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NONTRIVIAL COMPLEX SOLUTIONS FOR MAGNETIC SCHRÖDINGER EQUATIONS WITH CRITICAL NONLINEARITIES

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ABSTRACT. Using minimization arguments we establish the existence of a complex solution to the magnetic Schrödinger equation

$$-(\nabla + iA(x))^2 u + u = f(|u|^2)u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $A{:}\mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential and f satisfies some critical growth assumptions. First we obtain bounds from a real Pohozaev manifold. Then relate them to Sobolev imbedding constants and to the least energy level associated with the real equation in absence of the magnetic field (i.e., with $A(x) = 0$). We also apply the Lions Concentration Compactness Principle to the modula of the minimizing sequences involved.

1. INTRODUCTION

The aim of this article is to study the magnetic Schrödinger equation

$$-(\nabla + iA(x))^2 u + u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $u{:}\mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, i is the imaginary unit, $A = (A_1, \dots, A_N){:}\mathbb{R}^N \to \mathbb{R}^N$ is the magnetic (or vector) potential and the nonlinear term $f{:}\mathbb{R}^+ \to \mathbb{R}$ is a regular function satisfying suitable assumptions and having critical growth at infinity with critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \geq 3$. In the recent literature, magnetic Schrödinger equations have been studied in the critical case from different points of view but in few papers which we recall here in the following.

Esteban and Lions [11] (1989), found solutions to

$$(-i\nabla + A(x))^2 u + \lambda u = |u|^4 u \quad \text{in } \mathbb{R}^3$$

with $\lambda \in \mathbb{R}$ by solving constrained minimization problems with Concentration-Compactness methods. Arioli and Szulkin [3] (2003), found non-trivial solutions to

$$(-i\nabla + A(x))^2 u + V(x)u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N,$$

with A and V locally Lebesgue measurable by means of constrained minimization and Concentration-Compactness arguments under suitable assumptions on the

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spectrum of the operator $(-i\nabla + A(x))^2 + V$. Chabrowski-Szulkin [8] (2005) proved the existence of a non trivial solution to

$$(-i\nabla + A(x))^2 u + V(x)u = Q(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

when the electric potential V changes sign by a min-max type argument based on a topological linking. Certain regularity properties of solutions for a rather general class of equations involving the operator $(-i\nabla + A(x))^2$ are also established. Barile, Cingolani, Secchi [4] (2006) established existence results by abstract perturbation techniques to

$$(i\nabla + \varepsilon A(x))^2 u + \varepsilon^\alpha V(x)u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in [1, 2]$, $N > 4$ and the potentials A and V are bounded continuous and Lebesgue measurable. Han [12] (2006) showed the existence of a non-trivial complex solution to

$$(-i\nabla + A(x))^2 u - V(x)u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

with $N \geq 3$, have been established by a Mountain Pass Theorem under suitable assumptions on the integrability and the behaviour of the magnetic potential A and the electric potential V. Wang [21] (2008) established existence results for a nontrivial solution to

$$\begin{gathered}(-i\nabla + A(x))^2 u + \lambda V(x)u = K(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,\\ \lim_{|x|\to+\infty} u(x) = 0\end{gathered}$$

with $N \geq 3$, by means of Linking Theorem applied twice when $\lambda > 0$, the magnetic potential $A \in L^2_{\rm loc}(\mathbb{R}^N)$, the electric potential $V(x)$ is sign-changing, $K(x)$ is positive bounded and continuous and V, K satisfying suitable local assumptions. Liang and Zhang [15] (2011) studied standing waves solutions $\psi(x,t) = e^{-\frac{iEt}{\hbar}}u(x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $N \geq 3$, to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + iA(x))^2\psi + W(x)\psi - K(x)|\psi|^{2^*-2}\psi - h(x, |\psi|^2)\psi,$$

thus establishing the existence of at least one solution and, for any $m \in \mathbb{N}$, the existence of at least m pairs of solutions under suitable assumptions. Ding and Liu [10] (2013) proved then existence and have also described concentration phenomena of (ground states) solutions to

$$(-i\varepsilon\nabla + A(x))^2 u + V(x)u = W(x)(g(|u|) + |u|^{2^*-2})u \quad \text{in } \mathbb{R}^N$$

in the semiclassical limit (i.e. as $\varepsilon \to 0$) when $A \in C^1(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, V, W are positive and satisfy proper boundedness assumptions and $g(|u|)u$ is superlinear and subcritical. Liang and Song [14] (2014) treated

$$-\varepsilon^2(\nabla + iA(x))^2 u + V(x)u = |u|^{2^*-2}u + h(x, |u|^2)u \quad \text{in } \mathbb{R}^N$$

where $N \geq 3$ and $V(x)$ is a nonnegative potential by establishing for $\varepsilon > 0$ sufficiently small the existence of both one solution and m pairs of solutions for every $m \in \mathbb{N}$ by means of Lions' second Concentration-Compactness method and Concentration Compactness principle at infinity in order to recover a $(PS)_c$ condition. Alves and Figueiredo [1] (2014) studied the multiplicity of nontrivial solutions to

$$\begin{gathered}(-i\nabla - A(x))^2 u = \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \partial\Omega,\end{gathered}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N \geq 4$, A is continuous on $\overline{\Omega}$, $2 \leq q < 2^*$ thus relating the number of solutions with the topology of Ω by Ljusternik-Schnirelmann theory. Tang and Wang [24] (2015) studied

$$-(\nabla + iA(x))^2 u + (\lambda a(x) - \delta)u = |u|^{2^*-1}u \quad \text{in } \mathbb{R}^N$$

with $\delta > 0$ and the electric potential can be negative in some domain; specifically, by variational and Nehari methods they have established the existence of a least energy solution u_λ which localizes at the bottom of the potential well as $\lambda \to +\infty$.

Throughout this article, we use the following assumptions:

(A1) $A \in L^2_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$ and there exists $x_0 \in \mathbb{R}^N$ such that A is continuous at x_0;
(A2) $A(\frac{x}{\sigma}) = \sigma A(x)$ for every $x \in \mathbb{R}^N$ and for every $\sigma > 0$;
(A3) $A \in L^2_{rad}(\mathbb{R}^N, \mathbb{R}^N)$;
(A4) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\lim_{s\to 0^+} f(s) = 0$;
(A5) $0 < \limsup_{s\to+\infty} f(s)/s^{(2^*-2)/2} \leq 1$;
(A6) $f(s)s - F(s) \geq 0$ for every $s \in \mathbb{R}$ with $s \geq 0$ where $F(s) = \int_0^s f(t)\,dt$;
(A7) there exist $\lambda > 0$ and $q \in (2, 2^*)$ such that

$$f(s) \geq \lambda s^{(q-2)/2} \quad \text{for every } s \in \mathbb{R} \text{ with } s \geq 0.$$

A typical example of a function satisfying conditions (A1)–(A3) is $A(x) = A/|x|$, where A is a constant vector.

Note that by hypothesis (A2), the magnetic potential A is homogeneous of degree -1. This hypothesis is just used in Section 4 when we show that the limit of minimizing sequence is a nontrivial complex solution of Problem 1.1.

We shall prove the following result in the case $N \geq 3$. For the definition of the Sobolev embeddings constants S and c_A which appear in the next theorem see Section 2.

Theorem 1.1. *Suppose that $N \geq 3$ and (A1)–(A7) hold. Then, there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ problem (1.1) admits a nontrivial complex solution. Precisely, the constant is*

$$\lambda^* = \Big[2^{(2-N)/2} S^{-N/2} N \big(\frac{2N}{N-2}\big)^{(N-2)/2}\Big]^{(q-2)/2} c_A^{(q-2)/2}.$$

From our point of view, the main contributions of this article are as follows:
(1) Inspired by recent results obtained by Alves, Souto and Montenegro [2] (see also Zhang and Zou [23]) for equation (1.1) when $A(x) = 0$, we aim to establish the existence of a complex solution to the magnetic equation (1.1) by means of Concentration Compactness Principle of Lions [18, 19] in the case $N \geq 3$. Concerning the case $N = 2$, we mention the paper by Barile and Figueiredo [5].
(2) Since we do not know the Pohozaev identity associated with the problem (P), we use Pohozaev's identity of real problem, causing a modification in the arguments that can be found in [2] (see also Zhang and Zou [23]).

Remark 1.2. Condition (A5) can be replaced by

$$0 < \limsup_{s\to+\infty} \frac{f(s)}{s^{(2^*-2)/2}} \leq \mu$$

with $\mu > 0$ or more in general by

$$0 < \limsup_{s\to+\infty} \frac{f(s)}{s^{(2^*-2)/2}} < +\infty.$$

For simplicity but without loosing in generality, we assume (A5) thus studying the case $\mu = 1$.

