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NONLINEAR ROBIN PROBLEMS WITH UNILATERAL CONSTRAINTS AND DEPENDENCE ON THE GRADIENT

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Abstract. We consider a nonlinear Robin problem driven by the p-Laplacian, with unilateral constraints and a reaction term depending also on the gradient (convection term). Using a topological approach based on fixed point theory (the Leray-Schauder alternative principle) and approximating the original problem using the Moreau-Yosida approximations of the subdifferential term, we prove the existence of a smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear elliptic differential inclusion with Robin boundary condition

$$\begin{gathered} -\Delta_p u(z) + \partial\varphi(u(z)) \ni f(z, u(z), \nabla u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega), \quad 2 \le p < +\infty.$$

Also $\varphi \in \Gamma_0(\mathbb{R})$, the cone of proper, convex and lower semicontinuous functions (see Section 2), and $\partial\varphi(\cdot)$ denotes the subdifferential in the sense of convex analysis. The presence of the subdifferential term, incorporates in our framework problems with unilateral constraints (differential variational inequalities). The forcing term $f(z, x, y)$ is a measurable function which is locally Hölder in the $(x, y) \in \mathbb{R} \times \mathbb{R}^N$ variables. The dependence on the gradient implies that we can not use directly on (1.1) variational methods. For this reason our approach is topological based on the fixed point theory (the Leray-Schauder alternative principle). In the boundary condition, $\frac{\partial u}{\partial n_p}$ is the conormal derivative of u defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to |\nabla u|^{p-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

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By a solution of (1.1) we understand a function $u \in W^{1,p}(\Omega)$ such that there exists $g \in L^2(\Omega)$ satisfying $g(z) \in \partial\varphi(u(z))$ for a.a. $z \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh d\sigma + \int_\Omega gh\, dz = \int_\Omega f(z, u, \nabla u)h\, dz$$

for all $h \in W^{1,p}(\Omega)$.

Existence theorems for nonlinear elliptic equations with convection were proved by de Figueiredo-Girardi-Matzeu [2], Girardi-Matzeu [5] (semilinear Dirichlet problems driven by the Laplacian), Ruiz [16], Faraci-Motreanu-Puglisi [1], Huy-Quan-Khanh [7], Iturriaga-Lorca-Sánchez [8] (nonlinear Dirichlet problems driven by the p-Laplacian) and Gasiński-Papageorgiou [4], Papageorgiou-Rădulescu-Repovš [15] (Neumann problems driven by a differential operator of the form $\operatorname{div}(a(u)\nabla u)$ with $a(\cdot)$ continuous, bounded and strictly positive). Of the aforementioned works, only Papageorgiou-Rădulescu-Repovš [15] consider problems with unilateral constraint (that is, with a subdifferential term $\partial\varphi(u)$). Their conditions on the convection term are different and they employ a suitable variant of the classical Nagumo-Hartman condition. Their differential operator is of the form $\operatorname{div}(a(u)\nabla u)$ with $a : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous and $0 < c_1 \le a(x) \le c_2$ for all $x \in \mathbb{R}$. This particular form of the differential operator is essential for their proofs to work and their method can not accommodate a nonlinear differential operator like the p-Laplacian.

The presence of the subdifferential term $\partial\varphi(u)$ introduces a multivalued unilateral constraint in the problem which complicates the study of (1.1). Our aim is to prove an existence theorem for problem (1.1). In fact we show the existence of a smooth solution for problem. The method of proof passes through a regularization of the subdifferential map. The regularization is based on the Moreau-Yosida approximation of φ. Exploiting the properties of the Moreau-Yosida approximation we solve the regularized problem, using topological tools based on the fixed point theory. Using the nonlinear regularity theory, we derive uniform a priori bounds for the solutions of the approximate problems and then we pass to the limit to obtain the desired solution of (1.1).

2. Mathematical background and hypotheses

Let X be a reflexive Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the dual pair (X^*, X). A map $A : D(A) \subseteq X \to 2^{X^*}$ is said to be "monotone", if

$$\langle x^* - u^*, x - u \rangle \ge 0 \quad \text{for all } (x, x^*),\ (u, u^*) \in \operatorname{Gr} A.$$

Here $\operatorname{Gr} A$ is the "graph of $A(\cdot)$" defined by

$$\operatorname{Gr} A = \{(x, x^*) \in X \times X^* : x^* \in A(x)\}$$

and $D(A)$ is the "domain of $A(\cdot)$" defined by

$$D(A) = \{x \in X : A(x) \ne \emptyset\}.$$

We say that $A(\cdot)$ is "strictly monotone", if it is monotone and

$$\langle x^* - u^*, x - u \rangle = 0 \ \Rightarrow\ x = u.$$

A monotone map is "maximal monotone", if its graph is maximal among the graphs of monotone maps. This means that

$$\langle u^* - x^*, u - x \rangle \ge 0 \quad \text{for all } (x, x^*) \in \operatorname{Gr} A \ \Rightarrow\ (u, u^*) \in \operatorname{Gr} A.$$

The importance of maximal monotone maps, comes from their remarkable surjectivity properties. More precisely we have (see, for example, Gasiński-Papageorgiou [3, p. 319]).

Proposition 2.1. *If* $A{:}X \to 2^{X^*}$ *is maximal monotone and coercive (that is,* $\|A(x)\|_* \to +\infty$ *as* $\|x\| \to +\infty$*), then* A *is surjective.*

By $\Gamma_0(X)$ we denote the cone of all functions $\varphi{:}X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are convex, lower semicontinuous and proper (that is, the effective domain of φ, $\operatorname{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}$ is nonempty).

