad t > 0,\\ \zeta(0) = \alpha^0 + \|u_0\|_\infty.\end{gathered} \tag{2.12}$$

It is known that $\zeta(t)$ decreases for small t and then converges to $P(t)$ as $t \to \infty$. Hence for any small $\epsilon > 0$, there exists $\tau_1 > 0$ such that $\zeta(t) \le P(t)+\epsilon$ when $t \ge \tau_1$. By comparison we have

$$u(t,x) \le \zeta(t) \le P(t) + \epsilon, \quad x \in [g(t),h(t)],\ t \ge \tau_1.$$

Combining with (2.8) we prove (1.7). This conclusion and Lemma 2.2 complete the proof. $\square$

3. Using semi-wave to characterize the spreading phenomena

In this section we first construct a time-periodic traveling semi-wave U propagating rightward with speed $r(t)$, and then prove the boundedness of $|h(t)-\int_0^t r(s)ds|$ and $|g(t)+\int_0^t r(s)ds|$ by using the method of lower and upper solutions as in [8]. At last, we characterize the fronts of spreading solutions by the semi-wave and prove Theorem 1.2.

3.1. **Time-periodic traveling semi-wave.** In this subsection we construct a traveling semi-wave which is periodic in time and is used to characterize the spreading solutions near the boundaries. Our approach is similar as that in [15]. For readers' convenience, we present the details.

Let $\mathcal{P}$ be the set of periodic functions defined as in section 1, $P(t)$ be the largest periodic solution to $u_t = f(t,u)$. First, we consider the following problem

$$\begin{gathered} v_t = v_{zz} + k(t)v_z + f(t,v), \quad t\in[0,T],\ z\in(0,l), \\ v(t,0)=0, v(t,l)=P^0 := \max_{0\le t\le T} P(t), \quad t\in[0,T], \\ v(0,z)=v(T,z), \quad z\in[0,l]. \end{gathered} \tag{3.1}$$

Lemma 3.1. *For any $k\in\mathcal{P}$ and any $l>0$, the problem (3.1) has a maximal solution $v=U_1(t,z;k,l)$, which is strictly increasing in both $z\in[0,l]$ and $k\in\mathcal{P}$, strictly decreasing in $l>0$.*

Proof. Consider the equation and the boundary condition in (3.1) with initial data $v(0,z) := P^0\cdot\chi_{[0,l]}(z)$, which is the characteristic function on the interval $[0,l]$. This initial boundary value problem has a unique solution $v(t,z;k,l)$. Using the maximum principle we see that $v(t,z;k,l)$ is strictly increasing in $z\in[0,l]$ and $k\in\mathcal{P}$, strictly decreasing in $l>0$, and $v(t,z;k,l)\le P^0$. Using the zero number argument in a similar way as in the proof of [1, Theorem 1] one can show that $||v(t,\cdot;k,l)-U_1(t,\cdot;k,l)||_{C^2([0,l])}\to 0$ as $t\to\infty$, where $U_1(t,z;k,l)\in C^{1+\gamma/2,2+\gamma}([0,T]\times[0,l])$ is a time-periodic solution of (3.1). By the maximum principle again, we see that U_1 has the same monotonic properties as v in z, k and l. □

Next, we consider the problem on the half line,

$$\begin{gathered} v_t = v_{zz} + k(t)v_z + f(t,v), \quad t\in[0,T], z>0, \\ v(t,0)=0, \quad t\in[0,T], \\ v(0,z)=v(T,z), \quad z\ge 0. \end{gathered} \tag{3.2}$$

Lemma 3.2. *For each $k\in\mathcal{P}$, problem (3.2) has a maximal bounded and nonnegative solution $U(t,z;k)$ with $U_z(t,z;k)\ge 0$ in $[0,T]\times[0,\infty)$. $U_z(t,0;k)$ is continuous in k in the sense that, for $\{k_1,k_2,\dots\}\subset\mathcal{P}$, $U_z(t,0;k_n)\to U_z(t,0;k)$ in $C^{\gamma/2}([0,T])$ if $k_n\to k$ in $C^{\gamma/2}([0,T])$.*

Assume further that $k\ge 0$. Then $U_z(t,z;k)>0$ in $[0,T]\times[0,\infty)$, $U(t,z;k)-P(t)\to 0$ as $z\to\infty$. $U_z(t,0;k)$ has a positive lower bound δ (independent of t), and it is strictly increasing in k: $U_z(t,0;k_1)<U_z(t,0;k_2)$ for $k_1,k_2\in\mathcal{P}$ satisfying $0\le k_1\le,\ne k_2$.

