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CHARACTERIZATION OF DOMAINS OF SYMMETRIC AND SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

AIPING WANG, ANTON ZETTL

Abstract. We characterize the two point boundary conditions which determine symmetric ordinary differential operators of any order, even or odd, with complex coefficients and arbitrary deficiency index, in a Hilbert space. The self-adjoint characterizations are a special case.

1. Introduction

We consider the equation

$$My = \lambda w y \quad \text{on } J = (a,b), \quad -\infty \le a < b \le \infty, \tag{1.1}$$

where M is a general symmetric ordinary quasi-differential expression of any order, even or odd.

For the case when M is regular Möller and Zettl [10] characterized the two-point boundary conditions which generate symmetric operator realizations of equation (1.1) in the Hilbert space $H = L^2(J,w)$. Here we extend this result to singular M of even or odd order with complex coefficients and arbitrary deficiency index. Self-adjoint operators have recently been characterized by Wang et al in [14] when one endpoint is regular and by Hao et al in [6, 7] when both endpoints are singular. The symmetric characterizations in [10], and the self-adjoint characterizations in [6, 7] are a special case of our main result.

Our proof is in the spirit of the proofs in [6, 10, 14], but there are some significant differences between even and odd order differential operators and real and complex coefficients. In particular, although our construction of the symmetric operators uses LC solutions for real values of the spectral parameter λ, these solutions cannot be chosen to be real valued in contrast to the even order case with real coefficients. Also the extension of the heavy dose of linear algebra analysis using nonsquare matrices introduced in [10] for regular problems is extended to singular problems. In particular, this involves an extension of the Naimark Patching Lemma and the use of Lagrange brackets in place of quasi-derivatives.

The organization of the paper is as follows: This Introduction is followed by a brief discussion of the basic theory of first order systems of differential equations and their relationship to very general n-th order scalar equations in Section 2. Section 3 discusses the minimal and maximal operators, Section 4 the Lagrange Identity, Section 5 the construction of LC solutions and the decomposition of the

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maximal domain. The characterization of symmetric operators is given in Section 6 and illustrated with examples in Section 7.

2. Preliminaries

In this section we summarize some basic facts about general symmetric quasi-differential equations of even and odd order with real or complex coefficients for the convenience of the reader. For a comprehensive discussion of these equations and their relationship to the classical symmetric (formally self-adjoint) case discussed in the well known books by Coddington and Levinson [1], Dunford and Schwartz [2] and [3, 4, 11, 19] and for the 'special' symmetric quasi-differential expressions studied in Naimark [12], as well as additional references, historical remarks and other comments, notation, definitions, etc., the reader is referred to the recent survey article by Sun and Zettl [21].

These expressions generate symmetric differential operators in the Hilbert space $L^2(J,w)$ and it is these operators which are studied here. Let $J=(a,b)$ be an interval with $-\infty \le a < b \le \infty$ and let $n>1$ be a positive integer (even or odd).

Notation. Let $\mathbb{R}$ denote the real numbers, $\mathbb{N}_2 = \{2,3,4,\dots\}$, $\mathbb{C}$ the complex numbers, $M_{n,k}(X)$ the $n\times k$ matrices with entries from X, $M_n(X)=M_{n,k}(X)$ when $n=k$, $M_{n,1}(X)$ be also denoted by X^n, $M_{n,k}(X)$ be abbreviated by $M_{n,k}$ when $X=\mathbb{C}$; $L(J,\mathbb{R})$ and $L(J,\mathbb{C})$ the Lebesgue integrable real and complex valued functions on J, respectively, $L_{\rm loc}(J,\mathbb{R})$ and $L_{\rm loc}(J,\mathbb{C})$ the real and complex valued functions which are Lebesgue integrable on all compact subintervals of J, respectively. We also use $L_{\rm loc}(J)=L_{\rm loc}(J,\mathbb{C})$ and $L(J)=L(J,\mathbb{C})$. $AC_{\rm loc}(J)$ denotes the complex valued functions which are absolutely continuous on compact subintervals of J and $AC(J)$ denotes the absolutely continuous functions on J. $D(S)$ denotes the domain of the operator S.

Definition 2.1. For $w\in L_{\rm loc}(J,\mathbb{R})$, $w>0$ a.e. in J, $L^2(J,w)$ denotes the Hilbert space of functions $f:J\to\mathbb{C}$ satisfying $\int_J |f|^2 w<\infty$ with inner product $(f,g)_w = \int_J f\overline{g}w$. Such a w is called a 'weight function'.

Let
$$
\begin{aligned}
Z_n(J) := \Big\{ Q=(q_{rs})_{r,s=1}^n &: q_{r,r+1}\neq 0 \text{ a.e. on } J,\ q_{r,r+1}^{-1}\in L_{\rm loc}(J),\\
&1\le r\le n-1,\ q_{rs}=0 \text{ a.e. on } J,\ 2\le r+1<s\le n;\\
&q_{rs}\in L_{\rm loc}(J),\ s\neq r+1,\ 1\le r\le n-1\Big\}.
\end{aligned}
\tag{2.1}
$$
For $Q\in Z_n(J)$, define
$$
V_0 := \{y : J\to\mathbb{C}, y \text{ is measurable}\},
$$
$$
y^{[0]} := y \quad (y\in V_0).
$$
Inductively, for $r=1,\dots,n$, we define
$$
V_r = \{y\in V_{r-1} : y^{[r-1]}\in AC_{\rm loc}(J)\},
$$
$$
y^{[r]} = q_{r,r+1}^{-1}\{y^{[r-1]\prime} - \sum_{s=1}^{r} q_{rs}y^{[s-1]}\} \quad (y\in V_r),
$$
where $q_{n,n+1}:=1$. Then we set
$$
My = M_Q y := i^n y^{[n]} \quad \text{on } J \quad (y\in V_n,\ i=\sqrt{-1}). \tag{2.2}
$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the notations $D(M_Q)$ and $D(Q)$. The function $y^{[r]}$ $(0 \le r \le n)$ is called the r-th quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$.

Remark 2.2. Note that the operator $M : D(Q) \to L_{\rm loc}(J)$ is linear. Also note that the differential expression M_Q in equation (2.2) requires only local integrability assumptions on the coefficients (2.1).

The initial value problem associated with $Y' = QY + F$ has a unique solution.

Proposition 2.3. *Suppose $Q \in Z_n(J)$. For each $F \in (L_{\rm loc}(J))^n$, each α in J and each $C \in \mathbb{C}^n$ there is a unique $Y \in (AC_{\rm loc}(J))^n$ such that*

$$Y' = QY + F \quad \textit{and} \quad Y(\alpha) = C.$$

For a proof of the above proposition, see [20, Chapter 1]. From Proposition 2.3, we immediately infer the following result.

Corollary 2.4. *For each $f \in L_{\rm loc}(J)$, each $\alpha \in J$ and $c_0, \dots, c_{n-1} \in \mathbb{C}$ there is a unique $y \in D(Q)$ such that*

$$y^{[n]} = f \quad \textit{and} \quad y^{[r]}(\alpha) = c_r \quad (r = 0, \dots, n-1).$$

If $f \in L(J)$, J is bounded and all components of Q are in $L(J)$, then $y \in AC(J)$.