For $\varphi \in \Gamma_0(X)$, the subdifferential of φ at $x \in X$, is defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x+h) - \varphi(x) \text{ for all } h \in X\}.$$

Note that $\partial\varphi(x) \subseteq X^*$ is w-closed, convex and possibly empty. If $\varphi(\cdot)$ is continuous at $x \in X$, then $\partial\varphi(x)$ is nonempty, w-compact and convex. If $\varphi(\cdot)$ is Gâteaux differentiable at $x \in X$, then

$$\partial\varphi(x) = \{\varphi'_G(x)\}$$

with $\varphi'_G(x)$ being the Gâteaux derivative of φ at x. The map $\partial\varphi{:}X \to 2^{X^*}$ is maximal monotone.

If $X = H =$ a Hilbert space and $\varphi \in \Gamma_0(H)$, then for every $\lambda > 0$, the "Moreau-Yosida approximation" φ_λ of φ, is defined by

$$\varphi_\lambda(x) = \inf[\varphi(u) + \frac{1}{2\lambda}\|x-u\|^2 : u \in H] \quad \text{for all } x \in H.$$

This functional has the following properties:

- φ_λ is convex and $\operatorname{dom}\varphi_\lambda = H$.
- φ_λ is Fréchet differentiable and the Fréchet derivative φ'_λ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$.
- If $\lambda_n \to 0^+$, $x_n \to x$ in H, $\varphi'_{\lambda_n}(u_n) \xrightarrow{w} x^*$ in H^*, then $x^* \in \partial\varphi(x)$.

Let V and Z be Banach spaces. We say that $g : V \to Z$ is "compact" if it is continuous and maps bounded sets in V to relatively compact sets in Z.

As we already mentioned, our approach is topological and will make use of the "Leray-Schauder alternative principle" (see Gasiński-Papageorgiou [3, p. 827]).

Theorem 2.2. *If* V *is a Banach space,* $\varphi : V \to V$ *is a compact map and*

$$T(\varphi) = \{x \in V : \textit{ there exists } t \in (0,1) \textit{ such that } x = t\varphi(x)\},$$

then either $T(\varphi)$ *is unbounded or* φ *has a fixed point.*

The main space of our analysis is the Sobolev space $W^{1,p}(\Omega)$. Endowed with the norm

$$\|u\| = [\|u\|_p^p + \|\nabla u\|_p^p]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega),$$

this Sobolev space becomes a separable reflexive Banach space. The nonlinear regularity theory will bring into play the Banach space $C^1(\overline{\Omega})$ and the Robin boundary condition the "boundary" Lebesgue spaces $L^r(\partial\Omega)$, $1 \le r \le +\infty$. To define the latter, on $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure denoted by $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^r(\partial\Omega)$. From the theory of Sobolev spaces, we know that there exists a unique linear continuous map $\gamma_0{:}W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Hence, we understand the trace map as representing the boundary values of an arbitrary Sobolev function $u \in W^{1,p}(\Omega)$. We know that γ_0 is a compact operator into $L^r(\partial\Omega)$ for $1 \le r < \frac{p(N-1)}{N-p}$ when $N > p$ and into $L^r(\partial\Omega)$ for $1 \le r < +\infty$, when $N \le p$. Moreover,

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \big(\frac{1}{p} + \frac{1}{p'} = 1\big) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces. We introduce the following condition on the boundary coefficient $\beta(\cdot)$:

(H1) $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

Note that if $\beta \equiv 0$, then we recover the Neumann problem. We consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of the Robin p-Laplacian, if problem (2.1) admits a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$, known as an "eigenfunction" corresponding to the eigenvalue $\widehat{\lambda}$. We know that (2.1) admits a smallest eigenvalue $\widehat{\lambda}_1$ which has the following properties (see Papageorgiou-Rădulescu [13]):

- $\widehat{\lambda}_1 \ge 0$ (in fact, if $\beta \equiv 0$ (Neumann problem), then $\widehat{\lambda}_1 = 0$, while if $\beta \not\equiv 0$ then $\widehat{\lambda}_1 > 0$).
- $\widehat{\lambda}_1$ is isolated in the spectrum $\widehat{\sigma}(p)$ of the Robin p-Laplacian (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).
- If $\gamma(u) = \|\nabla u\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$, then

$$\widehat{\lambda}_1 = \inf\big[\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \ne 0\big]. \tag{2.2}$$

The infimum in (2.2) is realized on the corresponding one dimensional eigenspace. It is clear from the above properties that the elements of this eigenspace do not change sign. By $\widehat{u}_1$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$), eigenfunction corresponding to $\widehat{\lambda}_1$. The nonlinear regularity theory of Lieberman [9] implies that $\widehat{u}_1 \in C^1(\overline{\Omega})$. Moreover, the nonlinear strong maximum principle (see Gasiński-Papageorgiou [3, p. 738]), implies that $\widehat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$. Using these properties one can prove easily the following lemma (see Mugnai-Papageorgiou [12, Lemma 4.11]).

Lemma 2.3. *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \le \widehat{\lambda}_1$ for a.a. $z \in \Omega$ and this inequality is strict on a set of positive measure, then there exists $\widehat{c} > 0$ such that*

$$\widehat{c}\|u\|^p \le \gamma(u) - \int_\Omega \vartheta(z)|u|^p\,dz \quad \textit{for all } u \in W^{1,p}(\Omega).$$

We say that a function $f:\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is an L^∞-locally Hölder function, if for all $\rho > 0$, there exists $\eta_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x,y) - f(z,x',y')| \leq \eta_\rho(z)\,[|x-x'|^\mu + |y-y'|^\mu]$$

for a.a. $z \in \Omega$, all $|x|$, $|x'|$, $|y|$, $|y'| \leq \rho$, with $0 < \mu \leq 1$.