Proof. Let $U_1(t,z;k,l)$ be the solution of (3.1) obtained in the previous lemma. Since it is decreasing in l, by taking limit as $l\to\infty$ we see that $U_1(t,z;k,l)$

converges to some function $U(t,z;k)$, which is non-decreasing in z and in k since U_1 is so. By standard regularity argument, U is a classical solution of (3.2). The continuous dependence in k can be proved in a similar way as [4, Theorem 2.4].

By Lemma 2.1, we know that for any fixed $\beta \in (\bar{\theta}, \alpha_\varepsilon)$, there exists a unique positive solution $V(z;\beta)$ of (1.4). Since $\beta < P^0$, for any $k \geq 0$, it follows from the comparison principle for parabolic equations that

$$U_1(t,z;k,l) \geq V(z - L(\beta);\beta) \text{ for } l > \frac{L(\beta)}{2}.$$

Hence, $U(t,z;k) \geq V(z-L(\beta);\beta)$. It means that $U_z(t,0;k) \geq \delta := V'(-L(\beta);\beta) > 0$.

Using the strong maximum principle to U_z we conclude that $U_z(t,z;k) > 0$ in $[0,T)\times[0,\infty)$. Thus $P_1 := \lim_{z\to\infty} U(t,z;k)$ exists. In a similar way as in the proof of [4, Proposition 2.1] one can show that $P_1(t)$ is nothing but the maximal positive periodic solution $P(t)$ of $u_t = f(t,u)$. Since $U(t,z;k)$ is non-decreasing in k we have $U_z(t,0;k_1) \leq U_z(t,0;k_2)$ when $k_1 \leq k_2$. The strict inequality $U_z(t,0;k_1) < U_z(t,0;k_2)$ follows from the Hopf Lemma and the assumption $k_1 \leq, \neq k_2$. □

For each $k \in \mathcal{P}$, let $U(t,z;k)$ be the solutions obtained in the above lemma, denote

$$A[k](t) := \mu U_z(t,0;k),$$

where μ is the constant in the Stefan condition in (1.1). From Lemma 3.2 we see that $A[k](t)$ is non-decreasing in $k \in \mathcal{P}$. The solution of $r = A[-r]$ can lead to the traveling semi-wave as follows.

Proposition 3.3. *Assume* (H1)*. Then there exists a function $r(t) \in \mathcal{P}_+$ such that $u(t,x) = U(t,R(t)-x;-r)$ (with $R(t) := \int_0^t r(s)ds$) solves the equation* $(1.1)_1$ *for $t \in \mathbb{R}$, $x < R(t)$ and $r(t) = -\mu u_x(t,R(t)) = A[-r](t)$.*

Proof. By Lemma 3.2, for any $r \in \mathcal{P}$, the problem (3.2) with $k = -r$ has a bounded and nonnegative solution $U(t,z;-r)$, and $A[-r](t) = \mu U_z(t,0;-r)$ is non-increasing in r. When $r = 0$ we have $A[0] = \mu U_z(t,0;0) > 0$. When $r = A[0]$ we have $A[-A[0]] = \mu U_z(t,0;-A[0]) \geq 0$ and $A[-A[0]] \leq A[0]$. Set $\mathcal{R} := [0, A[0]]$, then as in the proof of [4, Theorem 2.4] one can show that the mapping $A[-\cdot]$ maps $\mathcal{R}$ continuously into a precompact set in $\mathcal{R}$. Using the Schauder fixed point theorem we see that there exists $r(t) \in \mathcal{R}$ such that $r(t) = A[-r](t)$. Clearly, $r(t) \geq 0$. Obviously, $r(t) \equiv 0$ is impossible since $A[0] > 0$. If $r(t) \geq, \neq 0$, the strong maximum principle and Hopf Lemma tells us that $U_z(t,0;-r) > 0$, so it contradicts to $r(t) = A[-r](t)$. This yields $r(t) \in P_+$. Finally, a direct calculation shows that the function $u = U(t,R(t)-x;-r)$ with $R(t) := \int_0^t r(s)ds$ solves the equation $(1.1)_1$ in $\mathbb{R} \times (-\infty, R(t))$. □

3.2. Boundedness for $|h(t)-R(t)|$ and $|g(t)+R(t)|$.

Let $U(t,R(t)-x;-r)$ be the rightward periodic traveling semi-wave with speed $r(t)$, where $R(t) := \int_0^t r(s)ds$. We show that $|h(t)-R(t)|$ and $|g(t)+R(t)|$ are both bounded for all $t \geq 0$.