Definition 2.5 (Regular endpoints)**.** Let $Q \in Z_n(J)$, $J = (a, b)$. The expression $M = M_Q$ is said to be regular at a if for some c, $a < c < b$, we have

$$\begin{gathered} q_{r,r+1}^{-1} \in L(a,c), \quad r = 1, \dots, n-1; \\ q_{rs} \in L(a,c), \quad 1 \le r, s \le n, \ s \ne r+1. \end{gathered}$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$\begin{gathered} q_{r,r+1}^{-1} \in L(c,b), \ r = 1, \dots, n-1; \\ q_{rs} \in L(c,b), \ 1 \le r, \ s \le n, \ s \ne r+1. \end{gathered}$$

Note that, from (2.1) it follows that if the above hold for some $c \in J$ then they hold for any $c \in J$. We say that M is regular on J, or just M is regular, if M is regular at both endpoints.

An endpoint is called singular if it is not regular.

Remark 2.6. In much of the literature when an endpoint of the underlying interval is infinite the problem is automatically classified as singular; note that in Definition 2.5 $a = -\infty$ or $b = \infty$ is allowed. For any J observe that M is regular on any compact subinterval of J. Although we focus on the singular case because there the results are new but the results hold when each endpoint is either regular or singular.

Next we give the definition of symmetric quasi-differential expressions. For examples and illustrations see [21].

Remark 2.7. The symplectic matrix

$$E_k = ((-1)^r \delta_{r,k+1-s})_{r,s=1}^k, \ k \in \mathbb{N}_2 \tag{2.3}$$

plays an important role in the construction of symmetric quasi-differential expressions as well as in the characterization of symmetric differential operators.

Definition 2.8. Let $Q \in Z_n(J)$ and let $M = M_Q$ be defined as in (2.2). Assume that

$$Q = -E_n^{-1} Q^* E_n. \tag{2.4}$$

Then we call Q a Lagrange symmetric matrix and $M = M_Q$ is called a symmetric differential expression.

3. Minimal and maximal operators

In this section we recall the minimal and maximal operators and their basic properties.

Definition 3.1. Let $Q \in Z_n(J)$ satisfy (2.4) and let $M = M_Q$ be the corresponding symmetric differential expression. The maximal operator $S_{\max}$ generated by M is defined by

$$\begin{aligned} D_{\max} = \Big\{ & y \in L^2(J,w) : y^{[0]}, y^{[1]}, \dots, y^{[n-1]} \text{ are absolutely continuous} \\ & \text{in } J, \text{ and } w^{-1}My \in L^2(J,w) \Big\}, \\ & S_{\max} y = w^{-1}My, \quad y \in D_{\max}. \end{aligned}$$

The minimal operator $S_{\min}$ is defined by

$$S_{\min} = S_{\max}^*.$$

Lemma 3.2. *Suppose M is regular at c. Then for any $y \in D_{\max}$ the limits*

$$y^{[r]}(c) = \lim_{t \to c} y^{[r]}(c)$$

exist and are finite, $r = 0, \dots, n-1$. In particular this holds at any regular endpoint and at each interior point of J. At an endpoint the limit is the appropriate one sided limit.

For a proof of the above lemma see [12, Lemma 2, p.63].

Let $a < c < b$. Below we will also consider (2.2) and the operators generated by it on the intervals (a,c) and (c,b). Note that if $Q \in Z_n(J)$, then it follows that $Q \in Z_n(a,c)$, $Q \in Z_n(c,b)$ and we can study equation (2.2) on (a,c) and (c,b) as well as on $J = (a,b)$. Also (2.4) holds on (a,c) and on (c,b). In particular, the minimal and maximal operators are defined on these two subintervals and we can also study the operator theory generated by (2.2) in the Hilbert spaces $L^2((a,c),w)$ and $L^2((c,b),w)$. Below we will use the notation $S_{\min}(I)$, $S_{\max}(I)$ for the minimal and maximal operators on the interval I for $I = (a,c)$, $I = (c,b)$, $I = (a,b) = J$. The interval $J = (a,b)$ may be omitted when it is clear from the context. So we make the following definition.

Definition 3.3. Let $a < c < b$. Let d_a^+, d_b^+ denote the dimension of the solution space of $My = i\,wy$ lying in $L^2((a,c),w)$ and $L^2((c,b),w)$, respectively, and let d_a^-, d_b^- denote the dimension of the solution space of $My = -i\,wy$ lying in $L^2((a,c),w)$ and $L^2((c,b),w)$, respectively. Then d_a^+ and d_a^- are called the positive deficiency index and the negative deficiency index of $S_{\min}(a,c)$, respectively. Similarly for d_b^+ and d_b^-. Also d^+, d^- denote the deficiency indices of $S_{\min}(a,b)$; these are the dimensions of the solution spaces of $My = iwy$, $My = -iwy$ lying in $L^2((a,b),w)$. If $d_a^+ = d_a^-$, then the common value is denoted by d_a and is called the deficiency index of $S_{\min}(a,c)$, or the deficiency index at a. Similarly for d_b. Note that d_a, d_b are independent of c. If $d^+ = d^-$, then we denote the common value by d and call it the deficiency index of $S_{\min}(a,b)$ or just of $S_{\min}$.

The relationships between d_a, d_b and d are well known and given in the next lemma which is well known, see for example the book [18].

Lemma 3.4. *For $d_a^+, d_b^+, d_a^-, d_b^-, d^+, d^-, d_a, d_b$ defined as Definition 3.3, we have*

(1) $d^+ = d_a^+ + d_b^+ - n$, $d^- = d_a^- + d_b^- - n$;
(2) *if* $d_a^+ = d_a^- = d_a, d_b^+ = d_b^- = d_b$, *then* $[\frac{n+1}{2}] \le d_a, d_b \le n$;
(3) *the minimal operator $S_{\min}$ has self-adjoint extensions in H if and only if $d = d^+ = d^-$. If $d = 0$ then $S_{\min}$ is self-adjoint with no proper self-adjoint extension. In all other cases $S_{\min}$ has an uncountable number of self-adjoint extensions, i.e. there are an uncountable number of operators S in H satisfying*
$$S_{\min} \subset S = S^* \subset S_{\max}.$$

4. Lagrange identity

In the study of boundary value problems the Lagrange identity is fundamental.

Lemma 4.1 (Lagrange identity [11])**.** *Let $Q \in Z_n(J)$ satisfy (2.4) and let $M = M_Q$ be the corresponding differential expression. Let the quasi-derivatives y, $y^{[1]}$, ..., $y^{[n-1]}$ be defined as above. Then for any $y, z \in D(Q)$, we have*
$$\overline{z}My - (\overline{Mz})y = [y,z]', \tag{4.1}$$
where
$$[y,z] = i^n \sum_{r=0}^{n-1} (-1)^{n+1-r} \overline{z}^{[n-r-1]} y^{[r]}.$$
Here $[y,z]$ or just $[\cdot,\cdot]$ is called a Lagrange bracket.

Lemma 4.2. *For any y, z in $D_{\max}$ we have*
$$\int_a^b \{\overline{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a),$$
where $[y,z](b) = \lim_{t\to b^-}[y,z](t)$, and $[y,z](a) = \lim_{t\to a^+}[y,z](t)$, $t \in (a,b)$.

The above lemma follows by integrating (4.1). The finite limits guaranteed by Lemma 4.2 play a fundamental role in the characterization of the symmetric and self-adjoint domains.