Our hypotheses on the reaction term $f(z,x,y)$ are:

(H2) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is an L^∞-locally Hölder function such that

- (i) $|f(z,x,y)| \leq a(z)[1 + |x|^{p-1} + |y|^q]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, $y \in \mathbb{R}^N$, with $2 \leq p$,
$$2(p-1) \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$$
and $q = \max\{\frac{p-1}{2}, 1\}$;
- (ii) there exists a function $\vartheta \in L^\infty(\Omega)$ such that $\vartheta(z) \leq \widehat{\lambda}_1$ for a.a. $z \in \Omega$, the above inequality is strict on a set of positive measure, $\limsup_{x\to\pm\infty} \frac{f(z,x,y)}{|x|^{p-2}x} \leq \vartheta(z)$ uniformly for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ on a bounded set.

Note that if $p \geq \frac{N}{2}$, then $2(p-1) \leq p^*$ (see hypothesis (H2)(i)). The following function satisfies hypotheses (H2) (for the sake of simplicity, we drop the z-dependence),

$$f(x,y) = \vartheta|x|^{p-2}x + g(x)|y|^q \quad \text{for all } x \in \mathbb{R},\ y \in \mathbb{R}^N,$$

with $\vartheta < \widehat{\lambda}_1$, $2 \leq p < +\infty$ with $2(p-1) \leq p^*$, $q = \max\{\frac{p-1}{2}, 1\}$ and $g : \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous and $\lim_{x\to\pm\infty} \frac{g(x)}{|x|^{p-2}x} = 0$.

The hypotheses on the function φ are:

(H3) $\varphi \in \Gamma_0(\mathbb{R})$ with $0 \in \partial\varphi(0)$.

These conditions mean that $\varphi \geq 0$ and $\varphi(0) = \inf \varphi$.

3. Existence of solutions

Recall that from hypothesis (H2)(ii), $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leq \widehat{\lambda}_1$ for a.a. $z \in \Omega$, $\vartheta \not\equiv \widehat{\lambda}_1$. Let $\lambda > 0$, $g \in L^\infty(\Omega)$ and $\widehat{\xi} > \|\vartheta\|_\infty$. We consider the auxiliary nonlinear Robin problem

$$\begin{gathered} -\Delta_p u(z) + \widehat{\xi}|u(z)|^{p-2}u(z) + \partial\varphi_\lambda(u(z)) = g(z) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega,\ \lambda > 0. \end{gathered} \tag{3.1}$$

Proposition 3.1. *If hypotheses* (H1) *and* (H3) *hold, then problem* (3.1) *has a unique solution $S_\lambda(g) \in C^1(\overline{\Omega})$ and the solution map $S_\lambda : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is compact.*

Proof. First we show the existence of a solution for problem (3.1). So, we consider the operator $E : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\begin{aligned} \langle E(u), h\rangle &= \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\,dz + \widehat{\xi}\int_\Omega |u|^{p-2}uh\,dz \\ &\quad + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh d\sigma + \int_\Omega \partial\varphi_\lambda(u)h\,dz \quad \text{for all } u,h \in W^{1,p}(\Omega). \end{aligned}$$

Clearly $E(\cdot)$ is continuous, monotone, thus maximal monotone (see [3, p. 310]). Also, for all $u \in W^{1,p}(\Omega)$ we have

$$\langle E(u), u\rangle \geq \|\nabla u\|_p^p + \widehat{\xi}\|u\|_p^p \geq c_0\|u\|^p \text{ with } c_0 = \min\{\widehat{\xi}, 1\}$$
$$\Rightarrow E(\cdot) \text{ is coercive;}$$

see hypothesis (H1) and recall that $\partial\varphi_\lambda$ is monotone, $\partial\varphi_\lambda(0) = 0$).

Invoking Proposition 2.1, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&E(\widehat{u}_\lambda) = g\\
&\Rightarrow \int_\Omega |\nabla \widehat{u}_\lambda|^{p-2}(\nabla \widehat{u}_\lambda, \nabla h)_{\mathbb{R}^N}\,dz + \widehat{\xi}\int_\Omega |\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda h d\sigma\\
&\quad + \int_\Omega \partial\varphi_\lambda(\widehat{u}_\lambda)h\,dz = \int_\Omega gh\,dz \quad \text{for all } h \in W^{1,p}(\Omega), \qquad (3.2)\\
&\Rightarrow \begin{cases} -\Delta_p \widehat{u}_\lambda(z) + \widehat{\xi}|\widehat{u}_\lambda(z)|^{p-2}\widehat{u}_\lambda(z) + \partial\varphi_\lambda(\widehat{u}_\lambda(z)) = g(z) & \text{for a.a. } z \in \Omega,\\ \frac{\partial \widehat{u}_\lambda}{\partial n_p} + \beta(z)|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda = 0 & \text{on } \partial\Omega; \end{cases} \qquad (3.3)
\end{aligned}$$

see Papageorgiou-Rădulescu [13].

