Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 147, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF INFINITELY SOLUTIONS FOR A MODIFIED NONLINEAR SCHRÖDINGER EQUATION VIA DUAL APPROACH

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Communicated by Vicentiu D. Radulescu

Abstract. In this article, we focus on the existence of infinitely many weak solutions for the modified nonlinear Schrödinger equation

$$-\Delta u+V(x)u-[\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{\frac{2-\alpha}{2}}}=f(x,u),\quad \text{in } \mathbb{R}^N,$$

where $1\le\alpha<2$, $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$. By using a symmetric mountain pass theorem and dual approach, we prove that the above equation has infinitely many high energy solutions.

1. Introduction

The quasilinear Schrödinger equation

$$-\Delta u+V(x)u-k\Delta(u^2)u=f(x,u),\quad x\in\mathbb{R}^N \tag{1.1}$$

is referred as a modified form of the nonlinear Schrödinger equation

$$iz_t+\Delta z-\omega(x)z+\kappa\Delta(h(|z|^2))h'(|z|^2)z+g(x,z)=0,\quad x\in\mathbb{R}^n, \tag{1.2}$$

where ω is a given potential, h and g are real functions and κ is a real constant. (1.1) is related to the existence of standing waves solutions of (1.2). In fact, let $z(t,x)=e^{-i\beta t}u(x)$, by exploring the Lorentz invariance equation (1.2), we can get a solitary traveling wave and a corresponding equation of elliptic type which has a formal variational structure like (1.1) for suitable ω,h and g.

Many researchers focus on the nonlinear Schrödinger equation (1.2) because it can model many important physical phenomena [16, 17, 32, 35]. If $h(s)=s$, it describes the time evolution of the condensate wave function in superfluid film for plasma physics in Kurihara [16], and if $h(s)=(1+s)^{1/2}$, the equation (1.2) models the self-channeling of a high-power ultrashort laser in matter [14] and the Heidelberg ferromagnetism [40].

Because of the strong physical background, (1.1) has attracted a lot of attention from mathematics science field. In the case of $k=0$, by using the mountain pass theorem (for the impact of the mountain pass theory in nonlinear analysis, we refer

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2010 *Mathematics Subject Classification.* 35J50, 35J92.
Key words and phrases. Modified nonlinear Schrödinger equation; dual approach; critical point theorem; multiplicity; variational methods.

Submitted March 8, 2018. Published July 31, 2018.

reader to see Pucci and Radulescu [36], Ghergu and Radulescu [8]), Bahrouni et al [1] established infinitely many solutions for the following nonlinear Schrödinger equation

$$-\Delta u + V(x)u = a(x)g(u),\ x \in \mathbb{R}^N (N \geq 3),$$

where V and a are functions changing sign and the nonlinearity g has a sublinear growth. Recently, a Schrödinger-Maxwell system involving sublinear terms was studied in [15], and the existence of at least two non-trivial solutions as well as the stability of system was established via to a recent Ricceri-type result. In addition, for the radial case of Schrödinger equations and systems, many excellent works have been reported, we refer the reader to [1, 6, 10, 13, 23, 45, 58, 68]. However, if the Schrödinger equation contains a quasilinear and non-convex diffusion term $\Delta(u^2)u$, some unpredictable difficulties will appear, such as no suitable space where the energy functional is well defined or the functional is not C^1-class except for $N = 1$ (see [34]). In order to overcome these difficulties, Liu, Wang and Wang [19] (see also [5]) introduced a technique of changing variables, i.e., dual approach to rewrite the energy functional with new variable and to find solutions of an auxiliary semilinear equation. Following this technique, many good results on various modified forms (1.1) of (1.2) have been reported, see [7, 31, 51, 52, 57, 59, 61, 63, 64, 65]. Recently, Cheng and Yang [4] studied the model of self-channeling of a high-power ultrashort laser in matter which has form of a nonlinear Schrödinger equation

$$\begin{gathered}-\Delta u + Ku - [\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{\frac{2-\alpha}{2}}} = |u|^{q-1}u + |u|^{p-1}u, \\ u \in H^1(\mathbb{R}^N), \quad K > 0, \quad N \geq 3, \quad \alpha \geq 1, \quad 2 < q+1 < p+1 < \alpha 2^*,\end{gathered} \tag{1.3}$$

by using a change of variables and Mountain pass theorem, the nontrivial solution of the equation (1.3) has been established. However, we notice that the potential $V(x) = K$ is bounded, and the infinitely many solutions with high energy have not been studied for a more general nonlinear term. Thus motivated by the above work, in this paper, we are concerned with the existence of infinitely many high energy solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{\frac{2-\alpha}{2}}} = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $1 \leq \alpha < 2$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $V(x)$ satisfies

(A1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V_0 := \inf V(x) > 0$ and for every $\Lambda > 0$

$$\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) < \Lambda\}) < +\infty,$$

where meas denotes Lebesgue measure in $\mathbb{R}^N$.

Our research is also closely related to some work by Sun et al [44, 46, 47, 48, 49], Mao et al [25, 26, 27, 28], Liu et al [18], Elisandra [9], Shao[42], Shi and Chen [43], Zhang et al [11, 56, 62] and variational methods for ordinary differential equations [20, 21, 22, 60, 66, 67] and partial differential equations [53, 12, 24, 29, 30, 37, 38, 39], where the authors obtained some interesting theoretical results.

At the end of this section, we state a version of symmetric mountain pass theorem due to Rabinowize [41], the proof of our main result will depend on it.