Remark 1.3. From (A4) and (A5), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon + C_\varepsilon |s|^{(2^*-2)/2} \quad \text{for all } s \geq 0$$

and by integration

$$|F(s)| \leq \varepsilon |s| + \overline{C}_\varepsilon |s|^{2^*/2} \quad \text{for all } s \geq 0,$$

where $\overline{C}_\varepsilon = (2C_\varepsilon)/2^*$.

Remark 1.4. From hypothesis (A7), by integration we obtain

$$F(s) \geq \frac{2}{q}\lambda |s|^{q/2} \quad \text{for every } s \in \mathbb{R} \text{ with } s \geq 0.$$

This article is organized as follows. In Section 2, we fix notation and variational tools. In Section 3 we establish some preliminary results which will be useful in Section 4 for proving Theorem 1.1.

2. Notation and variational tools

To introduce the variational structure of the problem, we set

$$H^1_A(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |\nabla_A u|^2 dx < +\infty \right\}$$

with $\nabla_A u = (\nabla + iA(x))u$. The space $H^1_A(\mathbb{R}^N, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$(u, v)_A = \mathrm{Re} \int_{\mathbb{R}^N} \left(\nabla_A u \cdot \overline{\nabla_A v} + u\overline{v} \right) dx \quad \text{for any } u, v \in H^1_A(\mathbb{R}^N, \mathbb{C})$$

where Re and the bar denote the real part of a complex number and the complex conjugation respectively. The norm induced by this inner product is

$$\|u\|_A = \Big(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2)\, dx \Big)^{1/2} \quad \text{for } u \in H^1_A(\mathbb{R}^N, \mathbb{C})$$

and $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $H^1_A(\mathbb{R}^N, \mathbb{C})$ with respect to the norm $\|\cdot\|_A$ (see [11, Section 2] and [16, Theorem 7.22]). We denote by $H^{-1}_A(\mathbb{R}^N, \mathbb{C})$ the dual space of $H^1_A(\mathbb{R}^N, \mathbb{C})$. Recall that for every $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ one has

$$\int_{\mathbb{R}^N} |\nabla_A u|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |A(x)|^2 |u|^2 dx - 2\,\mathrm{Re} \int_{\mathbb{R}^N} \nabla u \cdot iA(x)\overline{u}\, dx.$$

Since there is no relation between $H^1_A(\mathbb{R}^N, \mathbb{C})$ and $H^1(\mathbb{R}^N, \mathbb{C})$; that is, $H^1_A(\mathbb{R}^N, \mathbb{C}) \not\subset H^1(\mathbb{R}^N, \mathbb{R})$ and $H^1(\mathbb{R}^N, \mathbb{C}) \not\subset H^1_A(\mathbb{R}^N, \mathbb{C})$, we will frequently use in this paper the following diamagnetic inequality (see [16, Theorem 7.21])

$$|\nabla |u|(x)| \leq |\nabla_A u(x)| \quad \text{for almost every } x \in \mathbb{R}^N. \tag{2.1}$$

This implies that, if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Therefore, $u \in L^p(\mathbb{R}^N, \mathbb{C})$ for any $p \in [2, 2^*]$.

By adapting standard variational arguments exploited in existing literature and by exploiting radial assumptions it is not difficult to prove that there exists ϕ a solution to

$$-(\nabla + iA(x))^2 \phi + \phi = |\phi|^{q-2}\phi \quad \text{in } \mathbb{R}^N,\ \phi \in H^1_A(\mathbb{R}^N, \mathbb{C}). \tag{2.2}$$

Note that if I_q is the functional associated with problem 2.2, then $I_q(\phi) = c_A$, where

$$c_A = \inf_{\gamma\in\Gamma_A} \max_{t\in[0,1]} I_q(\gamma(t)) > 0,$$

$$\Gamma_A = \big\{\gamma \in C\left([0,1], H^1_{A,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})\right) : \gamma(0) = 0 \text{ and } I_q(\gamma(1)) < 0\big\},$$

$$\|\phi\|_A^2 = \int_{\mathbb{R}^N} |\phi|^q dx, \tag{2.3}$$

$$\|\phi\|_A^2 = \frac{2q}{q-2} c_A. \tag{2.4}$$

Moreover, we consider the space

$$\mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C}) = \{u \in L^{2^*}(\mathbb{R}^N,\mathbb{C}) : \int_{\mathbb{R}^N} |\nabla_A u|^2 dx < +\infty\},$$

which is the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C})} = \Big(\int_{\mathbb{R}^N} |\nabla_A u|^2 dx\Big)^{1/2} \quad \text{for } u \in \mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C})$$

corresponding to the inner product

$$(u,v)_{\mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C})} = \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}\, dx \quad \text{for } u,v \in \mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C}).$$

Recall that $\mathcal{D}^{1,2}_A(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{C})$. It is also useful to define

$$\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}) = \big\{u \in L^{2^*}(\mathbb{R}^N,\mathbb{R}) : \int_{\mathbb{R}^N} |\nabla u|^2 dx < +\infty\big\}$$

which is the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R})} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2} \quad \text{for } u \in \mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R})$$

corresponding to the inner product

$$(u,v)_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R})} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx \quad \text{for } u,v \in \mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}).$$

Recall that $\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$ and we denote by $S_0 > 0$ the best constant of Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$; that is,

$$S_0\Big(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\Big)^{2/2^*} \le \int_{\mathbb{R}^N} |\nabla u|^2\, dx \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}).$$

If S denotes the best constant of the imbedding $D^{1,2}_A(\mathbb{R}^N,\mathbb{C}) \to L^{2^*}(\mathbb{R}^N,\mathbb{C})$, that is,

$$S = \inf_{u\in D^{1,2}_A(\mathbb{R}^N,\mathbb{C})} \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}},$$

we have that $S = S_0$, for details see [3, Theorem 1.1].

The energy functional $I_A : H^1_A(\mathbb{R}^N,\mathbb{C}) \to \mathbb{R}$ associated with (1.1) is defined as

$$I_A(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2)\, dx \quad \text{for } u \in H^1_A(\mathbb{R}^N,\mathbb{C}).$$

Under assumptions (A4), (A5) (see in particular Remark 1.3 and a direct application of [20, Theorem 1.22]), we obtain $I_A \in C^1(H^1_A(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ with Gâteaux differential

$$I'_A(u)v = \mathrm{Re}\int_{\mathbb{R}^N} (\nabla_A u \cdot \overline{\nabla_A v} + u\overline{v})\,dx - \mathrm{Re}\int_{\mathbb{R}^N} f(|u|^2)u\overline{v}\,dx$$

for all $u, v \in H^1_A(\mathbb{R}^N,\mathbb{C})$, and its critical points are the weak solutions to (1.1).

We denote by $H^1_{A,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ and $H^1_{rad}(\mathbb{R}^N,\mathbb{R})$ the subspaces of $H^1_A(\mathbb{R}^N,\mathbb{C})$ and $H^1(\mathbb{R}^N,\mathbb{R})$ formed by the radial functions, that is

$$H^1_{A,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}) = \{u \in H^1_A(\mathbb{R}^N,\mathbb{C}) : u(x) = u(|x|) \text{ for } x \in \mathbb{R}^N\},$$
$$H^1_{rad}(\mathbb{R}^N,\mathbb{R}) = \{u \in H^1(\mathbb{R}^N,\mathbb{R}) : u(x) = u(|x|) \text{ for } x \in \mathbb{R}^N\}.$$

Now, for finding a nontrivial complex solution to (1.1) let

$$D_A = \inf\big\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx : u \in \mathcal{M}_A\big\} \tag{2.5}$$

where

$$\mathcal{M}_A = \big\{u \in H^1_A(\mathbb{R}^N,\mathbb{C}) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)\,dx = 1\big\} \quad \text{with } N \geq 3$$

with $g(u) = f(|u|^2)u - u$ and $G(u) = \frac{1}{2}\big(F(|u|^2) - |u|^2\big)$ for $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$. Hereafter, we can denote by

$$D_A = \inf_{u\in\mathcal{M}_A} T_A(u)$$

where for simplicity of notation

$$T_A(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx \quad \text{for } u \in H^1_A(\mathbb{R}^N,\mathbb{C}).$$

Moreover, we set

$$J(u) = \int_{\mathbb{R}^N} G(u)\,dx = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|u|^2) - |u|^2\big)\;dx \quad \text{for } u \in H^1_A(\mathbb{R}^N,\mathbb{C}),$$
$$J'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} g(u)\overline{v}\,dx = \mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u|^2)u - u\big)\overline{v}\,dx \quad \text{for } u, v \in H^1_A(\mathbb{R}^N,\mathbb{C}).$$