We know that $|\partial\varphi_\lambda(x)| \leq \frac{1}{\lambda}|x|$ for all $x \in \mathbb{R}$ (see Hu-Papageorgiou [6, p. 350]). Hence from (3.3) and Papageorgiou-Rădulescu [14], we have $u_\lambda \in L^\infty(\Omega)$. Invoking Lieberman [9, Theorem 2], we infer that

$$u_\lambda \in C^1(\overline{\Omega}).$$

So, we have proved the existence of a smooth solution for problem (3.1). Next we show the uniqueness of this solution. So, suppose that $\widehat{v}_\lambda \in W^{1,p}(\Omega)$ is another solution. Again we have $\widehat{v}_\lambda \in C^1(\overline{\Omega})$ and

$$\begin{aligned}
&\int_\Omega |\nabla \widehat{v}_\lambda|^{p-2}(\nabla \widehat{v}_\lambda, \nabla h)_{\mathbb{R}^N}\,dz + \widehat{\xi}\int_\Omega |\widehat{v}_\lambda|^{p-2}\widehat{v}_\lambda h\,dz\\
&+ \int_{\partial\Omega} \beta(z)|\widehat{v}_\lambda|^{p-2}\widehat{v}_\lambda h d\sigma + \int_\Omega \partial\varphi_\lambda(\widehat{v}_\lambda)h\,dz \qquad (3.4)\\
&= \int_\Omega gh\,dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}$$

In both (3.4) and (3.2) we choose $h = \widehat{u}_\lambda - \widehat{v}_\lambda \in W^{1,p}(\Omega)$ and then subtract (3.4) from (3.2). We obtain

$$\begin{aligned}
&\int_\Omega (|\nabla \widehat{u}_\lambda|^{p-2}\nabla \widehat{u}_\lambda - |\nabla \widehat{v}_\lambda|^{p-2}\nabla \widehat{v}_\lambda, \nabla \widehat{u}_\lambda - \nabla \widehat{v}_\lambda)_{\mathbb{R}^N}\,dz\\
&+ \widehat{\xi}\int_\Omega (|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda - |\widehat{v}_\lambda|^{p-2}\widehat{v}_\lambda)(\widehat{u}_\lambda - \widehat{v}_\lambda)\,dz\\
&+ \int_{\partial\Omega} \beta(z)[|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda - |\widehat{v}_\lambda|^{p-2}\widehat{v}_\lambda](\widehat{u}_\lambda - \widehat{v}_\lambda)d\sigma\\
&+ \int_\Omega (\partial\varphi_\lambda(\widehat{u}_\lambda) - \partial\varphi_\lambda(\widehat{v}_\lambda))(\widehat{u}_\lambda - \widehat{v}_\lambda)\,dz = 0.
\end{aligned}$$

Recalling that $\mathbb{R}^N \ni y \to |y|^{p-2}y$ and $\mathbb{R} \ni x \to \partial\varphi_\lambda(x)$ are monotone and since $\beta \geq 0$ (see hypothesis (H1)), we obtain

$$\widehat{\xi}\int_\Omega (|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda - |\widehat{v}_\lambda|^{p-2}\widehat{v}_\lambda)(\widehat{u}_\lambda - \widehat{v}_\lambda)\,dz \leq 0 \;\Rightarrow\; \widehat{u}_\lambda = \widehat{v}_\lambda,$$

since $\mathbb{R} \ni x \to |x|^{p-2}x$ is strictly monotone. This proves the uniqueness of the solution of (3.1). Therefore the solution map $S_\lambda : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is well-defined. We show that $S_\lambda(\cdot)$ is continuous. To this end, let $g_n \to g$ in $L^\infty(\Omega)$ and set $u_n = S_\lambda(g_n)$ for all $n \in \mathbb{N}$ and $u = S_\lambda(g)$. We have

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}(\nabla u_n, \nabla h)_{\mathbb{R}^N}\,dz + \widehat{\xi}\int_\Omega |u_n|^{p-2}u_n h\,dz \\
&+ \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h d\sigma + \int_\Omega \partial\varphi_\lambda(u_n)h\,dz \\
&= \int_\Omega g_n h\,dz \quad \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}.
\end{aligned} \tag{3.5}$$

In (3.5) we choose $h = u_n \in W^{1,p}(\Omega)$. Using (H1), the monotonicity of $\partial\varphi_\lambda(\cdot)$ and the fact that $\partial\varphi_\lambda(0) = 0$, we obtain

$$\begin{aligned}
&\|\nabla u_n\|_p^p + \widehat{\xi}\|u_n\|_p^p \le c_1\|u_n\| \quad \text{for some } c_1 > 0, \text{ all } n \in \mathbb{N}, \\
&\Rightarrow\ \{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\end{aligned} \tag{3.6}$$

For every $n \in \mathbb{N}$, we have

$$\begin{gathered}
-\Delta_p u_n(z) + \widehat{\xi}|u_n(z)|^{p-2}u_n(z) + \partial\varphi_\lambda(u_n(z)) = g_n(z) \quad \text{for a.a. } z \in \Omega, \\
\frac{\partial u_n}{\partial n_p} + \beta(z)|u_n|^{p-2}u_n = 0 \quad \text{on } \partial\Omega;
\end{gathered} \tag{3.7}$$

see (3.5) and Papageorgiou-Rădulescu [13].