Corollary 4.3. *If $M\,y = \lambda wy$ and $M\,z = \overline{\lambda} wz$ on some interval (a,b), then $[y,z]$ is constant on (a,b). In particular, if λ is real and $M\,y = \lambda w\,y$, $Mz = \lambda wz$ on some interval (a,b), then $[y,z]$ is constant on (a,b).*

The above corollary follows directly from (4.1). For real λ, the solutions of (1.1) are not, in general, real-valued. However, the Lagrange bracket of two linearly independent solutions of (1.1) for real λ is a constant. For n even and real coefficients, if there are d linearly independent solutions of (1.1) in H, then there are d linearly independent real-valued solutions in H. This is one of the important differences between the equation (1.1) studied here and the equations studied in [14, 6].

Following Everitt and Zettl [3] we call the next lemma, the Naimark Patching Lemma or just the Patching Lemma.

Lemma 4.4. *Let $Q \in Z_n(J)$ and assume that M is regular on J. Let*
$$\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{C}.$$
Then there is a function $y \in D_{\max}$ such that
$$y^{[r]}(a) = \alpha_r, \quad y^{[r]}(b) = \beta_r \quad (r = 0, \dots, n-1).$$

Corollary 4.5. *Let $a < c < h < b$ and $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{C}$. Then there is a $y \in D_{\max}$ such that y has compact support in J and satisfies:*
$$y^{[r]}(c) = \alpha_r, \quad y^{[r]}(h) = \beta_r \quad (r = 0, \dots, n-1).$$

Proof. The proof in Naimark [12] can easily be adapted to prove the above corollary. □

Corollary 4.6. *Let $a_1 < \dots < a_k \in J$, where a_1 and a_k can also be regular endpoints. Let $\alpha_{jr} \in \mathbb{C}$ $(j = 1, \dots, k; r = 0, \dots, n-1)$. Then there is a $y \in D_{\max}$ such that*
$$y^{[r]}(a_j) = \alpha_{jr} \quad (j = 1, \dots, k;\ r = 0, \dots, n-1).$$

The above corollary follows from repeated applications of Corollary 4.5.

Lemma 4.7. *For d_a, d_b given in Definition 3.3, we have*

(1) *If $r_a(\lambda)$ denotes the number of linearly independent solutions of* (1.1) *lying in $L^2((a, c), w)$ for $\lambda \in \mathbb{R}$, then $r_a(\lambda) \le d_a$. Similarly $r_b(\lambda) \le d_b$.*
(2) *If $r_a(\lambda) < d_a$ or $r_b(\lambda) < d_b$ for some $\lambda \in \mathbb{R}$, then λ is in the essential spectrum of every self-adjoint extension of $S_{\min}$.*

For a proof of the above lemma see [7, 8].

5. LC solutions and the decomposition of the maximal domain

In this section we recall some properties of the maximal and minimal operators, construct limit-circle (LC) solutions and discuss the decomposition of the maximal domain used below in Section 6 to prove our main theorem. The next theorem is well known.

Theorem 5.1. *Let $M = M_Q$, $Q \in Z_n(J)$, $n > 1$, satisfy* (2.4) *and let w be a weight function. Then $D_{\max}(Q)$ is dense in H. Let $S_{\min} = S_{\min}(Q) = S^*_{\max}(Q) = S^*_{\max}$. Then $S_{\min}$ is a closed symmetric operator in H with dense domain and $S^*_{\min} = S_{\max}$.*

Proof. The method of Naimark [12, Chapter V] can be adapted to prove this theorem with minor modifications. See also [3]. □

For the rest of this article we assume that the hypothesis holds.

(H1) Let $a < c < b$ and assume that the equation (1.1) on (a, c) has d_a linearly independent solutions, denoted by $u_1, u_2, \dots, u_{d_a}$, in $L^2((a, c), w)$ for some real $\lambda = \lambda_a$ and that (1.1) has d_b linearly independent solutions, denoted by $v_1, v_2, \dots, v_{d_b}$, in $L^2((c, b), w)$ for some real $\lambda = \lambda_b$. Note that d_a and d_b are independent of c.

Regarding hypothesis (H1), note that $d_a^+ = d_a^- = d_a$, $d_b^+ = d_b^- = d_b$ and $d^+ = d^- = d$. Recall that $r_a(\lambda)$ denotes the number of linearly independent solutions of (1.1) on (a, c) which lie in $L^2((a, c), w)$ for real λ. For any real λ it is known [7, 8] that $r_a(\lambda) \le d_a$ and if $r_a(\lambda) < d_a$ then λ is in the essential spectrum of every self-adjoint extension of $S_{\min}(a, c)$ and of $S_{\min}(a, b)$. Thus if there does not exist a real λ_a such that (1.1) on (a,c) has d_a linearly independent solutions in $L^2((a,c),w)$ then the essential spectrum of all self-adjoint extensions $S_{\min}(a,c)$ and of $S_{\min}(a,b)$ covers the whole real line. Similarly for the endpoint b. If the essential spectrum of every self-adjoint realization of (1.1) in $L^2((a,b),w)$ covers the whole real line then any eigenvalue, if there is one, is embedded in the essential spectrum. In this case the dependence of such eigenvalues on the boundary condition seems to be 'coincidental' and nothing seems to be known, aside from examples, about this dependence.

The next theorem constructs LC solutions at each endpoint.

Theorem 5.2. *Suppose that $Q \in Z_n(J,\mathbb{C})$, $J=(a,b)$, $-\infty \le a < b \le \infty$, is Lagrange symmetric, $M = M_Q$ and w is a weight function. Let $a < c < b$ and assume* (H1) *holds. Consider the equation*

$$My = \lambda w y \quad \text{on } J.$$

Then

(1) *For $m_a = 2d_a - n$ the solutions $u_1, \dots, u_{d_a}$ can be ordered such that the $m_a \times m_a$ matrix $\widehat{U} = ([u_i,u_j](a))_{1\le i,j\le m_a}$ is given by*

$$\widehat{U} = \begin{bmatrix} [u_1,u_1](a) & \dots & [u_{m_a},u_1](a) \\ \dots & \dots & \dots \\ [u_1,u_{m_a}](a) & \dots & [u_{m_a},u_{m_a}](a) \end{bmatrix} = -i^n E_{m_a}$$

and is therefore nonsingular.

(2) *For $m_b = 2d_b - n$ the solutions $v_1, \dots, v_{d_b}$ on (c,b) can be ordered such that the $m_b \times m_b$ matrix $\widehat{V} = ([v_i,v_j](b))_{1\le i,j\le m_b}$ is given by*

$$\widehat{V} = \begin{bmatrix} [v_1,v_1](b) & \dots & [v_{m_b},v_1](b) \\ \dots & \dots & \dots \\ [v_1,v_{m_b}](b) & \dots & [v_{m_b},v_{m_b}](b) \end{bmatrix} = -i^n E_{m_b}$$

and is therefore nonsingular.