Lemma 2.1. *Suppose* (A1), (A4), (A5), (A7) *hold. Then, the functional I_A has a mountain pass geometry, that is*

(i) $I_A(0) = 0$,
(ii) *there exist $\rho_0, \delta_0 > 0$ such that $I_A(u) \geq \delta_0$ for all $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ with $\|u\|_A = \rho_0$;*
(iii) *there exists $u_0 \in H^1_A(\mathbb{R}^N,\mathbb{C})$ such that $\|u_0\|_A > \rho_0$ and $I_A(u_0) \leq 0$.*

Proof. (i) follows easily by Remark 1.3. (ii) By exploiting Remark 1.3 again and Sobolev embeddings, for any $\varepsilon > 0$ there exists $\overline{C}_\varepsilon > 0$ such that

$$\begin{aligned} I_A(u) &= \frac{1}{2}\|u\|^2_A - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2)\,dx \\ &\geq \frac{1}{2}\|u\|^2_A - \frac{1}{2}\varepsilon\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{2}\,\overline{C}_\varepsilon\int_{\mathbb{R}^N} |u|^{2^*}\,dx \\ &\geq \frac{1}{2}(1-\varepsilon)\|u\|^2_A - \overline{C}'_\varepsilon\|u\|^{2^*}_A\,. \end{aligned}$$

If we take $\|u\|_A = \rho$ and choose $\varepsilon > 0$ and $\rho > 0$ small enough, we obtain (ii) is satisfied for suitable $\rho_0, \delta_0 > 0$.

(iii) Let $v_0 \in H^1_A(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ arbitrary and $t > 0$. By Remark 1.4 it is

$$I_A(tv_0) \le \frac{1}{2}\, t^2 \|v_0\|_A^2 - \frac{1}{q} \lambda t^q \int_{\mathbb{R}^N} |u|^q \, dx.$$

Since $q > 2$, we obtain $I_A(t\, v_0) \to -\infty$ as $t \to +\infty$ thus, taken $u_0 = tv_0$ for t sufficiently large (iii) is proved. $\square$

Therefore, we can define the following minimax value or mountain pass level of I_A, i.e.

$$b_A = \inf_{\gamma \in \Gamma_A} \max_{t \in [0,1]} I_A(\gamma(t)) > 0,$$

where

$$\Gamma_A = \{\gamma \in C\left([0,1], H^1_A(\mathbb{R}^N, \mathbb{C})\right) : \gamma(0) = 0 \text{ and } I_A(\gamma(1)) < 0\}.$$

At this point, it is useful to consider the real scalar problem

$$\begin{gathered} -\Delta u + u = f(|u|^2)u \quad \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N, \mathbb{R}) \end{gathered} \tag{2.6}$$

with $f{:}\mathbb{R}^+ \to \mathbb{R}$ satisfying assumptions (A4), (A5)-(A7) when $N \ge 3$. Here below we give a brief description of the results which we will exploit in next sections and which have been established in Alves, Souto and Montenegro [2] by using the ideas in Berestycki and Lions [6], Coleman, Glazer and Martin [9] and in Jeanjean and Tanaka [13]. The functional $I_0 \in C^1(H^1(\mathbb{R}^N, \mathbb{R}), \mathbb{R})$ associated with (2.6) is

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2)\, dx \quad \text{for } u \in H^1(\mathbb{R}^N, \mathbb{R}),$$

where

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\, dx\Big)^{1/2} \quad \text{for } u \in H^1(\mathbb{R}^N, \mathbb{R}).$$

The authors investigated the existence of a ground state solution to (2.6), which means a solution to $u \in H^1(\mathbb{R}^N, \mathbb{R})$ such that $I_0(u) \le I_0(v)$ for every nontrivial solution $v \in H^1(\mathbb{R}^N, \mathbb{R})$ of (2.6). Denoting

$$m_0 = \inf\{I_0(u) : u \text{is a nontrivial solution to (2.6)}\}$$

and taking into consideration the set of non-zero critical point of I_0, namely

$$Sigma_0 = \{u \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : I_0'(u) = 0\},$$

it follows that

$$m_0 = \inf_{u \in \Sigma_0} I_0(u).$$

Let

$$D_0 = \inf_{u \in \mathcal{M}_0} T_0(u) \quad \text{with} \quad T_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \text{for } u \in H^1(\mathbb{R}^N, \mathbb{R})$$

and the C^1 manifold

$$\mathcal{M}_0 = \big\{u \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)\, dx = 1\big\} \quad \text{with } N \ge 3.$$

It has been taken into account the Pohozaev identity manifold

$$\begin{aligned}\mathcal{P}_0 &= \big\{u \in H^1(\mathbb{R}^N,\mathbb{R}) \setminus \{0\} : \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx = N\int_{\mathbb{R}^N} G(u)\, dx\big\} \\ &= \big\{u \in H^1(\mathbb{R}^N,\mathbb{R}) \setminus \{0\} : (N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx = N\int_{\mathbb{R}^N} \big(F(|u|^2) - |u|^2\big)\, dx\big\}.\end{aligned}$$

and $p_0 = \inf_{u\in\mathcal{P}_0} I_0(u)$. Since I_0 has a mountain pass geometry, we define

$${}_0 = \inf_{\overline{\gamma}\in\Gamma}\max_{t\in[0,1]} I_0(\overline{\gamma}(t)),$$

where

$$\Gamma_0 = \{\overline{\gamma} \in C\left([0,1], H^1(\mathbb{R}^N,\mathbb{R})\right) : \overline{\gamma}(0) = 0 \text{ and } I_0(\overline{\gamma}(1)) < 0\}.$$

In [2] the authors showed that D_0 is attained in $H^1_{rad}(\mathbb{R}^N,\mathbb{R})$ and the following least energy characterizations holds

$$m_0 = b_0 = p_0 = \frac{1}{N}\big(\frac{N-2}{2N}\big)^{(N-2)/2}(2D_0)^{N/2} \quad \text{if } N \geq 3\,;$$

so that (2.6) has a nontrivial ground state solution.

These existence results have been established without assuming two widely used conditions, that is the monotonicity condition

$$\frac{f(s^2)s}{s} \quad \text{is increasing in } (0,+\infty)$$

and the Ambrosetti-Rabinowitz condition: there exists a constant $\theta > 2$ such that

$$0 < \theta F(s^2) \leq f(s^2)s^2 \quad \text{for any } s \in \mathbb{R}\setminus\{0\}.$$

Therefore, the paper [2] complements the results obtained in the subcritical case by Jeanjean and Tanaka [13] and improves previous results established under the two previous conditions. Furthermore, we stress that assumption (A7) ensures that there exists $s > 0$ such that $G(s) > 0$, which is a necessary condition for the existence of a solution to (2.6) since it allows to exploit Pohozaev's identity then Pohozaev identity manifold $\mathcal{P}_0$ as done by Berestycki and Lions in [6, Proposition 1].

As in [2] which we follow, from the moment we extend the existence of a solution to (2.6) to the magnetic case ($A(x) \neq 0$), we also improve previous results obtained under the two previous conditions in absence of a magnetic field. Moreover, we complement all the papers in literature treating equation (1.1) in the subcritical case.

3. Preliminary results

Here we establish some preliminary results which will be used for proving Theorem 1.1 in Section 4.

Lemma 3.1. *Under hypothesis* (A1), (A4), (A5)–(A7)*, the following assertions hold:*

(a) $\mathcal{M}_A$ *is not empty;*
(b) $\mathcal{M}_A$ *is a* C^1 *manifold.*

Proof. (a) By remarks 1.3 and 1.4, for $u \neq 0$ one has $J(tu) < 0$ if $t > 0$ is small and $J(tu) \to +\infty$ if $t \to +\infty$. Hence, $J(t_0)$=1, for some $t_0 > 0$.
(b) By the definitions of J and J' given in Section 2, for every $u \in \mathcal{M}_A$ we have

$$\begin{aligned} J'(u)u &= \int_{\mathbb{R}^N} \left(f(|u|^2)|u|^2 - |u|^2\right) dx \\ &= \int_{\mathbb{R}^N} \left(f(|u|^2)|u|^2 - F(|u|^2)\right) dx + \int_{\mathbb{R}^N} \left(F(|u|^2) - |u|^2\right) dx. \end{aligned}$$

By (A6) and $J(u) = 1$, it follows that

$$J'(u)u \geq \int_{\mathbb{R}^N} \left(F(|u|^2) - |u|^2\right) dx = 2\int_{\mathbb{R}^N} G(u)\, dx = 2 > 0$$

for all $u \in \mathcal{M}_A$. Then, $J'(u)u \neq 0$ for any $u \in \mathcal{M}_A$. □

Lemma 3.2. *Let assumptions* (A1)–(A6) *be satisfied. Then, any minimizing sequence* $\{u_n\}$ *for* D_A *is bounded in* $H^1_A(\mathbb{R}^N, \mathbb{C})$. *The same assertion holds in particular in* $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$.