From (3.6), (3.7) and Papageorgiou-Rădulescu [14, Proposition 7] we infer that there exists $c_2 > 0$ such that

$$\|u_n\|_\infty \le c_2 \quad \text{for all } n \in \mathbb{N}.$$

Then Lieberman [9, Theorem 2] implies that there exist $\tau \in (0,1)$ and $c_3 > 0$ such that

$$u_n \in C^{1,\tau}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\tau}(\overline{\Omega})} \le c_3 \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

The space $C^{1,\tau}(\overline{\Omega})$ is embedded compactly in $C^1(\overline{\Omega})$. So, from (3.8) and by passing to a subsequence if necessary, we may assume that

$$u_n \to \widehat{u} \text{ in } C^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.9}$$

Passing to the limit as $n \to +\infty$ in (3.5) and using (3.9), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\widehat{u}|^{p-2}(\nabla\widehat{u}, \nabla h)_{\mathbb{R}^N}\,dz + \widehat{\xi}\int_\Omega |\widehat{u}|^{p-2}\widehat{u}h\,dz \\
&+ \int_{\partial\Omega} \beta(z)|\widehat{u}|^{p-2}\widehat{u}h d\sigma + \int_\Omega \partial\varphi_\lambda(\widehat{u})h\,dz \\
&= \int_\Omega gh\,dz \quad \text{for all } h \in W^{1,p}(\Omega), \\
&\Rightarrow\ \widehat{u} = S_\lambda(g) = u.
\end{aligned}$$

So, for the original sequence we have that $u_n \to u$ in $C^1(\overline{\Omega})$ which implies $S_\lambda(\cdot)$ is continuous.

To show the compactness of $S_\lambda(\cdot)$ we need to show also that it maps bounded sets in $L^\infty(\Omega)$ into relatively compact sets in $C^1(\overline{\Omega})$. So, let $B \subseteq L^\infty(\Omega)$ be bounded.

Then as above, using Papageorgiou-Rădulescu [14, Proposition 7], we have that

$$S_\lambda(B) \subseteq L^\infty(\Omega) \text{ is bounded.}$$

Then Lieberman [9, Theorem 2] implies that there exist $\eta \in (0,1)$ and $c_4 > 0$ such that

$$u \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u\|_{C^{1,\eta}(\overline{\Omega})} \le c_4 \quad \text{for all } u \in S_\lambda(B).$$

Exploiting the compact embedding of $C^{1,\eta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we obtain that

$$\begin{aligned}
&\{u : u \in S_\lambda(B)\} \subseteq C^1(\overline{\Omega}) \text{ is relatively compact,}\\
&\Rightarrow\ S_\lambda : L^\infty(\Omega) \to C^1(\overline{\Omega}) \text{ is a compact map.}
\end{aligned}$$

□

Let $\widehat{N} : C^1(\overline{\Omega}) \to L^\infty(\Omega)$ be defined by

$$\widehat{N}(u) = N_f(u) + \widehat{\xi}|u|^{p-2}u \quad \text{for all } u \in C^1(\overline{\Omega})$$

with $N_f(\cdot)$ being the Nemitsky map corresponding to f and defined by

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in C^1(\overline{\Omega}).$$

We have the following result concerning this map.

Proposition 3.2. *If hypothesis* (H2)(i) *holds, then* $\widehat{N} : C^1(\overline{\Omega}) \to L^\infty(\Omega)$ *is continuous and bounded (that is, maps bounded sets to bounded sets).*

Proof. It is clear from hypothesis (H2)(i), that $\widehat{N}(\cdot)$ is well-defined, namely it maps $C^1(\overline{\Omega})$ into $L^\infty(\Omega)$. Suppose that $u_n \to u$ in $C^1(\overline{\Omega})$. Then we can find $\rho > 0$ such that $\|u_n\|_{C^1(\overline{\Omega})} \le \rho$.

Since $f(z,\cdot,\cdot)$ is L^∞-locally Hölder continuous, we can find $\mu \in (0,1]$ and $c_5 > 0$ such that

$$\begin{aligned}
&|f(z, u_n(z), \nabla u_n(z)) - f(z, u(z), \nabla u(z))|\\
&\le c_5[|u_n(z) - u(z)|^\mu + |\nabla u_n(z) - \nabla u(z)|^\mu] \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N},\\
&\Rightarrow\ \widehat{N}(u_n) \to \widehat{N}(u) \quad \text{in } L^\infty(\Omega),\\
&\Rightarrow\ \widehat{N}(\cdot) \text{ is continuous.}
\end{aligned}$$

From (H2)(i) it is clear that $\widehat{N}(\cdot)$ is bounded. □

On account of Propositions 3.1 and 3.2, the map $S_\lambda \circ \widehat{N} : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$, $\lambda > 0$, is compact. We set

$$T_\lambda = \Big\{u \in C^1(\overline{\Omega}) : u = tS_\lambda(\widehat{N}(u)),\ 0 < t < 1\Big\}.$$

Proposition 3.3. *If hypotheses* (H1)–(H3) *hold, then for every* $\lambda > 0$, $T_\lambda \subseteq C^1(\overline{\Omega})$ *is bounded.*

Proof. Let $u \in T_\lambda$. We have

$$
\begin{aligned}
& u = t(S_\lambda \circ \widehat{N})(u), \\
& \Rightarrow \ \frac{1}{t} u = S_\lambda(\widehat{N}(u)), \\
& \Rightarrow \ \frac{1}{t^{p-1}} \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla h)_{\mathbb{R}^N}\, dz + \frac{\widehat{\xi}}{t^{p-1}} \int_\Omega |u|^{p-2} u h\, dz \\
& \quad + \frac{1}{t^{p-1}} \int_{\partial\Omega} \beta(z)|u|^{p-2} u h d\sigma + \int_\Omega \partial\varphi_\lambda\big(\frac{1}{t}u\big) h\, dz \\
& \quad = \int_\Omega [f(z,u,\nabla u) + \widehat{\xi}|u|^{p-2}u] h\, dz \quad \text{for all } h \in W^{1,p}(\Omega) \\
& \Rightarrow \ \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz + \widehat{\xi} \int_\Omega |u|^{p-2} u h\, dz \\
& \quad + \int_{\partial\Omega} \beta(z)|u|^{p-2} u h d\sigma + t^{p-1} \int_\Omega \partial\varphi_\lambda\big(\frac{1}{t}u\big) h\, dz \\
& \quad = t^{p-1} \int_\Omega [f(z,u,\nabla u) + \widehat{\xi}|u|^{p-2}u] h\, dz \quad \text{for all } h \in W^{1,p}(\Omega)\,.
\end{aligned}
\tag{3.10}
$$