Lemma 3.4. *Assume that* (H1), (H2)*. There exists $C > 0$ such that*

$$|h(t) - R(t)|,\ |g(t) + R(t)| \leq C \quad \textit{for all } t \geq 0. \tag{3.3}$$

We will show the boundedness of $|h(t)-R(t)|$ only, since the situation for $|g(t)+R(t)|$ can be proved similarly. For convenience, write $v(t,x) := \frac{u(t,x)}{P(t)}$, $\nu(t) := \mu P(t)$ to normalize the problem (1.1) into

$$\begin{gathered} v_t = v_{xx} + F(t,v), \quad t>0,\ g(t)<x<h(t), \\ v(t,h(t)) = 0, h'(t) = -\nu(t)v_x(t,h(t)), \quad t>0, \\ v(t,g(t)) = 0, g'(t) = -\nu(t)v_x(t,g(t)), \quad t>0, \\ v(0,x) = u_0(x)/P(0), \quad -h_0 \le x \le h_0, \end{gathered} \tag{3.4}$$

where $F(t,v) = \frac{1}{P(t)}[f(t,P(t)v) - f(t,P(t))v]$ satisfying $F(t,v) \in C^{\gamma/2,1}_{\rm loc}([0,T]\times\mathbb{R})$ for some $\gamma \in (0,1)$, T-periodic in t, $F(t,0) \equiv F(t,1) \equiv 0$.

Then by assumption (H2) and the fact that $P(t) > 0$, there exists $\delta > 0$ such that $F_v(t,1) = \frac{P(t)f_u(t,P(t)) - f(t,P(t))}{P(t)} < -2\delta$, so we can find some $\epsilon > 0$ such that

$$F_v(t,v) \le -\delta \text{ for } t \in [0,T],\ v \in [1-\epsilon, 1+\epsilon]. \tag{3.5}$$

Consider the solution $\xi(t)$ of ODE $\xi_t = F(t,\xi)$ with initial value $\xi(0) = M/P(0)+1$, where $M = \|u_0\|_{L^\infty([-h_0,h_0])} + 1$. Clearly, $\xi(t)$ decreases to 1 as $t \to \infty$ by the uniqueness of $P(t)$. Hence for $\epsilon > 0$ given in (3.5), one can choose a large integer m such that $1 < \xi(t) < 1+\epsilon$ and $\xi_t = F(t,\xi) \le \delta(1-\xi)$ for $t \ge mT$. So $\xi(t) \le 1 + \epsilon e^{\delta(mT-t)}$ for $t \ge mT$. Then by the comparison principle, we have

$$v(t,x) \le \xi(t) \le 1 + \epsilon e^{\delta(mT-t)} \quad \text{for } g(t) \le x \le h(t),\ t \ge mT.$$

Now we also normalize the periodic rightward semi-wave $U(t,z;-r)$ by setting $V(t,z) := \frac{U(t,z;-r)}{P(t)}$, then $V(t,z)$ satisfies

$$\begin{gathered} V_t = V_{zz} - r(t)V_z + F(t,V), \quad t \in [0,T], z > 0, \\ V(t,0) = 0, V(t,\infty) = 1, \quad t \in [0,T], \\ V(0,z) = V(T,z), V_z(t,z) > 0, \quad t \in [0,T], z > 0, \\ r(t) = \nu(t)V_z(t,0), \quad t \in [0,T]. \end{gathered} \tag{3.6}$$

Then we can find an integer $m_1 > m$ and a constant $X > 0$ large enough such that

$$(1 + M_1 e^{-\delta T_1})V(t,z) \ge 1 + \epsilon e^{\delta(mT - T_1)} \quad \text{for all } t \in [0,T],\ z \ge X. \tag{3.7}$$

where $M_1 = 2\epsilon e^{\delta mT}$, $T_1 = m_1 T$.