(3) *For every $y \in D_{\max}(a,b)$ we have $[y,u_j](a) = 0$ for $j = m_a+1, \dots, d_a$.*

(4) *For every $y \in D_{\max}(a,b)$ we have $[y,v_j](b) = 0$ for $j = m_b+1, \dots, d_b$.*

(5) *For $1 \le i,j \le d_a$, we have $[u_i,u_j](a) = [u_i,u_j](c)$.*

(6) *For $1 \le i,j \le d_b$, we have $[v_i,v_j](b) = [v_i,v_j](c)$.*

(7) *The solutions $u_1, \dots, u_{d_a}$ can be extended to (a,b) such that the extended functions, also denoted by $u_1, \dots, u_{d_a}$, satisfy $u_j \in D_{\max}(a,b)$ and u_j is identically zero in a left neighborhood of b, $j = 1, \dots, d_a$.*

(8) *The solutions $v_1, \dots, v_{d_b}$ can be extended to (a,b) such that the extended functions, also denoted by $v_1, \dots, v_{d_b}$, satisfy $v_j \in D_{\max}(a,b)$ and v_j is identically zero in a right neighborhood of a, $j = 1, \dots, d_b$.*

A proof of the above theorem can be found in [7, Theorem 1].

Definition 5.3. The solutions $u_1, \dots, u_{m_a}$ and $v_1, \dots, v_{m_b}$ are called LC solutions at a and b, respectively. The solutions $u_{m_{a+1}}, \dots, u_{d_a}$ and $v_{m_b+1}, \dots, v_{d_b}$ are called LP solutions at a and b, respectively. The definitions of LC solutions and LP solutions were proposed by Wang et al in [14].

Remark 5.4. Only the LC solutions are used in the construction of the boundary conditions which characterize the self-adjoint and symmetric operators in the Hilbert space $L^2(J,w)$. The LP solutions and the solutions not in this space make no contribution to the construction of the self-adjoint and symmetric boundary conditions.

Our proof of the symmetric operator characterization uses the decomposition of the maximal domain in terms of LC solutions given by the next theorem.

Theorem 5.5 ([7]). *Let the notation and hypotheses of Theorem 5.2 hold. Then*

$$D_{\max}(a,b) = D_{\min}(a,b) \dotplus \mathit{span}\{u_1,\dots,u_{m_a}\} \dotplus \operatorname{span}\{v_1,\dots,v_{m_b}\}. \tag{5.1}$$

6. Symmetric operators

In this section we state and prove our main result: the characterization of two-point boundary conditions which determine symmetric operators in the Hilbert space $L^2(J,w)$. The proof depends on several lemmas; some of these are stated as Theorems because we believe they are of independent interest.

Definition 6.1. Let the hypothesis and notation of Theorem 5.2 hold. For any $y \in D_{\max}$ define

$$Y_{a,b} = \begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix}, \quad Y(a) = \begin{pmatrix} [y,u_1](a) \\ \dots \\ [y,u_{m_a}](a) \end{pmatrix}, \quad Y(b) = \begin{pmatrix} [y,v_1](b) \\ \dots \\ [y,v_{m_b}](b) \end{pmatrix} \tag{6.1}$$

and recall that the Lagrange brackets $[y,u_j](a)$ and $[y,v_j](b)$ exist as finite limits by Lemma 4.2.

Definition 6.2. A matrix $U \in M_{l,2d}$ with rank l, $0 \le l \le 2d$, $2d = m_a + m_b$ is called a boundary condition matrix. And for $y \in D_{\max}$ and $Y_{a,b}$ given by (6.1) the equation

$$UY_{a,b} = 0 \tag{6.2}$$

is called a boundary condition. The null space of U is denoted by $\mathcal{N}(U)$ and $\mathcal{R}(U)$ denotes its range, U^* is its adjoint.

Note that any boundary condition (6.2) can be reduced by elementary matrix operations to the case that the rank of U is the number of its rows.

Definition 6.3. Suppose $U \in M_{l,2d}$ is a boundary condition matrix. Define an operator $S(U)$ in $L^2(J,w)$ by

$$\begin{gathered} D(S(U)) = \big\{y \in D_{\max} : UY_{a,b} = 0\big\}, \\ S(U)y = My \quad \text{for } y \in D(S(U)). \end{gathered} \tag{6.3}$$

Remark 6.4. If $l = 0$, then $U = 0$ and $S(U) = S_{\max}$. If $l = 2d$, and I_{2d} denotes the $2d \times 2d$ identity matrix, then $S(I_{2d}) = S_{\min}$ by Theorem 6.7 below and for any nonsingular boundary condition matrix U we have

$$S(U) = S(I_{2d}) = S_{\min}.$$

Hence for any boundary condition matrix U, $D(S(U))$ is a linear submanifold of $D_{\max}$ and we have

$$S_{\min} \subset S(U) \subset S_{\max}$$

and consequently, since $S_{\max}$ is a closed finite dimensional extension of $S_{\min}$, it follows that every operator $S(U)$ is a closed finite dimensional extension of $S_{\min}$. For which matrices U is $S(U)$ a symmetric operator in $L^2(J,w)$? This is the question we answer below.

We start by recalling the well known abstract von Neumann charaterization of the domain of the adjoint of a densely defined closed symmetric operator in Hilbert space.

Lemma 6.5. *Let T be a closed densely defined symmetric operator on a complex Hilbert space H, and let N_+ and N_- be the deficiency spaces of T. Then we have*

$$D(T^*) = D(T) \dotplus N_+ \dotplus N_-$$

An operator S is a closed symmetric extension of T if and only if there exist closed subspaces F_+ of N_+ and F_- of N_- and an isometric mapping V of F_+ onto F_- such that

$$D(S) = D(T) + \{g + Vg : \ g \in F_+\}.$$

Furthermore, S is self-adjoint if and only if $F_+ = N_+$ and $F_- = N_-$.

Proof. For the definition of deficiency spaces and a proof of the lemma see any classical book on operator theory, e.g. [2, 12, 17]. □

When applied to the minimal operator $S_{\min} = S_{\min}(Q)$, where $Q \in Z_n(J)$ is Lagrange symmetric, the von Neumann formula yields the following result.

Lemma 6.6.

$$D(S_{\max}) = D(S_{\min}) \dotplus N_\lambda \dotplus N_{\overline{\lambda}}, \quad \mathrm{Im}(\lambda) \neq 0,$$

where

$$N_\lambda = \{y \in D(S_{\max}) : M_Q y = \lambda w\, y, \mathrm{Im}(\lambda) \neq 0\}.$$

Since the solution bases of $M_Q\, y = \lambda w\, y$ have dimension d when $\mathrm{Im}(\lambda) \neq 0$, where d is the deficiency index, it is clear that $D_{\max}$ is a $2d$ dimensional extension of $D_{\min}$. Therefore $S_{\min}$ has self-adjoint extensions and every self-adjoint extension is a d dimensional extension. Furthermore, every d dimensional symmetric extension of $S_{\min}$ is self-adjoint. Moreover, every symmetric extension of $S_{\min}$ is an m dimensional extension with

$$0 \leq m \leq d$$

and an $l = 2d - m$ dimensional restriction of $S_{\max}$ with

$$d \leq l \leq 2d.$$

The decomposition of $D_{\max}$ given by Lemma 6.6 is well known [12, 17], and the furthermore and moreover statements follow from Lemma 6.5.

By Lemma 6.6 the operator $S(U)$ is not symmetric if $l < d$. But its adjoint operator $(S(U))^*$ may be symmetric. For example, when $d \neq 0$, $S_{\max}$ is not symmetric but its adjoint $S_{\min} = S^*_{\max}$ is symmetric. When $d = 0$, then $S_{\min} = S_{\max}$ and $S_{\max}$ is symmetric and self-adjoint. So we will continue to study $S(U)$ for $U \in M_{l,2d}$ with rank l for $0 \leq l \leq 2d$.