Hypothesis (H2)(ii) implies that given $\varepsilon, \eta > 0$, we can find $M > 0$ such that

$$
f(z,x,y)x \le [\vartheta(z) + \varepsilon]|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M, \text{ all } |y| \le \eta. \tag{3.11}
$$

On the other hand, from hypothesis (H2)(i) we see that we can find $c_6 > 0$ such that

$$
f(z,x,y)x \le c_6(1 + |y|^q) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| < M, \text{ all } y \in \mathbb{R}^N. \tag{3.12}
$$

Since $\vartheta \in L^\infty(\Omega)$, combining (3.11) and (3.12), we obtain

$$
f(z,x,y)x \le [\vartheta(z) + \varepsilon]|x|^p + c_7(1 + |y|^q) \tag{3.13}
$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$, some $c_7 > 0$.

In (3.10) we choose $h = u \in W^{1,p}(\Omega)$ and use the fact that $\partial\varphi_\lambda(x)x \ge 0$ for all $x \in \mathbb{R}$ (recall that $\partial\varphi_\lambda(\cdot)$ is monotone and $\partial\varphi_\lambda(0) = 0$). We obtain

$$
\begin{aligned}
& \|\nabla u\|_p^p + \widehat{\xi}\|u\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \\
& \le t^{p-1} \int_\Omega [f(z,u,\nabla u)u + \widehat{\xi}|u|^p]\, dz \\
& \le t^{p-1} \int_\Omega [(\vartheta(z) + \varepsilon)|u|^p + c_7(1 + |\nabla u|^q) + \widehat{\xi}|u|^p]\, dz \quad \text{(see (3.13))} \\
& \le \int_\Omega [(\vartheta(z) + \varepsilon)|u|^p + c_7(1 + |\nabla u|^q) + \widehat{\xi}|u|^p]\, dz \quad (\text{recall that } \widehat{\xi} > \|\vartheta\|_\infty), \\
& \Rightarrow \ \|\nabla u\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \vartheta(z)|u|^p\, dz - \varepsilon\|u\|^p \le c_8(1 + \|u\|^q) \\
& \quad \text{for some } c_8 > 0, \\
& \Rightarrow \ (\widehat{c} - \varepsilon)\|u\|^p \le c_8(1 + \|u\|^q) \quad \text{(see Lemma 2.3)}.
\end{aligned}
$$

By choosing $\varepsilon \in (0, \widehat{c})$ we have that

$$
\|u\|^p \le c_9(1 + \|u\|^q) \quad \text{for some } c_9 > 0.
$$

Since $q < p$ (see hypothesis (H2)(i)), we conclude that

$$T_\lambda \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.14}$$

For every $u \in T_\lambda$ we have

$$\begin{aligned} &-\Delta_p u(z) + \widehat{\xi}|u(z)|^{p-2}u(z) \\ &= t^{p-1}[f(z,u(z),\nabla u(z)) + \widehat{\xi}|u(z)|^{p-2}u(z)] - t^{p-1}\partial\varphi_\lambda\big(\frac{1}{t}u(z)\big) \\ &\quad \text{for a.a. } z \in \Omega, \\ &\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.15}$$

Note that

$$t^{p-1}\big|\partial\varphi_\lambda\big(\frac{1}{t}u(z)\big)\big| \le t^{p-2}|u(z)| \le |u(z)| \quad \text{(recall that } 2 \le p,\ 0 < t < 1). \tag{3.16}$$

Then (3.14), (3.15), (3.16) imply that there exists $c_{10} > 0$ such that

$$\|u\|_\infty \le c_{10} \quad \text{for all } u \in T_\lambda \text{ (see [14]).}$$

Invoking Lieberman [9, Theorem 2], we can find $s \in (0,1)$ and $c_{11} > 0$ such that

$$\begin{aligned} &u \in C^{1,s}(\overline{\Omega}) \quad \text{and} \quad \|u\|_{C^{1,s}(\overline{\Omega})} \le c_{11} \quad \text{for all } u \in T_\lambda, \\ &\Rightarrow\ T_\lambda \subseteq C^1(\overline{\Omega}) \quad \text{is bounded.} \end{aligned}$$

□

We consider the following approximation of problem (1.1):

$$\begin{gathered} -\Delta_p u(z) + \partial\varphi_\lambda(u(z)) = f(z,u(z),\nabla u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega,\ \lambda > 0. \end{gathered} \tag{3.17}$$

On account of Propositions 3.2 and 3.3, we can use Theorem 2.2 (the Leray-Schauder alternative principle) and have the following existence result for problem 3.17.

Proposition 3.4. *If hypotheses (H1)–(H3) hold, then for every $\lambda > 0$ problem 3.17 admits a solution $u_\lambda \in C^1(\overline{\Omega})$.*

Finally we pass to the limit as $\lambda \to 0^+$ to produce a solution of the original problem (1.1). So, we can state the following existence theorem for problem (1.1).