We construct a supersolution (v^+, g, h^+) to (3.4) as follows: let

$$\begin{gathered} h^+(t) := \int_{T_1}^t r(s)ds + h(T_1) + KM_1(e^{-\delta T_1} - e^{-\delta t}) + X, \\ v^+(t,x) := (1 + M_1 e^{-\delta t})V(t, h^+(t) - x). \end{gathered}$$

where K is a positive constant that can be chosen sufficiently large. By direct computations, one can easily check that

$$\begin{gathered} v^+_t - v^+_{xx} \ge F(t,v^+) \quad \text{for } t \ge T_1,\ g(t) < x < h^+(t), \\ v^+ \ge v \quad \text{for } t \ge T_1,\ x = g(t), \\ v^+ = 0, (h^+)'(t) > -\gamma(t)v^+(t,x) \quad \text{for } t \ge T_1,\ x = h^+(t), \\ h(T_1) \le h^+(T_1), v(T_1,x) \le v^+(T_1,x) \quad \text{for } x \in [g(T_1), h(T_1)]. \end{gathered}$$

Proposition 3.5. *For sufficiently large $K > 0$, the function $h(t)$ satisfies*

$$h(t) < R(t) + H_r \quad \text{for all } t \geq 0, \tag{3.8}$$

where $H_r := h(T_1) + X + KM_1$ and T_1, X, M_1 are defined as above.

To give the lower bounds for $h(t)$ and $v(t,x)$, we need the property that $v(t,\cdot) \to 1$ exponentially near $x = 0$.

Proposition 3.6. *For any $\delta > 0$ given above, there exists some $c > 0$ and $K_1 > 0$ such that*

$$\|u(t,\cdot) - P(t)\|_{L^\infty([-ct,ct])} \leq K_1 e^{-\delta t} \quad \text{for } t \gg 1. \tag{3.9}$$

Proof. For $c > 0$ small enough and some suitable ϵ_1 to be determined below, by simple phase plane analysis, the problems

$$\begin{gathered} q_{zz} \pm c q_z + \rho(q) = 0, \quad z \in [-L, L], \\ q(\pm L) = 0, \quad \max q(z) = \alpha_\varepsilon - \epsilon_1, \end{gathered} \tag{3.10}$$

have the solutions $q_\pm(z)$ respectively. Therefore, $w(t,x) = q_+(x - ct)$ and $w(t,x) = q_-(x + ct)$ are two compactly supported *traveling wave* solutions to the problems

$$\begin{gathered} w_t = w_{xx} + \rho(w), \quad x \in [\pm ct - L, \pm ct + L],\ t \in \mathbb{R}, \\ w(t, \pm ct + L) = 0, \quad w(t, \pm ct - L) = 0, \quad t \in \mathbb{R}, \\ \max w(t,x) = \alpha_\varepsilon - \epsilon_1, \quad t \in \mathbb{R}. \end{gathered} \tag{3.11}$$

Since spreading happens for the solution (u, g, h), there exists a large integer m_0 such that

$$u(mT, x) > \alpha_\varepsilon - \epsilon_1 \quad \text{for all } m \geq m_0,\ x \in [-cT - L, cT + L].$$

Thus, we have

$$u(mT, x) > q_\pm(x + x_0) \quad \text{for all } m \geq m_0,\ x \in [-cT - L, cT + L],\ x_0 \in [-cT, cT].$$

Then by the comparison principle,

$$u(mT + t, x) \geq q_+(x + x_0 - ct),\ q_-(x + x_0 + ct)$$

for all $t > 0$ and $x \in [-L - cT \pm ct, L + cT \pm ct]$. Using this inequality it is easily to show that

$$u(mT + t, x) \geq \alpha_\varepsilon - \epsilon_1 \quad \text{for } t > 0,\ x \in [-ct - L, ct + L].$$

Since ϵ and ϵ_1 can be chosen sufficiently small, using the same argument for normalized function v as in the proof of [6, Lemma 6.5], one can check that

$$|v(t,x) - 1| \leq k_1 e^{-\delta t} \quad \text{for } x \in [-ct, ct],\ t \geq T_2 := m_2 T,$$

where $m_2 > m_0$ is an integer and $k_1 > 0$ is a constant sufficiently large. This reduces to (3.9).

Let c, K_1 and δ be the constant as before, we define

$$\begin{gathered} g^-(t) := 0, \quad h^-(t) := \int_{T_2}^t r(s)ds + h(T_1) - K_2 K_1 (e^{-\delta T_2} - e^{-\delta t}) + cT_2, \\ v^-(t,x) := (1 - K_1 e^{-\delta t}) V(t, h^-(t) - x). \end{gathered}$$

Then for a suitable constant $K_2 > 0$, by the similar argument as in the construction of supersolution, one can show that (v^-, g^-, h^-) is a subsolution. Hence

$$h(t) \geq h^-(t) - \max_{t \in [0, T_2]} |h(t) - h^-(t)| \geq R(t) - H_l \quad \text{for all } t \geq 0,$$

where $H_l := \max_{t\in[0,T_2]} |h(t) - h^-(t)| + cT_2 + K_2K_1$. Then we obtain (3.3). $\square$

Proof of Theorem 1.2. We only prove (1.9) and (1.11), since the proof for (1.10) and (1.12) are similar.