Proof. Taken $\{u_n\}$ a minimizing sequence for D_A in $H^1_A(\mathbb{R}^N, \mathbb{C})$, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx \to D_A \quad \text{as } n \to +\infty$$

and

$$\int_{\mathbb{R}^N} G(u_n)\, dx = 1, \quad \text{that is } \frac{1}{2}\int_{\mathbb{R}^N} \left(F(|u_n|^2) - |u_n|^2\right) dx = 1.$$

It follows that

$$\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx \leq C \quad \text{for all } n \in \mathbb{N} \text{ and for some constant } C > 0 \tag{3.1}$$

and

$$\int_{\mathbb{R}^N} F(|u_n|^2)\, dx = 2 + \int_{\mathbb{R}^N} |u_n|^2 dx.$$

By Remark 1.3 with $\varepsilon = \frac{1}{2}$ we obtain

$$2 + \int_{\mathbb{R}^N} |u_n|^2 dx \leq \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 dx + \overline{C}_{1/2}\int_{\mathbb{R}^N} |u_n|^{2^*} dx.$$

Then, by (3.1), for every $n \in \mathbb{N}$,

$$\frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 dx \leq \overline{C}_{1/2}\int_{\mathbb{R}^N} |u_n|^{2^*} dx \leq \frac{\overline{C}_{1/2}}{S^{2^*/2}}\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx\Big)^{2^*/2} \leq \overline{C}.$$

Consequently, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N, \mathbb{C})$ and this implies that $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^N, \mathbb{C})$. Without any difficulty, these arguments work also in $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$. □

Clearly, by Sobolev imbeddings we obtain any minimizing sequence $\{u_n\}$ for D_A is bounded also in $L^m(\mathbb{R}^N, \mathbb{C})$ for every $m \in [2, 2^*]$. It is useful to establish some lemmas involving the level D_0 associated with (2.6) and the min-max level b_A of I_A.

Lemma 3.3. *Under assumptions* (A1), (A4)–(A7), *it holds*

$$\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2}(2D_0)^{N/2} \leq b_A.$$

Proof. First let us prove that, by diamagnetic inequality (2.1), we have $b_0 \le b_A$, where b_0 and b_A are respectively the min-max levels for the functionals I_0 and I_A, that is

$$b_0 = \inf_{\overline{\gamma}\in\Gamma_0} \max_{t\in[0,1]} I_0(\overline{\gamma}(t)) \quad \text{and} \quad b_A = \inf_{\gamma\in\Gamma_A} \max_{t\in[0,1]} I_A(\gamma(t)).$$

To do this, we take an arbitrary $\gamma \in \Gamma_A$. Then, since $\gamma(t) \in H^1_A(\mathbb{R}^N, \mathbb{C})$ for every $t \in [0,1]$, by diamagnetic inequality (2.1) we obtain $|\gamma(t)| \in H^1(\mathbb{R}^N, \mathbb{R})$ and

$$\int_{\mathbb{R}^N} |\nabla(|\gamma(t)|)|^2 dx \le \int_{\mathbb{R}^N} |\nabla_A(\gamma(t))|^2 dx$$

which implies

$$I_0(|\gamma(t)|) \le I_A(\gamma(t)) \tag{3.2}$$

for any $t \in [0,1]$. Therefore,

$$\max_{t\in[0,1]} I_0(|\gamma(t)|) \le \max_{t\in[0,1]} I_A(\gamma(t)). \tag{3.3}$$

Now, since $\gamma \in C\big([0,1], H^1_A(\mathbb{R}^N, \mathbb{C})\big)$ we obtain $|\gamma| \in C\big([0,1], H^1(\mathbb{R}^N, \mathbb{R})\big)$; moreover, $\gamma(0) = 0$ implies $|\gamma(0)| = 0$ and by $I_A(\gamma(1)) < 0$ and (3.2) we obtain $I_0(|\gamma(1)|) < 0$. Consequently, $|\gamma| \in \Gamma_0$ which easily gives

$$b_0 \le \max_{t\in[0,1]} I_0(|\gamma(t)|).$$

Now, taking the infimum over all $\gamma \in \Gamma_A$ by (3.3) we conclude $b_0 \le b_A$. Therefore, to obtain the claim, it is sufficient to prove that

$$\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2}(2D_0)^{N/2} \le b_0.$$

By exploiting the results obtained in [2] (see Section 2) we know in particular that

$$\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2}(2D_0)^{N/2} = p_0 = b_0.$$

For the reader's convenience, we sketch here the proof of $p_0 \le b_0$. Indeed, from [13], for each $\overline{\gamma} \in \Gamma_0$ with

$$\Gamma_0 = \{\overline{\gamma} \in C\big([0,1], H^1(\mathbb{R}^N, \mathbb{R})\big) : \overline{\gamma}(0) = 0 \text{ and } I_0(\overline{\gamma}(1)) < 0\}$$

it results $\overline{\gamma}([0,1]) \cap \mathcal{P}_0 \ne \emptyset$. Then, there exists $t_0 \in [0,1]$ such that $\overline{\gamma}(t_0) \in \mathcal{P}_0$. So

$$p_0 \le I(\overline{\gamma}(t_0)) \le \max_{t\in[0,1]} I(\overline{\gamma 0}(t))$$

implies $p_0 \le b_0$. Consequently, since $b_0 \le b_A$ we obtain the result. $\square$

Lemma 3.4. *Suppose* (A1), (A4)–(A6) *hold. Then the number D_A given in* (2.5) *is positive, namely, $D_A > 0$.*

Proof. By definition $D_A \ge 0$. Suppose, by contradiction, that $D_A = 0$. If $\{u_n\}$ is a minimizing sequence for $D_A = 0$ in $H^1_A(\mathbb{R}^N, \mathbb{C})$, from (A3), without loss of generality, we can suppose that $\{u_n\}$ is a minimizing sequence for $D_A = 0$ in $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$. Then

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx \to 0 \quad \text{as } n \to +\infty$$

and

$$1 = \int_{\mathbb{R}^N} G(u_n)\, dx = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|u_n|^2) - |u_n|^2\big)\, dx.$$

Then, by Remark 1.3 it follows that

$$2+\int_{\mathbb{R}^N}|u_n|^2dx=\int_{\mathbb{R}^N}F(|u_n|^2)\,dx\le\varepsilon\int_{\mathbb{R}^N}|u_n|^2dx+\overline{C}_\varepsilon\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx$$

so we obtain

$$2+(1-\varepsilon)\int_{\mathbb{R}^N}|u_n|^2dx\le\overline{C}_\varepsilon\int_{\mathbb{R}^N}|u_n|^{2^*}\,dx\le\frac{\overline{C}_\varepsilon}{S^{2^*/2}}\Big(\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx\Big)^{2^*/2}$$

By choosing $\varepsilon=\frac12$ we obtain

$$2\le\frac{\overline{C}_{1/2}}{S^{2^*/2}}\Big(\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx\Big)^{2^*/2}\to0\text{ as }n\to+\infty$$

which is an absurd. □

Remark 3.5. Assume (A1), (A4)–(A6) hold. By Ekeland Variational Principle stated in [22, Theorem 8.5], we can suppose that the minimizing sequence $\{u_n\}\subset\mathcal{M}_A$ to D_A is a Palais-Smale sequence, namely, there exists a Lagrange multipliers sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$\begin{gathered}\frac12\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx\to D_A\quad\text{as }n\to+\infty,\\ T_A'(u_n)-\lambda_nJ'(u_n)\to0\text{ in }H_A^{-1}(\mathbb{R}^N,\mathbb{C})\text{ as }n\to+\infty,\end{gathered}$$

where we recall that, for simplicity of notation, in Section 2 we set $T_A(u)=\frac12\int_{\mathbb{R}^N}|\nabla_Au|^2dx$ for any $u\in H_A^1(\mathbb{R}^N,\mathbb{C})$.