Theorem 3.5. *If hypotheses* (H1)–(H3) *hold, then problem* (1.1) *admits a solution $\widehat{u} \in C^1(\overline{\Omega})$.*

Proof. Let $\lambda_n \to 0^+$ and let $u_n = u_{\lambda_n} \in C^1(\overline{\Omega})$ be a solution of problem $(3.17)_n$ (see Proposition 3.4). We have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}(\nabla u_n, \nabla h)_{\mathbb{R}^N}\,dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\,d\sigma + \int_\Omega \partial\varphi_{\lambda_n}(u_n)h\,dz \\ &= \int_\Omega f(z,u_n,\nabla u_n)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.18}$$

In (3.18) we choose $h = u_n \in W^{1,p}(\Omega)$ and have

$$\|\nabla u_n\|_p^p + \int_{\partial\Omega} \beta(z)|u_n|^p d\sigma$$

$$
\begin{aligned}
&\le \int_\Omega f(z,u_n,\nabla u_n)u_n\,dz \\
&\le \int_\Omega [(\vartheta(z)+\varepsilon)|u_n|^p + c_7(1+|\nabla u_n|^q)]\,dz \quad \text{for all } n\in\mathbb{N} \text{ (see (3.13))},\\
&\Rightarrow\ \|\nabla u_n\|_p^p + \int_{\partial\Omega}\beta(z)|u_n|^p d\sigma - \int_\Omega \vartheta(z)|u_n|^p\,dz - \varepsilon\|u_n\|^p \le c_{12}(1+\|u_n\|^q)\\
&\quad \text{for some } c_{12}>0, \text{ all } n\in\mathbb{N},\\
&\Rightarrow\ (\widehat{c}-\varepsilon)\|u_n\|^p \le c_{12}(1+\|u_n\|^q) \quad \text{for all } n\in\mathbb{N} \text{ (see Lemma 2.3)}.
\end{aligned}
$$

Choose $\varepsilon\in(0,\widehat{c})$ and recall that $q<p$. We infer that

$$
\{u_n\}_{n\ge1}\subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.19}
$$

So, we may assume that

$$
u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n\to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.20}
$$

Since $\partial\varphi_{\lambda_n}(\cdot)$ is Lipschitz continuous, from Marcus-Mizel [10] (see also Gasiński-Papageorgiou [3, Proposition 2.4.24, p. 194], we have

$$
\partial\varphi_{\lambda_n}(u_n(\cdot))\in W^{1,p}(\Omega) \quad \text{for all } n\in\mathbb{N}.
$$

Hence we can choose $h=\partial\varphi_{\lambda_n}(u_n)\in W^{1,p}(\Omega)$ in (3.18). We obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}(\nabla u_n,\nabla(\partial\varphi_{\lambda_n}(u_n)))_{\mathbb{R}^N}\,dz\\
&+\int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n\partial\varphi_{\lambda_n}(u_n)d\sigma + \|\partial\varphi_{\lambda_n}(u_n)\|_2^2\\
&=\int_\Omega f(z,u_n,\nabla u_n)\partial\varphi_{\lambda_n}(u_n)\,dz \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{3.21}
$$

From the chain rule for Sobolev functions of Marcus-Mizel [10] (see also Gasiński-Papageorgiou [3, p. 194]), we have

$$
\nabla(\partial\varphi_{\lambda_n}(u_n)) = (\partial\varphi_{\lambda_n})'(u_n)\nabla u_n \quad \text{for all } n\in\mathbb{N}.
$$

Hence

$$
\int_\Omega |\nabla u_n|^{p-2}(\nabla u_n,\nabla(\partial\varphi_{\lambda_n}(u_n)))_{\mathbb{R}^N}\,dz = \int_\Omega |\nabla u_n|^p(\partial\varphi_{\lambda_n})'(u_n)\,dz \tag{3.22}
$$

for all $n\in\mathbb{N}$. Since $\partial\varphi_{\lambda_n}(\cdot)$ is nondecreasing, we have $(\partial\varphi_{\lambda_n})'(u_n)\ge0$. Therefore

$$
0\le \int_\Omega |\nabla u_n|^{p-2}(\nabla u_n,\nabla(\partial\varphi_{\lambda_n}(u_n)))_{\mathbb{R}^N}\,dz \quad \text{for all } n\in\mathbb{N} \text{ (see (3.22))}. \tag{3.23}
$$

Recall that $\partial\varphi_{\lambda_n}(u_n)u_n\ge0$ for all $n\in\mathbb{N}$. So

$$
0\le \int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n\partial\varphi_{\lambda_n}(u_n)d\sigma \quad \text{for all } n\in\mathbb{N} \text{ (see hypothesis (H1))}. \tag{3.24}
$$

We return to (3.21) and use (3.23) and (3.24). We obtain

$$
\|\partial\varphi_{\lambda_n}(u_n)\|_2^2 \le \int_\Omega f(z,u_n,\nabla u_n)\partial\varphi_{\lambda_n}(u_n)\,dz \quad \text{for all } n\in\mathbb{N}. \tag{3.25}
$$