By using changing the coordinate $y := x - R(t)$, we set

$$
\begin{gathered}
h_1(t) := h(t) - R(t), \quad g_1(t) := g(t) - R(t) \quad \text{for } t \geq 0,\\
u_1(t,y) := u(t, y + R(t)) \quad \text{for } t > 0,\ y \in [g_1(t), h_1(t)].
\end{gathered}
$$

Also for any constant $y_0 \in R$, we define $V_1(t,y) := U(t, y_0 - y; -r)$, which is a rightward periodic traveling semi-wave with speed $r(t)$. Consider the zero number of function $\eta_1(t,y) := u_1(t,y) - V_1(t,y)$ in the moving area $J(t) := [g(t), \min(y_0, h(t))]$, and denote by $Z_{J(t)}[\eta_1]$ the zero number of $\eta_1(t,\cdot)$ in the interval $J(t)$. Then the zero number argument yields that $Z_{J(t)}[\eta_1]$ is finite and decreases strictly when $h_1(t)$ gets across y_0. So $h_1(t) - y_0$ changes sign at most finite times, namely, $h_1(t) > y_0$ or $h_1(t) < y_0$ or $h_1(t) \equiv y_0$ for t large enough. Since y_0 is arbitrary, we can get that there exists a constant $H_1 \in R$ such that $\lim_{t\to\infty}[h(t) - R(t)] = H_1$. Meanwhile, by the parabolic estimate, for any $\tau > 0$,

$$
\|h'(t)\|_{C^{\frac{\gamma}{2}}([\tau,\tau+1])} \leq C,
$$

where $C > 0$ is independent of τ. Combining with the convergence of $h_1(t)$ we have $h_1'(t) \to 0$ as $t \to \infty$, that is, $\lim_{t\to\infty}[h'(t) - r(t)] = 0$.

Next, we prove (1.11). We use the variable substitution $z := x - h(t)$ to set

$$
\begin{gathered}
g_2(t) := g(t) - h(t) \quad \text{for } t \geq 0,\\
u_2(t,z) := u(t, z + h(t)) \quad \text{for } t > 0,\ z \in [g_2(t), 0].
\end{gathered}
$$

Then the rightward free boundary of u is fixed at $z = 0$ and (u_2, g_2) satisfies

$$
\begin{gathered}
u_{2t} = u_{2zz} + h'(t)u_{2z} + f(t, u_2), \quad t > 0,\ g_2(t) < z < 0,\\
u_2(t,z) = 0, g_2'(t) = -\mu u_{2z}(t,z) - h'(t), \quad t > 0,\ z = g_2(t),\\
u_2(t,0) = 0, h'(t) = -\mu u_{2z}(t,0), \quad t > 0.
\end{gathered}
\tag{3.12}
$$

By L^p theory and Soblev embedding theorem, for any constant $K > 0$, there exists a sequence m_n with $m_n \to \infty$ such that

$$
\|u_2(m_nT + t, z)\|_{C^{1+\frac{\gamma}{2},2+\gamma}([-K,K]\times[-K,0])} \leq C,
$$

where $C > 0$ is a constant independent of n. By using Cantor's diagonal argument, there is a function $w(t,z) \in C^{1+\frac{\gamma}{2},2+\gamma}(R \times (-\infty, 0])$ and a subsequence of m_n, denote again by m_n, such that

$$
\lim_{n\to\infty} \|u_2(m_nT + t, z) - w(t,z)\|_{C^{1,2}_{\mathrm{loc}}(R\times(-\infty,0])} = 0.
$$

Replacing t by $m_nT + t$ in (3.12) and taking limit as $n \to \infty$, we obtain

$$
\begin{gathered}
w_t = w_{zz} + r(t)w_z + f(t,w), \quad -\infty < z < 0,\ t \in \mathbb{R},\\
w(t,0) = 0, \quad r(t) = -\mu w_z(t,0), \quad t \in R.
\end{gathered}
$$