The next theorem extends the well known characterization of the domain of the minimal operator

$$D_{\min} = \{y \in D_{\max} : y^{[i]}(a) = 0 = y^{[i]}(b), \ i = 0, 1, 2, \dots, n-1\}$$

for regular problems to singular ones.

Theorem 6.7. *Let the notation and hypotheses of Theorem 5.2 hold. Then*

$$\begin{aligned} D_{\min} = \Big\{ y \in D_{\max} : [y, u_j](a) = 0,\ for\ j = 1, \dots, m_a; \\ [y, v_j](b) = 0,\ for\ j = 1, \dots, m_b \Big\}. \end{aligned}$$

Proof. Recall that $S_{\min} = S^*_{\max}$ and $S^*_{\min} = S_{\max}$ and that in the decomposition of $D_{\max}$ given by Theorem 5.5 the u_j are identically 0 in a neighborhood of b and the v_j are identically zero in a neighborhood of a. From the definitions of the maximal and minimal domains and the Lagrange Identity we get

$$\begin{aligned} [y, z](b) - [y, z](a) = 0 \quad \text{for } z \in D_{\max} \text{ and all } y \in D_{\min}, \\ [y, z](b) - [y, z](a) = 0 \quad \text{for } y \in D_{\min} \text{ and all } z \in D_{\max}. \end{aligned}$$

Suppose that $y \in D_{\max}$ with $[y, u_j](a) = 0$, for $j = 1, \dots, m_a$ and $[y, v_j](b) = 0$, for $j = 1, \dots, m_b$. Let $z = z_0 + c_1u_1 + \dots + c_{m_a}u_{m_a} + h_1v_1 + \dots + h_{m_b}v_{m_b}$ where $z_0 \in D_{\min}$. Then

$$[y, z](b) - [y, z](a) = \sum_{j=1}^{m_b} \bar{h}_j [y, v_j](b) - \sum_{j=1}^{m_a} \bar{c}_j [y, u_j](a) = 0$$

and hence $y \in D_{\min}$.

For the converse we assume that $y \in D_{\min}$, then for all $z \in D_{\max}$, $[y, z](b) - [y, z](a) = 0$. Therefore for the functions u_j, $j = 1, 2, \dots, m_a$, $[y, u_j](b) - [y, u_j](a) = 0$, i.e. $[y, u_j](a) = 0$. Similarly, $[y, v_j](b) = 0$, for $j = 1, \dots, m_b$. □

The next lemma extends the 'Naimark Patching Lemma' 4.4 from regular to singular problems. It says that in our search for solutions of the algebraic equation $UY_{a,b} = 0$ the whole space $\mathbb{C}^{2d}$ is available, i.e. the range of $Y_{a,b}$ as y runs through $D_{\max}$ is the whole space $\mathbb{C}^{2d}$.

Lemma 6.8 (Singular patching lemma)**.** *For any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_{m_a}$, $\beta_1, \beta_2, \dots, \beta_{m_b}$, there exists $y \in D_{\max}$ such that*

$$\begin{gathered} (a) = \alpha_1, \quad [y, u_2](a) = \alpha_2, \quad \dots, \quad [y, u_{m_a}](a) = \alpha_{m_a}, \\ [y, v_1](b) = \beta_1, \quad [y, v_2](b) = \beta_2, \quad \dots, \quad [y, v_{m_b}](b) = \beta_{m_b}. \end{gathered} \tag{6.4}$$

Proof. Consider the equation

$$\begin{pmatrix} [u_1, u_1](a) & \dots & [u_{m_a}, u_1](a) \\ \dots & \dots & \dots \\ [u_1, u_{m_a}](a) & \dots & [u_{m_a}, u_{m_a}](a) \end{pmatrix} \begin{pmatrix} c_1 \\ \dots \\ c_{m_a} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_{m_a} \end{pmatrix},$$

namely

$$\widehat{U} \begin{pmatrix} c_1 \\ \dots \\ c_{m_a} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_{m_a} \end{pmatrix}. \tag{6.5}$$

Since the $\widehat{U}$ defined in Theorem 5.2 is nonsingular, 6.5 has a unique solution $c_1, \dots, c_{m_a}$. Similarly, since $\widehat{V}$ is nonsingular, the following equation

$$\begin{pmatrix} [v_1, v_1](b) & \dots & [v_{m_b}, v_1](b) \\ \dots & \dots & \dots \\ [v_1, v_{m_b}](b) & \dots & [v_{m_b}, v_{m_b}](b) \end{pmatrix} \begin{pmatrix} h_1 \\ \dots \\ h_{m_b} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \dots \\ \beta_{m_b} \end{pmatrix}, \tag{6.6}$$

i.e.

$$\widehat{V}\begin{pmatrix} h_1 \\ \dots \\ h_{m_b} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \dots \\ \beta_{m_b} \end{pmatrix}$$

has a unique solution $h_1, \dots, h_{m_b}$. Set

$$y = y_0 + c_1 u_1 + \dots + c_{m_a} u_{m_a} + h_1 v_1 + \dots + h_{m_b} v_{m_b},$$

where $y_0 \in D_{\min}$. Obviously $y \in D_{\max}(a,b)$ and then

$$\begin{gathered} [y,u_1](a) = c_1[u_1,u_1](a) + c_2[u_2,u_1](a) + \dots + c_{m_a}[u_{m_a},u_1](a) = \alpha_1, \\ [y,u_2](a) = c_1[u_1,u_2](a) + c_2[u_2,u_2](a) + \dots + c_{m_a}[u_{m_a},u_2](a) = \alpha_2, \\ \dots \\ [y,u_{m_a}](a) = c_1[u_1,u_{m_a}](a) + c_2[u_2,u_{m_a}](a) + \dots + c_{m_a}[u_{m_a},u_{m_a}](a) = \alpha_{m_a}. \end{gathered}$$

Similarly,

$$[y,v_1](b) = \beta_1, \quad [y,v_2](b) = \beta_2, \quad \dots, \quad [y,v_{m_b}](b) = \beta_{m_b}.$$

This completes the proof. □

For the benefit of the reader, we include the next two lemmas that show some basic results from linear algebra which are used below. We do not have specific references, but the discussions on pages 7-17 of Horn and Johnson [5] are helpful, and so is Kato [9, Chapter 1].

Lemma 6.9. *If S is a subset of $\mathbb{C}^n$, $n \in \mathbb{N}_2$, then*

(1) *$S^\perp$ is a subspace of $\mathbb{C}^n$.*
(2) *$(S^\perp)^\perp$ = span of S.*
(3) *$(S^\perp)^\perp = S$, if S is a subspace.*
(4) *$n = \dim S^\perp + \dim (S^\perp)^\perp$.*
(5) *Suppose $A \in M_{l,m}$. Then $\mathcal{R}(A) = (\mathcal{N}(A^*))^\perp$ i.e. $Ax = y$ has a solution (not necessarily unique) if and only if $y^* z = 0$ for all $z \in \mathbb{C}^l$ such that $A^* z = 0$.*

Lemma 6.10. *Let G be any invertible $p \times p$ matrix and F an $l \times p$ matrix with $\operatorname{rank} F = l$. Then the following assertions are equivalent:*

(i) *$\mathcal{N}(F) \subset \mathcal{R}(GF^*)$;*
(ii) *$\operatorname{rank}(FGF^*) \le 2l - p$;*
(iii) *$\operatorname{rank}(FGF^*) = 2l - p$;*
(iv) *$\mathcal{N}(F) = GF^*\big(\mathcal{N}(FGF^*)\big)$.*

The next lemma 'connects' the Lagrange identity with the boundary condition 6.2).