Lemma 3.6. *Let* (A1), (A4)–(A6) *be satisfied. Then, the sequence* $\{\lambda_n\}$ *of Lagrange multipliers (see Remark 3.5) is bounded from above and*

$$0<\liminf_{n\to+\infty}\lambda_n\le\limsup_{n\to+\infty}\lambda_n\le D_A.$$

Proof. Since $T_A'(u_n)-\lambda_nJ'(u_n)\to0$ in $H_A^{-1}(\mathbb{R}^N,\mathbb{C})$ as $n\to+\infty$, we obtain

$$\begin{aligned}&T_A'(u_n)u_n-\lambda_nJ'(u_n)u_n\\&=\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx-\lambda_n\operatorname{Re}\int_{\mathbb{R}^N}g(u_n)\overline{u_n}\,dx\\&=\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx-\lambda_n\int_{\mathbb{R}^N}\big(f(|u_n|^2)|u_n|^2-|u_n|^2\big)\,dx=o_n(1).\end{aligned}$$

This is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx&=\lambda_n\int_{\mathbb{R}^N}\big(f(|u_n|^2)|u_n|^2-F(|u_n|^2)\big)\,dx\\&\quad+\lambda_n\int_{\mathbb{R}^N}\big(F(|u_n|^2)-|u_n|^2\big)\,dx+o_n(1).\end{aligned}$$

By (A6) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla_Au_n|^2dx&\ge\lambda_n\int_{\mathbb{R}^N}\big(F(|u_n|^2)-|u_n|^2\big)\,dx+o_n(1)\\&=2\lambda_n\int_{\mathbb{R}^N}G(u_n)\,dx+o_n(1)=2\lambda_n+o_n(1).\end{aligned}$$

Then, from Remark 3.5

$$\limsup_{n\to+\infty} \lambda_n \le \frac{1}{2} \limsup_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx = D_A$$

and the right-hand inequality in the thesis follows.
Now, we prove that $\liminf_{n\to+\infty} \lambda_n > 0$. First observe that, by Remark 1.3, Sobolev imbeddings and the boundedness of the minimizing sequence $\{u_n\}$ stated in Lemma 3.2, we obtain

$$\begin{aligned} |J'(u_n)u_n| &\le \int_{\mathbb{R}^N} \Big(|f(|u_n|^2|)\,|u_n|^2 + |u_n|^2 \Big)\,dx \\ &\le \int_{\mathbb{R}^N} \Big((\varepsilon+1)|u_n|^2 + C_\varepsilon |u_n|^{2^*} \Big)\,dx \le C. \end{aligned}$$

Then, we conclude that

$$2T_A(u_n) = T_A'(u_n)u_n = \lambda_n J'(u_n)u_n + o_n(1)$$

and $2T_A(u_n) \to 2D_A > 0$ where the positivity of D_A has been established in Lemma 3.4. □

As observed in the following remark, to study the compactness of a minimizing sequence $\{u_n\}$ to D_A, it is sufficient to consider the sequence $\{|u_n|\}$ of the modula of u_n. So we can exploit the diamagnetic inequality (2.1) and apply to $\{|u_n|\}$ the Concentration-Compactness technique of Lions [18] which is based on a measure representation.

Remark 3.7. If a sequence $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^N,\mathbb{C})$, then by diamagnetic inequality (2.1) the sequence $\{|u_n|\}$ of its modula is bounded in $H^1(\mathbb{R}^N,\mathbb{R})$. Then, by Lions Concentration Compactness principle [17, Lemma 1.2], there are a countable index set Λ, nonnegative finite measures μ and ν and families $\{\mu_i\}, \{\nu_i\} \subset (0,+\infty)$ and $\{x_i\} \subset \mathbb{R}^N$ such that

(j) $|\nabla|u_n||^2 \rightharpoonup \mu \ge |\nabla|u||^2 + \sum_{i\in\Lambda} \delta_{x_i}\mu_i$ (weak* sense of measures);
(jj) $|u_n|^{2^*} \rightharpoonup \nu = |u|^{2^*} + \sum_{i\in\Lambda} \delta_{x_i}\nu_i$ (weak* sense of measures)
(jjj) $\mu_i \ge S\,\nu_i^{2/2^*}$ for every $i \in \Lambda$.

where $|u|$ is the weak limit of $|u_n|$ and δ_{x_i} are Dirac measures at x_i. This remark can be employed in the next lemma.

Lemma 3.8. *Suppose that* (A1), (A4)–(A6) *are satisfied. If $\nu_i > 0$ for some index i, then*

$$\nu_i \ge \Big(\frac{S}{D_A}\Big)^{N/2}.$$

Proof. Let $\{u_n\}$ be a minimizing sequence to D_A which we know it is bounded by Lemma 3.2. Let $\varphi \in C_0^\infty(\mathbb{R}^N,[0,1])$ be such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1(0) \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_2(0) \end{cases}$$

and $|\nabla\varphi(x)| \le 2$ for every $x \in \mathbb{R}^N$. Then, we can consider

$$\varphi_{\varepsilon,x_i}(x) = \varphi\Big(\frac{x-x_i}{\varepsilon}\Big)$$

for $\varepsilon > 0$ and x_i a singular point of the measures $\sum_{i\in\Lambda} \delta_{x_i}\mu_i$ and $\sum_{i\in\Lambda} \delta_{x_i}\nu_i$ whose existence is ensured in Remark 3.7 by Lions Concentration Compactness Principle. Clearly, $\varphi_{\varepsilon,x_i} \in C_0^\infty(\mathbb{R}^N, [0,1])$ satisfies

$$\varphi_{\varepsilon,x_i}(x) = \begin{cases} 1 & \text{if } x \in B_\varepsilon(x_i) \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{2\varepsilon}(x_i) \end{cases}$$

and $|\nabla\varphi_{\varepsilon,x_i}(x)| \le 2/\varepsilon$ for every $x \in \mathbb{R}^N$. Since $\{u_n\varphi_{\varepsilon,x_i}\}$ is bounded in $H_A^1(\mathbb{R}^N, \mathbb{C})$, by Remark 3.5 we obtain

$$T_A'(u_n)(u_n\varphi_{\varepsilon,x_i}) = \lambda_n J'(u_n)(u_n\varphi_{\varepsilon,x_i}) + o_n(1);$$

that is,

$$\begin{aligned} &\mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u_n \cdot \overline{\nabla_A(u_n\varphi_{\varepsilon,x_i})}\, dx \\ &= \lambda_n\ \mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u_n|^2)|u_n|^2 - |u_n|^2\big)\, \overline{\varphi_{\varepsilon,x_i}}\, dx + o_n(1). \end{aligned} \tag{3.4}$$

Since $\varphi_{\varepsilon,x_i}$ takes real values and by direct calculations,

$$\overline{\nabla_A(u_n\varphi_{\varepsilon,x_i})} = \overline{\nabla_A u_n}\varphi_{\varepsilon,x_i} + \overline{u_n}\nabla\varphi_{\varepsilon,x_i}\,.$$

Then we can write the term on the left-hand side in (3.4) as

$$\begin{aligned} &\mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u_n \cdot \overline{\nabla_A(u_n\varphi_{\varepsilon,x_i})}\, dx \\ &= \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u_n \cdot \big(\overline{\nabla_A u_n}\varphi_{\varepsilon,x_i} + \overline{u_n}\nabla\varphi_{\varepsilon,x_i}\big)\, dx \\ &= \int_{\mathbb{R}^N} |\nabla_A u_n|^2\varphi_{\varepsilon,x_i}\, dx + \mathrm{Re}\int_{\mathbb{R}^N} \overline{u_n}\nabla_A u_n \cdot \nabla\varphi_{\varepsilon,x_i}\, dx. \end{aligned} \tag{3.5}$$

Now, observe that

$$\mathrm{Re}(\overline{u}_n\nabla_A u_n) = |u_n|\nabla|u_n|. \tag{3.6}$$

Indeed,

$$\begin{aligned} \mathrm{Re}(\overline{u}_n\nabla_A u_n) &= \mathrm{Re}\left(\overline{u}_n\left(\nabla u_n + iA(x)u_n\right)\right) \\ &= \mathrm{Re}\left(\overline{u}_n\nabla u_n + iA(x)|u_n|^2\right) = \mathrm{Re}(\overline{u}_n\nabla u_n) \\ &= |u_n|\ \mathrm{Re}\Big(\frac{\overline{u}_n}{|u_n|}\nabla u_n\Big) = |u_n|\nabla|u_n|. \end{aligned}$$

By substituting (3.6) in (3.5) and by replacing in turn (3.6) in (3.4) we obtain by diamagnetic inequality

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla|u_n||^2\varphi_{\varepsilon,x_i}\, dx + \int_{\mathbb{R}^N} |u_n|\nabla|u_n| \cdot \nabla\varphi_{\varepsilon,x_i}\, dx \\ &\le \lambda_n\int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2\varphi_{\varepsilon,x_i}\, dx + \lambda_n\int_{\mathbb{R}^N} |u_n|^2\varphi_{\varepsilon,x_i}\, dx + o_n(1). \end{aligned} \tag{3.7}$$