Set $V_2(t,z) := U(t, -z; -r)$, then $V_2(t,z) \geq w(t,z)$ by the conclusions in Subsection 3.1. Set $\eta_2(t,z) := w(t,z) - V_2(t,z) \leq 0$. It follows that $w(t,z) \equiv V_2(t,z)$. For otherwise, $z = 0$ is a degenerate zero of $\eta_2(t,\cdot)$, contradicting to the Hopf Lemma. Combining this with the arbitrary of m_n, we obtain

$$
\|u_2(t + nT, z) - V_2(t,z)\|_{L^\infty([-K,K]\times[-K,0])} \to 0 \quad \text{as } n \to \infty.
$$

Namely,

$$\|u(t+nT,x) - U(t,h(t+nT)-x;-r)\|_{L^\infty([-K,K]\times[h(t+nT)-K,h(t+nT)])} \to 0$$

as $n \to \infty$. Note that $U(t,z;-r)$ is a T-periodic function in t. then we have

$$\|u(t,\cdot) - U(t,h(t)-\cdot;-r)\|_{L^\infty([h(t)-K,h(t)])} \to 0 \quad \text{as } t \to \infty.$$

Combing this with (1.9) we obtain

$$\|u(t,\cdot) - U(t,R(t)+H_1-\cdot;-r)\|_{L^\infty([h(t)-K,h(t)])} \to 0 \quad \text{as } t \to \infty.$$

This, and (1.7), yield that (1.11) holds. The proof is complete. □

Acknowledgments. This research was supported by the NSFC (Nos. 11701374, 11671262) and by the China Postdoctoral Science Foundation (No. 2017M611593).

References

[1] P. Brunovský, P. Poláčik, B. Sandstede; Convergence in general periodic parabolic equations in one space dimension, *Nonlinear Anal.*, **18** (1992), 209-215.
[2] G. Bunting, Y. Du, K. Krakowski; Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media.*, **7** (2012), 583-603.
[3] W. Ding, Y. Du, X. Liang; Spreading in space-time-periodic media governed by a monostable equation with free boundaries, part 1: continuous initial functions, *J. Differential Equations*, **262** (2017), 4988-5021.
[4] Y. Du, Z. M. Guo, R. Peng, A diffusion logistic model with a free boundary in time-periodic environment, *J. Funct. Anal.*, **265** (2013), 2089-2142.
[5] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, **42** (2010), 377-405.
[6] Y. Du, B, Lou; Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J, Eur. Math. Soc.*, **17** (2015), 2673-2724.
[7] Y. Du, H. Matano; Convergence and sharp thresholds for propagation in nonlinear diffusion problems, *J. Eur. Math. Soc.*, **12** (2010), 279-312.
[8] Y. Du, H. Matsuzawa, M. Zhou; Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.*, **46** (2014), 375-396.
[9] Y. Du, P. Poláčik; Locally uniform convergence to an equilibrium for nonlinear parabolic equations on $\mathbb{R}^N$, *Indiana Univ. Math. J.*, **64** (2015), 787-824.
[10] A. Ducrot, T. Giletti, H. Matano; Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations, *Trans. AMS.*, **366** (2014), 5541-5566.
[11] P. Hess; Periodic-parabolic Boundary Value Problems and Positivity, *Pitman Res.,* Notes in Mathematica 247, Longman Sci. Tech. Harlow, 1991.
[12] F, Li, X. Liang, W. Shen; Diffusive KPP equations with free boundaries in time almost periodic environments: I. spreading and vanishing dichotomy, *Discrete Contin. Dyn. Syst.*, **36** (2016), 3317-3338.
[13] F, Li, X. Liang, W. Shen; Diffusive KPP equations with free boundaries in time almost periodic environments: II. spreading speeds and semi-wave solutions, *J. Differential Equations*, **261** (2016), 2403-2445.
[14] G. Nadin; Existence and uniqueness of the solution of a space-time periodic reaction-diffusion equation, *J. Differential Equations,* **249** (2010), 1288-1304.
[15] N. Sun, B. Lou, M. Zhou; A diffusive Fisher-KPP equation with free boundaries and time-periodic advection, *Calc. Var. Partial Differential Equations,* **56** (2017), Art. 61, 36 pp.

Fang Li
Mathematics & Science College, Shanghai Normal University, Shanghai 200234, China
E-mail address: lifwx@shnu.edu.cn

Junfan Lu (corresponding author)
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
E-mail address: 1410541@tongji.edu.cn