Lemma 6.11. *Assume that $U \in M_{l,2d}$, $\operatorname{rank} U = l$, $d \le l \le 2d$. Let $y, z \in D_{\max}$ and define $Y_{a,b}$, $Z_{a,b}$ by (6.1). Let*

$$P = i^n \begin{pmatrix} E_{m_a} & 0 \\ 0 & -E_{m_b} \end{pmatrix} \tag{6.7}$$

and note that $P^{-1} = -P = P^*$. Then $S(U)$ is symmetric if and only if

$$Z_{a,b}^* P Y_{a,b} = 0, \quad \text{for all } y, z \in D(S(U)). \tag{6.8}$$

Proof. By Lemma 4.2 for any $y, z \in D_{\max}$, we have
$$\int_a^b \{\overline{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a).$$
Therefore, it follows from the definition of $S(U)$ given in (6.3) that $S(U)$ is symmetric if and only if for all $y, z \in D(S(U))$,
$$\int_a^b \{\overline{z}S(U)y - y\overline{S(U)z}\} = \int_a^b \{\overline{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a) = 0.$$
By (5.1), functions $y, z \in D_{\max}$ can be represented as
$$\begin{aligned} y &= y_0 + c_1u_1 + c_2u_2 + \dots + c_{m_a}u_{m_a} + h_1v_1 + h_2v_2 + \dots + h_{m_b}v_{m_b}, \\ z &= z_0 + \widehat{c}_1u_1 + \widehat{c}_2u_2 + \dots + \widehat{c}_{m_a}u_{m_a} + \widehat{h}_1v_1 + \widehat{h}_2v_2 + \dots + \widehat{h}_{m_b}v_{m_b}, \end{aligned}$$
where $y_0, z_0 \in D_{\min}$ and $c_j, \widehat{c}_j \in \mathbb{C}$, $j = 1, \dots, m_a$; $h_j, \widehat{h}_j \in \mathbb{C}$, $j = 1, \dots, m_b$. From (6.6), Lemma 6.8 and the definition of $\widehat{V}$ it follows that
$$\begin{aligned} [y,z](b) &= (\overline{\widehat{h}}_1, \overline{\widehat{h}}_2, \dots, \overline{\widehat{h}}_{m_b})\widehat{V}\begin{pmatrix} h_1 \\ \dots \\ h_{m_b} \end{pmatrix} \\ &= \big(\overline{[z,v_1](b)}, \dots, \overline{[z,v_{m_b}](b)}\big)(\widehat{V}^{-1})^*\widehat{V}\widehat{V}^{-1}\begin{pmatrix} [y,v_1](b) \\ \dots \\ [y,v_{m_b}](b) \end{pmatrix} \\ &= -i^n\big(\overline{[z,v_1](b)}, \dots, \overline{[z,v_{m_b}](b)}\big)E_{m_b}\begin{pmatrix} [y,v_1](b) \\ \dots \\ [y,v_{m_b}](b) \end{pmatrix}. \end{aligned}$$
Similarly, (6.6), Lemma 6.8 and the definition of $\widehat{U}$ lead to
$$\begin{aligned} [y,z](a) &= (\overline{\widehat{c}}_1,\ \overline{\widehat{c}}_2, \dots, \overline{\widehat{c}}_{m_a})\widehat{U}\begin{pmatrix} c_1 \\ \dots \\ c_{m_a} \end{pmatrix} \\ &= -i^n(\overline{[z,u_1](a)}, \dots, \overline{[z,u_{m_a}](a)})E_{m_a}\begin{pmatrix} [y,u_1](a) \\ \dots \\ [y,u_{m_a}](a) \end{pmatrix}. \end{aligned}$$
Therefore,
$$\begin{aligned} &[y,z](b) - [y,z](a) = \\ &= \big(\overline{[z,u_1](a)}, \dots, \overline{[z,u_{m_a}](a)}, \overline{[z,v_1](b)}, \dots, \overline{[z,v_{m_b}](b)}\,\big)P\begin{pmatrix} [y,u_1](a) \\ \dots \\ [y,u_{m_a}](a) \\ [y,v_1](b) \\ \dots \\ [y,v_{m_b}](b) \end{pmatrix}. \end{aligned}$$
Hence, the operator $S(U)$ is symmetric if and only if
$$[y,z](b) - [y,z](a) = 0 \quad \text{for all } y, z \in D(S(U)),$$
i.e.
$$Z^*_{a,b}PY_{a,b} = 0 \quad \text{for all } y, z \in D(S(U)).$$
□

Lemma 6.12. *Each of the following statements is equivalent to* (6.8)*:*

(1) *For all* $Y, Z \in \mathcal{N}(U)$, $Z^*PY = 0$*;*
(2) $\mathcal{N}(U) \perp P(\mathcal{N}(U))$*;*
(3) $P(\mathcal{N}(U)) \subset \mathcal{N}(U)^{\perp} = \mathcal{R}(U^*)$*;*
(4) $\mathcal{N}(U) \subset \mathcal{R}(P^{-1}U^*) = \mathcal{R}(PU^*)$.

Proof. Statements (1) and (2) are the same statements, just written differently. The equivalence of (2) and (3) follows from Lemma 6.9. Whereas the equivalence of (3) and (4) immediately follows from the fact that P is an invertible matrix and $P^{-1} = -P$. □

Theorem 6.13. *Let* U *be an* $l \times 2d$ *matrix with* $\operatorname{rank} U = l$*, where* $d \le l \le 2d$, $d = d_a + d_b - n$*. Then the operator* $S(U)$ *is symmetric if and only if*

$$\mathcal{N}(U) \subset \mathcal{R}(PU^*),$$

where P *is defined by* (6.7)*.*

Proof. This follows from the Singular patching lemma 6.8, Lemma 6.11 and Lemma 6.12. □

The result given by the next lemma is not new, it is [7, Theorem 3]. The decomposition (5.1) of the maximal domain plays an important role in our proof of Theorem 6.18. It is based on the construction of LC solutions and the decomposition of the maximal domain due to Wang et al [14], which, in turn, was influenced by a method of Sun [13]. We give this lemma here because of its relationship to Theorem 6.18 and because our proof is different.

Lemma 6.14. *Suppose* $U \in M_{l,2d}$*. Let* $U = (A : B)$ *where* $A \in M_{l,m_a}$ *consists of the first* m_a *columns of* U *in the same order as they are in* U *and* $B \in M_{l,m_b}$ *consists of the other* m_b *columns in the same order as in* U *(recall that* $m_a + m_b = 2d$*) and assume that* $\operatorname{rank} U = l$*. Then the operator* $S(U)$ *is self-adjoint if and only if*

$$l = d \quad \text{and} \quad AE_{m_a}A^* - BE_{m_b}B^* = 0.$$

Proof. It follows from Lemma 6.6 and Theorem 6.13 that $S(U)$ is self-adjoint if and only if $S(U)$ is a d dimensional symmetric extension of the minimal operator $S_{\min}$, i.e. if and only if $l = d$ and $\mathcal{N}(U) \subset \mathcal{R}(PU^*)$. When $l = d$, one has $\dim(\mathcal{N}(U)) = d$ and $\dim(\mathcal{R}(PU^*)) = d$. Hence $\mathcal{N}(U) \subset \mathcal{R}(PU^*)$ is equivalent to $\mathcal{R}(PU^*) \subset \mathcal{N}(U)$, and this is equivalent to $UPU^* = 0$, i.e. $AE_{m_a}A^* - BE_{m_b}B^* = 0$. □

Next we study matrices U such that $(S(U))^*$ is symmetric.