From hypothesis (H2)(i) we have

$$
\begin{aligned}
&|f(z,u_n(z),\nabla u_n(z))| \le a(z)[1+|u_n(z)|^{p-1}+|\nabla u_n(z)|^q] \quad \text{for a.a. } z\in\Omega,\\
&\Rightarrow\ |f(z,u_n(z),\nabla u_n(z))|^2 \le c_{13}[1+|u_n(z)|^{2(p-1)}+|\nabla u_n(z)|^{2q}]\\
&\quad \text{for some } c_{13}>0, \text{ a.a. } z\in\Omega, \text{ all } n\in\mathbb{N},\\
&\Rightarrow\ N_f(u_n)\in L^2(\Omega) \text{ and } \|N_f(u_n)\|_2 \le c_{14} \text{ for some } c_{14}>0, \text{ all } n\in\mathbb{N};
\end{aligned}
\tag{3.26}
$$

see (3.19) and recall that $2(p-1)\le p^*$, $2q=\max\{p-1,2\}$. Then from (3.25) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\|\partial\varphi_{\lambda_n}(u_n)\|_2^2 \le c_{14}\|\partial\varphi_{\lambda_n}(u_n)\|_2 \quad \text{for all } n\in\mathbb{N} \text{ (see (3.26))}\\
&\Rightarrow\ \{\partial\varphi_{\lambda_n}(u_n)\}_{n\ge1}\subseteq L^2(\Omega) \text{ is bounded.}
\end{aligned}
\tag{3.27}
$$

We may assume that

$$
\partial\varphi_{\lambda_n}(u_n) \xrightarrow{w} e \text{ in } L^2(\Omega) \quad \text{as } n\to+\infty. \tag{3.28}
$$

In (3.18) we choose $h=u_n-u\in W^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (3.20), (3.26), (3.28). Then

$$
\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p-2}(\nabla u_n,\nabla(u_n-u))_{\mathbb{R}^N}\,dz=0 \ \Rightarrow\ u_n\to u \text{ in } W^{1,p}(\Omega) \tag{3.29}
$$

as $n\to+\infty$; see Motreanu-Motreanu-Papageorgiou [11, Proposition 2.72, p. 40]. We have

$$
N_f(u_n)\to N_f(u) \text{ in } L^{p'}(\Omega) \quad \text{(see hypothesis (H2)(i))}. \tag{3.30}
$$

Passing to the limit as $n\to+\infty$ in (3.18) and using (3.28), (3.29), (3.30), we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}(\nabla u,\nabla h)_{\mathbb{R}^N}\,dz+\int_{\partial\Omega}\beta(z)|u|^{p-2}uh d\sigma+\int_\Omega eh\,dz\\
&=\int_\Omega f(z,u,\nabla u)h\,dz
\end{aligned}
\tag{3.31}
$$

for all $h\in W^{1,p}(\Omega)$. Also we have

$$
\partial\varphi_{\lambda_n}(u_n(z))\in\partial\varphi(J_{\lambda_n}(u_n(z)) \quad \text{for a.a. } z\in\Omega, \text{ all } n\in\mathbb{N},
$$

with $J_{\lambda_n}=(\mathrm{id}-\lambda_n\partial\varphi)^{-1}$ being the resolvent map. So, we have

$$
J_{\lambda_n}(u_n(z))+\lambda_n\partial\varphi_{\lambda_n}(u_n(z))=u_n(z) \quad \text{for a.a. } z\in\Omega, \text{ all } n\in\mathbb{N}. \tag{3.32}
$$

If we set $\widehat{J}_{\lambda_n}(u_n(\cdot))=J_{\lambda_n}(u_n(\cdot))\in L^2(\Omega)$ for all $n\in\mathbb{N}$ (see Gasiński-Papageorgiou [3, p. 323]), from (3.27), (3.29) and (3.32) we have

$$
\begin{aligned}
&\widehat{J}_{\lambda_n}(u_n)-u_n\to 0 \text{ in } L^2(\Omega) \quad \text{(recall that } \lambda_n\to0^+),\\
&\Rightarrow\ \widehat{J}_{\lambda_n}(u_n)\to u \text{ in } L^2(\Omega),\\
&\Rightarrow\ e(z)\in\partial\varphi(u(z)) \quad \text{for a.a } z\in\Omega,
\end{aligned}
\tag{3.33}
$$

see (3.28) and use Gasiński-Papageorgiou [3, Proposition 3.2.15, p. 308]). Then from (3.31) and (3.33) we infer that $u\in W^{1,p}(\Omega)$ is a solution of (1.1). As before the nonlinear regularity theory of Lieberman [9] implies that $u\in C^1(\overline{\Omega})$. □

4. Example

In this section we see a particular case of problem (1.1), which corresponds to a variational inequality. Let

$$\varphi(x) = i_+(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0, \end{cases}$$

(the indicator function of the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$). Then $\varphi \in \Gamma_0(\mathbb{R})$ and we have

$$\partial\varphi(x) = N_{\mathbb{R}_+}(x) \quad \text{(the normal cone to } \mathbb{R}_+ \text{ at } x\text{);}$$

see Gasiński-Papageorgiou [3, p. 526]. We have

$$N_{\mathbb{R}_+}(x) = \{x^* \in \mathbb{R} : x^*(c - x) \leq 0 \text{ for all } c \geq 0\}.$$

Evidently $0 \in \partial\varphi(0)$. We consider a reaction term $f(z, x, y)$ satisfying (H2). Then according to Theorem 3.5 we can find

$$u \in C^1(\overline{\Omega}) \text{ with } u(z) \geq 0 \quad \text{for all } z \in \overline{\Omega}$$

which satisfies

$$\begin{gathered} -\Delta_p u(z) = f(z, u(z), \nabla u(z)) \quad \text{for a.a. } z \in \Omega_+ = \{z \in \Omega : u(z) > 0\}, \\ -\Delta_p u(z) \geq f(z, u(z), \nabla u(z)) \quad \text{for a.a. } z \in \Omega_0 = \{z \in \Omega : u(z) = 0\}, \\ (-\Delta_p u(z))u(z) = f(z, u(z), \nabla u(z))u(z) \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

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