Note that by (A4), there exists $\delta > 0$ such that

$$|f(s)| \le 1, \quad \text{for all } 0 \le s \le \delta.$$

From (A5), there exists $K > 0$ such that

$$|f(s)| \le |s|^{(2^*-2)/2}, \quad \text{for all } s \ge K.$$

Moreover, for $s \in [\delta, k]$, we obtain

$$|f(s)| \le K_1, \quad \text{for some} \quad K_1 > 0.$$

Then

$$|f(s)| \le (1 + K_1) + |s|^{(2^*-2)/2}, \quad \text{for every } s \ge 0.$$

In particular,

$$f(|u_n|^2)|u_n|^2 \le (1 + K_1)|u_n|^2 + |u_n|^{2^*}, \quad \text{for every } n \in \mathbb{N}.$$

Therefore, by (3.7) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla |u_n||^2 \varphi_{\varepsilon,x_i}\, dx + \int_{\mathbb{R}^N} |u_n| \nabla |u_n| \cdot \nabla \varphi_{\varepsilon,x_i}\, dx \\
&\le \lambda_n \Big((2 + K_1) \int_{\mathbb{R}^N} |u_n|^2 \varphi_{\varepsilon,x_i}\, dx + \int_{\mathbb{R}^N} |u_n|^{2^*} \varphi_{\varepsilon,x_i}\, dx \Big) + o_n(1).
\end{aligned}$$

Now, we prove that

$$\limsup_{\varepsilon\to 0} \limsup_{n\to+\infty} \int_{\mathbb{R}^N} |u_n| \nabla |u_n| \cdot \nabla \varphi_{\varepsilon,x_i}\, dx = 0. \tag{3.8}$$

Indeed, by Hölder's inequality,

$$\begin{aligned}
&\limsup_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |u_n| \nabla |u_n| \cdot \nabla \varphi_{\varepsilon,x_i}\, dx \Big) \\
&\le \limsup_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |\nabla |u_n||\, |u_n|\, |\nabla \varphi_{\varepsilon,x_i}|\, dx \Big) \\
&\le \limsup_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |\nabla |u_n||^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N} |u_n|^2 |\nabla \varphi_{\varepsilon,x_i}|^2 dx \Big)^{1/2} \\
&\le C_1 \limsup_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |u_n|^2 |\nabla \varphi_{\varepsilon,x_i}|^2 dx \Big)^{1/2} \\
&= C_1 \Big(\int_{\mathbb{R}^N} |u|^2 |\nabla \varphi_{\varepsilon,x_i}|^2 dx \Big)^{1/2}
\end{aligned}$$

where $C_1 = \sup_n \big(\int_{\mathbb{R}^N} |\nabla |u_n||^2 dx \big)^{1/2}$. By using Hölder's inequality again it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^2 |\nabla \varphi_{\varepsilon,x_i}|^2 dx &\le \Big(\int_{B_{2\varepsilon}(x_i)} |u|^{2^*}\, dx \Big)^{2/2^*} \Big(\int_{\mathbb{R}^N} |\nabla \varphi_{\varepsilon,x_i}|^N\, dx \Big)^{2/N} \\
&\le C_N \Big(\int_{B_{2\varepsilon}(x_i)} |u|^{2^*}\, dx \Big)^{2/2^*}
\end{aligned}$$

with $C_N > 0$ a suitable constant depending on N. Letting $\varepsilon \to 0$ we obtain (3.8) is satisfied. Now, since $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^N)$ it easily follows

$$\limsup_{\varepsilon\to 0} \limsup_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^2 \varphi_{\varepsilon,x_i}\, dx = 0$$

which together with the assertion

$$\limsup_{n\to+\infty} \lambda_n \le D_A,$$

in Lemma 3.6, allow us to have $\mu_i \le D_A\, \nu_i$. By Remark 3.7 (jjj) we have $S\nu_i^{2/2^*} \le \mu_i$. Then

$$S\nu_i^{2/2^*} \le D_A \nu_i$$

and $\nu_i \geq \left(\frac{S}{D_A}\right)^{N/2}$. This completes the proof. $\square$

Lemma 3.9. *Suppose that* (A1), (A4)–(A6) *are satisfied. If $\nu_i > 0$ for some index i, then $D_A \geq 2^{-2/N} S$.*

Proof. From $0 \leq \varphi_{\varepsilon, x_i} \leq 1$, it follows that

$$\int_{\mathbb{R}^N} |u_n|^{2^*} \varphi_{\varepsilon, x_i} \, dx \leq \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq S^{-2^*/2} \Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx \Big)^{2^*/2}.$$

Passing to the limit as $n \to +\infty$, we have

$$\nu_i \leq S^{-2^*/2} \, (2D_A)^{2^*/2}.$$

From Lemma 3.8, since $\nu_i \geq \left(\frac{S}{D_A}\right)^{N/2}$, it follows that $D_A \geq 2^{-2/N} S$; thus completing the proof. $\square$

Now, let us recall the next result relating the constants D_A and D_0.

Lemma 3.10. *Under assumptions* (A1), (A4)–(A6)*, we have $D_A = D_0$.*

Proof. We follows the arguments used in [3, 11], and for the sake of completeness, we give here the details of the proof. By diamagnetic inequality (2.1) we obtain

$$D_0 \leq \int_{\mathbb{R}^N} \left(|\nabla |u||^2 + |u|^2 \right) \, dx \leq \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + |u|^2 \right) \, dx,$$

which implies $D_0 \leq D_A$.

Now, we show the reversed inequality $D_A \leq D_0$ also holds. Taking $\varepsilon > 0$ an arbitrarily small constant, we consider $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ whose $\operatorname{supp}(\varphi_\varepsilon)$ is a (compact) neighborhood of $x_0 \in \mathbb{R}^N$ (for simplicity, we can assume $x_0 = 0$) satisfying

$$\int_{\mathbb{R}^N} G(\varphi_\varepsilon) \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \varphi_\varepsilon|^2 dx \leq D_0 + \varepsilon.$$

We can define a function $u_\varepsilon = e^{i\chi_{x_0}(x)} \varphi_\varepsilon$ with $\chi_{x_0}(x) = A(x_0) \cdot x$. Since $\varphi_\varepsilon \in H^1(\mathbb{R}^N, \mathbb{R})$, for a direct calculation, we obtain $u_\varepsilon \in H^1_A(\mathbb{R}^N, \mathbb{C})$ and

$$\begin{aligned}
\int_{\mathbb{R}^N} G(u_\varepsilon) \, dx &= \int_{\mathbb{R}^N} G(e^{i\chi_{x_0}(x)} \varphi_\varepsilon) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \Big(F(|e^{i\chi_{x_0}(x)} \varphi_\varepsilon|^2) - |e^{i\chi_{x_0}(x)} \varphi_\varepsilon|^2 \Big) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \Big(F(|\varphi_\varepsilon|^2) - |\varphi_\varepsilon|^2 \Big) \, dx \\
&= \int_{\mathbb{R}^N} G(\varphi_\varepsilon) \, dx = 1.
\end{aligned}$$

Now, by the continuity assumption in (A1) we obtain $|A(x) + A(x_0)|^2 \leq c$ in the $\operatorname{supp}(\varphi_\varepsilon)$, if we suppose $\|\varphi_\varepsilon\|_2 = o(\varepsilon)$, we deduce

$$\begin{aligned}
D_A \leq \int_{\mathbb{R}^N} |\nabla_A u_\varepsilon|^2 dx &= \int_{\mathbb{R}^N} |\nabla \varphi_\varepsilon|^2 dx + \int_{\mathbb{R}^N} |A(x) + A(x_0)|^2 |\varphi_\varepsilon|^2 dx \\
&\leq D_0 + o(\varepsilon)
\end{aligned}$$

which completes the proof. $\square$

From Lemmas 3.9 and 3.10 we obtain easily the next result.

Lemma 3.11. *Let* (A1), (A4)–(A6) *be satisfied. If* $\nu_i > 0$ *for some index* i, *then* $D_0 \geq D_A \geq 2^{-2/N} S$.