Theorem 6.15. *Let* $U \in M_{l,2d}$*,* $0 \le l \le 2d$ *and assume that* $\operatorname{rank} U = l$*. Then*

$$D((S(U))^*) = \{z \in D_{\max} : Z_{a,b} = \begin{pmatrix} [z, u_1](a) \\ \dots \\ [z, u_{m_a}](a) \\ [z, v_1](b) \\ \dots \\ [z, v_{m_b}](b) \end{pmatrix} \in \mathcal{R}(PU^*)\}.$$

Proof. Let $z \in D_{\max}$. Then $z \in D((S(U))^*)$ if and only if

$$(S_{\max}y, z) = (y, S_{\max}z), \quad \text{for all } y \in D(S(U)).$$

This is equivalent to $Z_{a,b}^* P Y_{a,b} = 0$ for all $y \in D(S(U))$. Therefore $z \in D((S(U))^*)$ if and only if $Y_{a,b}^* P^* Z_{a,b} = 0$, i.e. $P^* Z_{a,b} \in \mathcal{N}(U)^\perp = \mathcal{R}(U^*)$. This completes the proof. □

Lemma 6.16. *Let $U \in M_{l,2d}$ and assume $\operatorname{rank} U = l$ and $0 \le l \le d$. Then the following assertions are equivalent:*

1. *$(S(U))^*$ is symmetric;*
2. *$\mathcal{N}(U) \supset \mathcal{R}(PU^*)$;*
3. *$UPU^* = 0$.*

Proof. From Lemma 6.11 and Theorem 6.15, it follows that $(S(U))^*$ is symmetric if and only if

$$Z_{a,b}^* P Y_{a,b} = 0, \quad \text{for all } y, z \in D((S(U))^*), \tag{6.9}$$

where $Y_{a,b}$, $Z_{a,b} \in \mathcal{R}(PU^*)$ are defined as in (6.1). By Lemma 6.8 and Theorem 6.15, (6.9) is equivalent to $Z^* P Y = 0$ for all $Z, Y \in \mathcal{R}(PU^*)$. Since $P^2 = -I$, this is equivalent to $\mathcal{R}(PU^*) \perp \mathcal{R}(U^*)$. From Lemma 6.9 we know that $\mathcal{R}(U^*) = (\mathcal{N}(U))^\perp$, so that $\mathcal{R}(PU^*) \perp \mathcal{R}(U^*)$ is equivalent to (2), which proves (1) $\Longleftrightarrow$ (2). The equivalence of (2) and (3) can be obtained immediately. □

Lemma 6.17. *Let $U \in M_{l,2d}$ and assume that $\operatorname{rank} U = l$ and $d \le l \le 2d = m_a + m_b$. Then the following statements are equivalent:*

1. *$S(U)$ is a symmetric extension of the minimal operator $S_{\min}$;*
2. *$\mathcal{N}(U) \subset \mathcal{R}(PU^*)$;*
3. *There exists a $d \times 2d$ matrix $\tilde{U}$ satisfying rank $\tilde{U} = d$, $\mathcal{N}(U) \subset \mathcal{N}(\tilde{U})$ and $\tilde{U} P \tilde{U}^* = 0$;*
4. *There exists a $d \times l$ matrix $\tilde{V}$ satisfying $\operatorname{rank} \tilde{V} = d$ and $\tilde{V} UPU^* \tilde{V}^* = 0$;*
5. *$\operatorname{rank}(UPU^*) = 2l - (m_a + m_b) = 2(l-d)$;*
6. *$\operatorname{rank}(UPU^*) \le 2l - (m_a + m_b) = 2(l-d)$;*
7. *$\mathcal{N}(U) = PU^*(\mathcal{N}(UPU^*))$.*

Proof. The equivalence of (1) and (2) is given in Theorem 6.13.

(1) $\Rightarrow$ (3): Note that every symmetric extension of $S_{\min}$ is a restriction of a self-adjoint extension of $S_{\min}$. By (1), $S(U)$ is a symmetric extension of $S_{\min}$, and by Lemma 6.14, $S(\tilde{U})$ is self-adjoint. Therefore (3) holds.

(3) $\Rightarrow$ (2): By matrix algebra and condition (3), we obtain that $\mathcal{N}(U) \subset \mathcal{N}(\tilde{U}) = \mathcal{R}(P\tilde{U}^*)$. It follows from $\mathcal{N}(U) \subset \mathcal{N}(\tilde{U})$ that

$$\mathcal{R}(\tilde{U}^*) = \mathcal{N}(\tilde{U})^\perp \subset \mathcal{N}(U)^\perp = \mathcal{R}(U^*).$$

Thus $\mathcal{R}(P\tilde{U}^*) \subset \mathcal{R}(PU^*)$, and then it follows that $\mathcal{N}(U) \subset \mathcal{R}(PU^*)$. This shows that (2) holds.

(3) $\Rightarrow$ (4): Since $\mathcal{N}(U) \subset \mathcal{N}(\tilde{U})$, we have $\mathcal{R}(U^*) \supset \mathcal{R}(\tilde{U}^*)$. Therefore there exists a $d \times l$ matrix $\tilde{V}$ such that $\tilde{U}^* = U^* \tilde{V}^*$, i.e. $\tilde{U} = \tilde{V} U$. From $\tilde{U} P \tilde{U}^* = 0$, it follows that $\tilde{V} UPU^* \tilde{V}^* = \tilde{U} P \tilde{U}^* = 0$. By $\operatorname{rank} U = l$, one has $\operatorname{rank} \tilde{V} = \operatorname{rank}(\tilde{V} U) = \operatorname{rank} \tilde{U} = d$.

(4) $\Rightarrow$ (3): Set $\tilde{U} = \tilde{V} U$. Then $\tilde{U} P \tilde{U}^* = \tilde{V} UPU^* \tilde{V}^* = 0$. It follows from $\operatorname{rank} U = l$ that $\operatorname{rank} \tilde{U} = \operatorname{rank}(\tilde{V} U) = \operatorname{rank} \tilde{V} = d$. For any $Y \in N(U)$, $\tilde{U} Y = \tilde{V} U Y = 0$ which shows that $\mathcal{N}(U) \subset \mathcal{N}(\tilde{U})$.

The equivalence of (2), (5), (6) and (7) can be obtained by from the Linear Algebra Lemma 6.10. □

Based on the above lemmas and theorems we now obtain our main result: the characterization of symmetric operators in the Hilbert space $L^2(J,w)$ determined by two-point boundary conditions.

Theorem 6.18. *Suppose M is a symmetric differential expression on the interval (a,b), $-\infty \le a < b \le \infty$, of order $n \in \mathbb{N}_2$. Let $a < c < b$. Assume that the deficiency indices of M on (a,c), (c,b) are d_a, d_b, respectively, and hypothesis (H1) holds. Let $u_1, u_2, \dots, u_{m_a}$, $m_a = 2d_a - n$, and $v_1, v_2, \dots, v_{m_b}$, $m_b = 2d_b - n$, be LC solutions on (a,c), (c,b) as constructed by Theorem 5.2, respectively, and extended to maximal domain functions in $D_{\max} = D_{\max}(a,b)$ as in Theorem 5.2. Define $Y_{a,b}$ by (6.1). Assume $U \in M_{l,2d}$ has rank l, $0 \le l \le m_a + m_b = 2d$ and let $U = (A : B)$ with $A \in M_{l,m_a}$ consisting of the first m_a columns of U in the same order as they are in U and $B \in M_{l,m_b}$ consisting of the next m_b columns of U in the same order as they are in U. Define the operator $S(U)$ in $L^2(J,w)$ by (6.3) and let*

$$C = C(A,B) = AE_{m_a}A^* - BE_{m_b}B^*, \text{ and let } r = \operatorname{rank} C.$$

Then we have

1. *If $l < d_a + d_b - n = d$, then $S(U)$ is not symmetric.*
2. *If $l = d_a + d_b - n = d$, then $S(U)$ is self-adjoint (and hence also symmetric) if and only if $r = 0$.*
3. *Let $l = d + s$, $0 < s \le d$. Then $S(U)$ is symmetric if and only if $r = 2s$.*

Proof. Part (1) follows from the abstract von Neumann formula stated by Lemma 6.5 and Lemma 6.6.

Part (2) is given by Lemma 6.14.

Part (3): $d < l \le 2d$. From Lemma 6.17 it follows that $S(U)$ is symmetric if and only if $rankC = rankUPU^* = 2(l-d) = 2s$. □

7. Examples of symmetric operators

In this section, based on Theorem 6.18, we construct examples of symmetric operators for the symmetric expressions M of order 5 based on Section 2 above: Let $Q \in Z_5(J)$ satisfy (2.4) and let $M = M_Q$.

Let $l = \operatorname{rank} U$. By (1) of Theorem 6.18 $S(U)$ is not symmetric when $l < d$. When $l = d$ Lemma 6.14 characterizes the self-adjoint (and therefore also symmetric) operators $S(U)$.

Example 7.1. Let the hypotheses and notation of Theorem 6.18 hold. It follows from Lemma 3.4 that the deficiency indices d_a and d_b satisfy $3 \le d_a, d_b \le 5$.

Assume that $d_a = 4$, $d_b = 5$, then $d = 4$, $m_a = 3$ and $m_b = 5$. In this case, the endpoint a is singular and the endpoint b is regular or limit-circle (LC). The LC solutions at a are u_1, u_2, u_3 and the LC solutions at b are $v_1, v_2, \dots, v_5$. If b is a regular endpoint for this M then, in the discussion below, simply replace $[y,v_1](b)$, $[y,v_2](b)$, $[y,v_3](b)$, $[y,v_4](b)$, $[y,v_5](b)$ with $y(b)$, $y^{[1]}(b)$, $y^{[2]}(b)$, $y^{[3]}(b)$, $y^{[4]}(b)$.

It follows from Theorem 6.18 that if $l = d + s = 4 + s$, $0 < s < 4$, then $S(U)$ is symmetric if and only if $r = rank(AE_{m_a}A^* - BE_{m_b}B^*) = 2s$. We construct examples for each $s = 1, 2, 3$.

(1) If $s = 1$, then $l = 5$ and $r = 2$.

(i) Let

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix}.$$

Then $l=\operatorname{rank}U=\operatorname{rank}(A:B)=5$ and $r=\operatorname{rank}(AE_3A^*-BE_5B^*)=2$. Therefore, by Theorem 6.18, the operator $S(U)$ determined by the following boundary condition is symmetric:

$$\begin{gathered}[y,u_1](a)=0,\quad [y,u_2](a)=0,\\ [y,v_1](b)=0,\quad [y,v_2](b)=0,\quad [y,v_3](b)=0.\end{gathered}$$

Note that the boundary conditions are strictly separated.

(ii) Let

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}.$$

Then $l=\operatorname{rank}U=\operatorname{rank}(A:B)=5$ and $r=\operatorname{rank}(AE_3A^*-BE_5B^*)=2$. By Theorem 6.18, the operator $S(U)$ is symmetric with mixed boundary condition:

$$\begin{gathered}[y,u_2](a)=0,\quad [y,v_3](b)=0,\quad [y,v_4](b)=0,\\ [y,u_1](a)+[y,v_5](b)=0,\quad [y,u_3](a)+[y,v_1](b)=0.\end{gathered}$$

(2) If $s=2$, then $l=6$ and $r=4$.

(i) Let

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}.$$

A direct computation shows that $l=\operatorname{rank}U=\operatorname{rank}(A:B)=6$ and $r=\operatorname{rank}(AE_3A^*-BE_5B^*)=4$. Therefore, the following boundary conditions determine a symmetric operator $S(U)$:

$$\begin{gathered}[y,u_1](a)=0,\quad [y,u_2](a)=0,\quad [y,v_1](b)=0,\\ [y,v_2](b)=0,\quad [y,v_3](b)=0,\quad [y,v_4](b)=0.\end{gathered}$$

(ii) Choose

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}.$$

Then $l = \operatorname{rank} U = \operatorname{rank}(A : B) = 6$ and $r = \operatorname{rank}(AE_3A^* - BE_5B^*) = 4$. Therefore $S(U)$ determined by the following mixed boundary condition is symmetric:

$$\begin{gathered} [y, u_2](a) = 0, \quad [y, v_2](b) = 0, \\ [y, v_3](b) = 0, \quad [y, v_4](b) = 0, \\ [y, u_1](a) + [y, v_5](b) = 0, \quad [y, u_3](a) + [y, v_1](b) = 0. \end{gathered}$$

(3) If $s = 3$, then $l = 7$ and $r = 6$.
(i) Choose

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By a direct computation we have: $l = \operatorname{rank} U = 7$ and $r = \operatorname{rank}(AE_3A^* - BE_5B^*) = 6$. Therefore the operator $S(U)$ determined by the following boundary condition is symmetric:

$$\begin{gathered} [y, u_1](a) = 0, \quad [y, u_2](a) = 0, \\ [y, v_i](b) = 0, \quad i = 1, 2, 3, 4, 5. \end{gathered}$$

Note that this is a symmetric operator with strictly separated boundary conditions: there are 2 conditions at the endpoint a, 5 at b and no coupled condition.

Example 7.2. (ii) Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then one has $l = \operatorname{rank} U = 7$ and $r = \operatorname{rank}(AE_3A^* - BE_5B^*) = 6$. By Theorem 6.18, the following mixed boundary condition determines a symmetric operator:

$$\begin{gathered} [y, u_2](a) = 0, \quad [y, u_3](a) = 0, \\ [y, v_1](b) = 0, \quad [y, v_2](b) = 0, \\ [y, v_3](b) = 0, \quad [y, v_4](b) = 0, \\ [y, u_1](a) = i[y, v_5](b). \end{gathered}$$

Note that here there are 2 separated conditions at a; 4 separated conditions at b and 1 nonreal coupled condition.

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Aiping Wang
School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
E-mail address: wapxf@163.com

Anton Zettl
Math. Dept., Northern Illinois University, DeKalb, IL 60115, USA
E-mail address: zettl@msn.com