Lemma 3.12. *Suppose* (A1), (A4)–(A7) *be satisfied. If*

$$\lambda > \Big[2^{(2-N)/2} S^{-N/2} N \Big(\frac{2N}{N-2}\Big)^{(N-2)/2}\Big]^{(q-2)/2} c_A^{(q-2)/2}, \tag{3.9}$$

then

$$b_A < \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2} 2^{(N-2)/2} S^{N/2}.$$

Proof. Take $\varphi \in H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ a solution of (2.2). From the definition $b_A = \inf_{\gamma \in \Gamma_A} \max_{t \in [0,1]} I_A(\gamma(t))$, (2.3), (2.4) and (A7), it follows that

$$b_A \leq \max_{t \geq 0} I_A(t\varphi) \leq \max_{t \geq 0} \big\{\frac{t^2}{2} - \lambda \frac{t^q}{q}\big\} c_A \frac{2q}{q-2} = \frac{c_A}{\lambda^{2/(q-2)}}.$$

By using the lower bound on λ assumed in hypothesis (3.9) we obtain

$$b_A < \frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2} 2^{(N-2)/2} S^{N/2}.$$

This completes the. $\square$

Lemma 3.13. *Assume* (A1)–(A7) *are satisfied. If* (3.9) *holds, namely*

$$\lambda > \Big[2^{(2-N)/2} S^{-N/2} N \Big(\frac{2N}{N-2}\Big)^{(N-2)/2}\Big]^{(q-2)/2} c_A^{(q-2)/2},$$

then the weak limit u *of any minimizing sequence* $\{u_n\}$ *to* D_A *is nontrivial.*

Proof. Let $\{u_n\}$ be a minimizing sequence to D_A. Then Lemma 3.2 states $\{u_n\}$ is bounded in $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$. Then $\{|u_n|\}$ is bounded in $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$ and there exists $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ such that $|u_n| \rightharpoonup |u|$ in $H^1_A(\mathbb{R}^N, \mathbb{R})$. Suppose by contradiction that $|u| = 0$. By Remark 3.7 (jj) based on Lions Concentration Compactness principle we obtain

$$|u_n|^{2^*} \rightharpoonup d\nu = \sum_i \delta_{x_i} \nu_i \quad \text{(in the weak}^* \text{ sense of measures).} \tag{3.10}$$

Since $\{|u_n|\} \subset H^1_{rad}(\mathbb{R}^N, \mathbb{R})$, by [6, Radial Lemma A.II] there exist a radius $R = R(N) > 0$ and a constant $C = C(N) > 0$ both independent of n such that

$$|u_n(x)| \leq C\, |x|^{-(N-1)/2} \quad \text{for } |x| \geq R,$$

or equivalently

$$|u_n(r)| \leq C\, r^{-(N-1)/2} \quad \text{for } r \geq R.$$

Then the sequence $\{|u_n|\}$ is bounded in $L^\infty(B^c_R(0), \mathbb{R})$ for every $R > 0$ or equivalently there exists a constant $M > 0$ such that

$$\|u_n\|_{L^\infty(B^c_R(0), \mathbb{R})} \leq M \quad \text{for every } n \in \mathbb{N}. \tag{3.11}$$

This implies $\{|u_n|\}$ converges strongly to 0 in $L^m(B^c_R(0), \mathbb{R})$ for all $m > 2$ and for any $R > 0$. We prove that also $\nu_{i_0} = 0$. If on the contrary $\nu_{i_0} > 0$, by Lemma 3.11 we obtain $D_0 \geq D_A \geq 2^{-2/N} S$. Since by Lemma 3.3,

$$\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2} (2D_0)^{N/2} \leq b_A,$$

we obtain
$$\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{(N-2)/2}2^{(N-2)/2}S^{N/2} \le b_A.$$
But this last inequality contradicts Lemma 3.12. Then there is no $\nu_i > 0$ for every $i \in \Lambda$. Consequently, by (3.10),
$$|u_n|^{2^*} \rightharpoonup 0 \quad \text{(in the weak}^* \text{ sense of measures)},$$
hence
$$\int_{\mathbb{R}^N} |u_n|^{2^*}\varphi\, dx \to 0, \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{C}).$$
This implies
$$u_n \to 0 \quad \text{in } L^{2^*}_{\rm loc}(\mathbb{R}^N, \mathbb{C}). \tag{3.12}$$
Using the same argument, we have
$$u_n \to 0 \quad \text{in } L^{2^*}(B_R^c(0), \mathbb{C}) \quad \text{for any } R > 0,$$
which together with (3.12) implies $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N, \mathbb{C})$.

Now, we can follow again the arguments used in Lemma 3.4. Indeed, since $\{u_n\} \subset \mathcal{M}_A$ implies
$$1 = \int_{\mathbb{R}^N} G(u_n)\, dx = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|u_n|^2) - |u_n|^2\big)\, dx,$$
by Remark 1.3 it follows that
$$2 + \int_{\mathbb{R}^N} |u_n|^2 dx = \int_{\mathbb{R}^N} F(|u_n|^2)\, dx \le \varepsilon \int_{\mathbb{R}^N} |u_n|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*}\, dx$$
so
$$2 + (1-\varepsilon)\int_{\mathbb{R}^N} |u_n|^2 dx \le C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*}\, dx.$$
By choosing $\varepsilon = 1/2$ we obtain
$$2 \le C_{1/2}\int_{\mathbb{R}^N} |u_n|^{2^*}\, dx \to 0 \quad \text{as } n \to +\infty$$
which is an absurd. Thus, we can conclude that $u \neq 0$. □

4. Proof of Theorem 1.1

Under the assumptions in Theorem 1.1 we show that D_A is attained by u, where u is the non trivial weak limit of the minimizing sequence $\{u_n\}$ to D_A. Indeed, since $\{u_n\}$ is bounded by Lemma 3.2, we have $u_n \rightharpoonup u$ in $H^1_{A,\rm rad}(\mathbb{R}^N, \mathbb{C})$ and being the weak limit u not trivial thanks to Lemma 3.13, we deduce that
$$T_A(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx \le \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx = D_A. \tag{4.1}$$
Now, by Lemmas 3.3, 3.11 and 3.12 we obtain that $\nu_i = 0$ for every i. It remains to prove that $u \in \mathcal{M}$. To do this, first observe that the uniform decay at infinity of $\{u_n\} \subset H^1_{A,\rm rad}(\mathbb{R}^N, \mathbb{C})$ together with (A4) imply the existence of a radius $R > 0$ such that
$$|u_n|^2 - F(|u_n|^2) \ge 0 \quad \text{for any } n \in \mathbb{N} \text{ and in } \mathbb{R}^N \setminus B_R$$
where B_R denotes the ball of radius R centered in 0. Since $u_n \to u$ in $L^{2^*}(B_R, \mathbb{C})$, from [7, Theorem 4.9, Section 4], up to a subsequence, $u_n \to u$ a.e. in B_R and there exists $v \in L^{2^*}(B_R)$ such that $|u_n(x)| \le v(x)$ a.e. in B_R. Moreover we have

$F(|u_n(x)|^2) \to F(|u(x)|^2)$ a.e. and, by Remark 1.3, in correspondence of any $\varepsilon > 0$ we obtain the existence of $\overline{C}_\varepsilon > 0$ such that

$$|F(|u_n|^2)| \leq \varepsilon |u_n|^2 + \overline{C}_\varepsilon |u_n|^{2^*} \leq \varepsilon |v|^2 + \overline{C}_\varepsilon |v|^{2^*}$$

for all $n \in \mathbb{N}$. By the arbitrariness of ε and the Dominated Convergence Theorem, we obtain

$$\int_{B_R} F(|u_n|^2)\, dx \to \int_{B_R} F(|u|^2)\, dx.$$

Now, since

$$\frac{1}{2}\int_{B_R} F(|u_n|^2)\, dx = \frac{1}{2}\int_{B_R} |u_n|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N \setminus B_R} \big(|u_n|^2 - F(|u_n|^2)\big)\, dx + 1,$$

taking into account the above considerations, the properties of limit inferior with respect to the sum of sequences and Fatou's Lemma we infer that

$$\int_{B_R} F(|u|^2)\, dx \geq \frac{1}{2}\int_{B_R} |u^2|\, dx + \frac{1}{2}\int_{\mathbb{R}^N \setminus B_R} \big(|u|^2 - F(|u|^2)\big)\, dx + 1;$$

that is, $\int_{\mathbb{R}^N} G(u)\, dx \geq 1$. If we prove that also that

$$\int_{\mathbb{R}^N} G(u)\, dx \leq 1, \tag{4.2}$$

then we obtain $u \in \mathcal{M}_A$ and $T_A(u) = D_A$, or equivalently,

$$T_A(u) = D_A = \min\Big\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx : u \in H^1_A(\mathbb{R}^N, \mathbb{C}) \setminus \{0\},\ \int_{\mathbb{R}^N} G(u)\, dx = 1\Big\}.$$

To show (4.2), suppose by contradiction that

$$\int_{\mathbb{R}^N} G(u)\, dx > 1.$$

We define $h{:}[0,1] \to \mathbb{R}$ by

$$h(t) = \int_{\mathbb{R}^N} G(tu)\, dx = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|tu|^2) - |tu|^2\big)\, dx \quad \text{for every } t \in [0,1].$$

Now we show that $h(t) < 0$ for t close to 0. Indeed, by Remark 1.3 we obtain

$$\begin{aligned} h(t) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(F(t^2|u|^2) - t^2|u|^2\big)\, dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} \Big(\varepsilon t^2|u|^2 + \overline{C}_\varepsilon t^{2^*}|u|^{2^*}\Big)\, dx - \frac{1}{2}\int_{\mathbb{R}^N} t^2|u|^2 dx \\ &= \frac{1}{2}\overline{C}_\varepsilon t^{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\, dx - \frac{1}{2}\, t^2(1-\varepsilon)\int_{\mathbb{R}^N} |u|^2 dx. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, e.g. $\varepsilon < 1$, we obtain $h(t) < 0$ for $t > 0$ small enough. Clearly, $h(1) = \int_{\mathbb{R}^N} G(u)\, dx > 1$. Then, by the continuity of h, there exists $t_0 \in (0,1)$ such that $h(t_0) = 1$ which gives

$$\int_{\mathbb{R}^N} G(t_0 u)\, dx = 1 \Longleftrightarrow t_0 u \in \mathcal{M}_A.$$

Consequently, by (4.1),

$$D_A \leq T_A(t_0 u) = \frac{t_0^2}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx = t_0^2 T_A(u) \leq t_0^2 D_A < D_A$$

which is absurd. Thus, $T_A(u) = D_A$ and $u \in \mathcal{M}_A$; that is,

$$D_A = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A u|^2 dx, \quad J(u) = \int_{\mathbb{R}^N} G(u)\,dx = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|u|^2) - |u|^2\big)\,dx = 1.$$

By Lagrange Multipliers Theorem, there exists a multiplier $\theta \in \mathbb{R}$ such that

$$T_A'(u) = \theta J'(u);$$

in particular, for every $v \in H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ we obtain $T_A'(u)v = \theta J'(u)v$, namely

$$\mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}\,dx = \theta\,\mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u|^2)u - u\big)\,\overline{v}\,dx. \tag{4.3}$$

By adapting the arguments in Berestycki and Lions [6], we are able to prove that $\theta > 0$. Indeed, first remark that $\theta \neq 0$; if not, namely if $\theta = 0$ we would have $T_A'(u) = 0$ and in particular $\int_{\mathbb{R}^N} |\nabla_A u|^2 dx = 0$. Therefore, $u = 0$ which is impossible.

Specifically, it results that $\theta > 0$. Indeed, suppose by contradiction that $\theta < 0$. Moreover, observe that $J'(u) \neq 0$; otherwise,

$$J'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u|^2)u - u\big)\,\overline{v}\,dx = 0$$

would imply $f(|u|^2)u - u = 0$ then $F(|u|^2) - |u|^2 = 0$ which leads to a contradiction with $J(u) = 1$.

Now let us consider a test function w such that

$$J'(u)w = \mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u|^2)u - u\big)\,\overline{w}\,dx > 0.$$

Since $J(u + \varepsilon w) \cong J(u) + \varepsilon J'(u)w$ and

$$T_A(u + \varepsilon w) \cong T_A(u) + \varepsilon\theta J'(u)w \quad \text{for } \varepsilon \to 0 \text{ and } \theta < 0,$$

it is possible to choose $\varepsilon > 0$ small enough so that $v = u + \varepsilon w$ satisfies $J(v) > J(u) = 1$ and $T_A(v) < T_A(u) = D_A$. Now, by a scale change $v_\sigma(x) = v(x/\sigma)$, there exists $0 < \sigma < 1$ such that

$$J(v_\sigma) = \frac{1}{2}\int_{\mathbb{R}^N} \big(F(|v_\sigma|^2) - |v_\sigma|^2\big)\,dx = \sigma^N J(v) = 1$$

and, thanks to assumption (A2) we obtain

$$T_A(v_\sigma) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_A v_\sigma|^2 dx = \sigma^{N-2} T_A(v) < D_A$$

which is absurd. Then, $\theta > 0$. Then u in $H^1_{A,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ satisfies (in the weak sense)

$$-\Delta_A u = \theta\big(f(|u|^2)u - u\big). \tag{4.4}$$

Now, we aim to prove that by exploiting a suitable change of variable the re-scaled u, say u_θ, satisfies

$$\mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u_\theta \cdot \overline{\nabla_A v}\,dx = \mathrm{Re}\int_{\mathbb{R}^N} \big(f(|u_\theta|^2)u_\theta - u_\theta\big)\,\overline{v}\,dx \tag{4.5}$$

namely, u_θ satisfies (in the weak sense)

$$-\Delta_A u_\theta + u_\theta = f(|u_\theta|^2)u_\theta \tag{4.6}$$

so that u_θ is a solution to (1.1). Indeed, since

$$\mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}\,dx = \mathrm{Re}\int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v}\,dx + \mathrm{Re}\int_{\mathbb{R}^N} \nabla u \cdot \overline{iA(x)v}\,dx$$

$$+\operatorname{Re}\int_{\mathbb{R}^N} iA(x)u\cdot\overline{\nabla v}\,dx+\operatorname{Re}\int_{\mathbb{R}^N} iA(x)u\cdot\overline{iA(x)v}\,dx,$$

by substituting $u(x)=u_\theta(\sqrt{\theta}x)$ (that is, $u_\theta(x)=u\left(\frac{x}{\sqrt{\theta}}\right)$) in (4.3) and by exploiting the change of variable $y=\sqrt{\theta}x$, we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^N}\sqrt{\theta}\nabla_y u_\theta(y)\cdot\overline{\sqrt{\theta}\nabla_y v\big(\frac{y}{\sqrt{\theta}}\big)}\frac{1}{(\sqrt{\theta})^N}\,dy\\
&+\operatorname{Re}\int_{\mathbb{R}^N}\sqrt{\theta}\nabla_y u_\theta(y)\cdot\overline{iA\big(\frac{y}{\sqrt{\theta}}\big)v\big(\frac{y}{\sqrt{\theta}}\big)}\frac{1}{(\sqrt{\theta})^N}\,dy\\
&+\sqrt{\theta}\operatorname{Re}\int_{\mathbb{R}^N} iA\big(\frac{y}{\sqrt{\theta}}\big)u_\theta(y)\cdot\overline{\sqrt{\theta}\nabla_y v\big(\frac{y}{\sqrt{\theta}}\big)}\frac{1}{(\sqrt{\theta})^N}\,dy\\
&+\operatorname{Re}\int_{\mathbb{R}^N} iA\big(\frac{y}{\sqrt{\theta}}\big)u_\theta(y)\cdot\overline{iA\big(\frac{y}{\sqrt{\theta}}\big)v\big(\frac{y}{\sqrt{\theta}}\big)}\frac{1}{(\sqrt{\theta})^N}\,dy\\
&=\theta\operatorname{Re}\int_{\mathbb{R}^N}\big(f(|u_\theta(y)|^2)u_\theta(y)-u_\theta(y)\big)\overline{v\big(\frac{y}{\sqrt{\theta}}\big)}\frac{1}{(\sqrt{\theta})^N}\,dy.
\end{aligned}\tag{4.7}$$

If we simplify $1/(\sqrt{\theta})^N$ and put in evidence θ, it follows that

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^N}\Big(\nabla_y+i\frac{1}{\sqrt{\theta}}A\big(\frac{y}{\sqrt{\theta}}\big)\Big)u_\theta(y)\cdot\overline{\Big(\nabla_y+i\frac{1}{\sqrt{\theta}}A\big(\frac{y}{\sqrt{\theta}}\big)\Big)v\big(\frac{y}{\sqrt{\theta}}\big)}\,dy\\
&=\operatorname{Re}\int_{\mathbb{R}^N}\big(f(|u_\theta(y)|^2)u_\theta(y)-u_\theta(y)\big)\,\overline{v\big(\frac{y}{\sqrt{\theta}}\big)}\,dy.
\end{aligned}$$

By assumption (A2), and since $\frac{1}{\sqrt{\theta}}A\big(\frac{y}{\sqrt{\theta}}\big)=A(y)$ for every $y\in\mathbb{R}^N$, we have

$$\operatorname{Re}\int_{\mathbb{R}^N}\nabla_A u_\theta(y)\cdot\overline{\nabla_A v\big(\frac{y}{\sqrt{\theta}}\big)}\,dy=\operatorname{Re}\int_{\mathbb{R}^N}\big(f(|u_\theta(y)|^2)u_\theta(y)-u_\theta(y)\big)\,\overline{v\big(\frac{y}{\sqrt{\theta}}\big)}\,dy;$$

thus we can conclude u_θ satisfies (4.6).

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