Lemma 1.1. *Let E be an infinite dimensional Banach space and let $I \in C^1(E, \mathbb{R})$ be even, satisfy (PS)-condition, and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional and I satisfies*

(i) *there are constants* $\rho, \delta > 0$ *such that* $I|_{\partial B_\rho \cap X} \geq \delta$, *and*
(ii) *for each finite-dimensional subspace* $E' \subset E$, *there is an* $R = R(E')$ *such that* $I|_{E' \setminus B_R} \leq 0$.

Then I *possesses an unbounded sequence of critical values.*

2. Variational setting and main results

The following notation will be adopted in this article. $L^s(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm

$$\|u\|_s = \Big(\int_{\mathbb{R}^N} |u|^s dx\Big)^{1/s}, \quad 1 \leq s < \infty.$$

Let

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the norm and inner product, respectively,

$$\|u\|_{H^1} = \Big[\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx\Big]^{1/2}, \quad \langle u, v\rangle_{H^1} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) dx.$$

Now under the assumption (A1), we define our work space

$$E = \big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx < +\infty\big\}$$

with the norm and inner product, respectively,

$$\|u\| = \Big[\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx\Big]^{1/2}, \quad \langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + v(x)uv) dx.$$

It is well known that if the assumption (A1) holds, then the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2, 2^*]$ and there exists a constant $c_s > 0$, $2 \leq s \leq 2^*$ such that

$$\|u\|_s \leq c_s \|u\|, \quad \forall u \in E.$$

In addition, from [2, 3], we have the following compactness lemma.

Lemma 2.1. *Under assumption* (A1), *the embedding* $E \hookrightarrow L^s(\mathbb{R}^N)$ *is compact for* $s \in [2, 2^*)$.

Normally, the solutions of (1.4) are the critical points of the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \big[1 + \frac{\alpha^2 u^2}{2(1+u^2)^{2-\alpha}}\big] |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx,$$

where $F(x, s) = \int_0^s f(x, \xi) d\xi$. But the natural associated functional $J(u)$ may not be well defined and is not Gâteaux differentiable functional in the corresponding Sobolev space E. To avoid these obstacles, we introduce a new function so that the dual approach can be used for establishing our results. Let

$$g(t) = \sqrt{1 + \frac{\alpha^2 t^2}{2(1+t^2)^{2-\alpha}}}$$

and make a change of variable

$$v = G(u) = \int_0^u g(t) dt.$$

Clearly, $g(t)$ is monotonous on $|t|$, which implies that the inverse function $G^{-1}(t)$ of $G(t)$ exists, thus similar to [4], we have an equivalent functional for the natural associated functional $J(u)$

$$I(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx-\int_{\mathbb{R}^N}F(x,G^{-1}(v))dx. \tag{2.1}$$

Based on the properties of $G^{-1}(v)$ (see Lemma 2.3 below), $I(\cdot)$ is well defined on E and $I(v)\in C^1(E,\mathbb{R})$ if and only if $\int_{\mathbb{R}^N}F(x,G^{-1}(\cdot))dx$ has the same property as the functional I, i.e., if $\int_{\mathbb{R}^N}F(x,G^{-1}(\cdot))dx$ is continuously differential on E, then $I(v)\in C^1(E,\mathbb{R})$, and for any $w\in C_0^\infty(\mathbb{R})$, we have

$$\langle I'(v),w\rangle=\int_{\mathbb{R}^N}\nabla v\nabla w dx+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}wdx-\int_{\mathbb{R}^N}\frac{f\left(x,G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}wdx.$$

The critical points of I are then weak solutions of the semilinear Schrödinger equation

$$-\Delta v=-V(x)\frac{G^{-1}(v)}{g\big(G^{-1}(v)\big)}+\frac{f\big(x,G^{-1}(v)\big)}{g\big(G^{-1}(v)\big)},\quad x\in\mathbb{R}^N. \tag{2.2}$$

Thus to obtain the existence of the weak solutions for the quasilinear Schrödinger equation (1.4), it is sufficient to study the existence of the weak solutions for the equivalent form (2.2) of (1.4).

In this article, we assume that the nonlinearity f in problem (1.4) satisfies the following assumptions:

- (A2) $f(x,-t)=-f(x,t)$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$.
- (A3) there exists $c>0$ such that $|f(x,t)|\le c(1+|t|^{r-1})$ for some $2\alpha<r<2^*\alpha$, where $2^*=\frac{2N}{N-2}$ if $N\ge 3$ and $2^*=\infty$ if $N=2$.
- (A4) $f(x,t)=o(|t|)$ uniformly in x as $|t|\to 0$.
- (A5) $\lim_{|t|\to\infty}\frac{|F(x,t)|}{|t|^{2\alpha}}=+\infty$ uniformly for $x\in\mathbb{R}^N$, where $F(x,t)=\int_0^t f(x,s)ds$.
- (A6) there exists a constant $\mu>2\alpha$ such that
$$f(x,t)G(t)-\mu F(x,t)g(t)\ge 0,$$
for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$.

On the existence of infinitely many high energy solutions we have the following result.

Theorem 2.2. *Suppose that* (A1)–(A6) *are satisfied. Then the* (1.4) *admits a sequence of weak solutions* $\{u_n\}\subset E$ *such that* $\|u_n\|\to\infty$ *and* $J(u_n)\to\infty$ *as* $n\to\infty$.

To prove our main result, some properties of $G^{-1}(t)$ will be introduced so that we can discuss the geometric structure of I more conveniently.

Lemma 2.3. $g(t)$ *and* $G^{-1}(t)$ *satisfy the following properties:*

- (G1) $g(t)\ge 1,\forall t\in\mathbb{R}$;
- (G2) $|G^{-1}(t)|\le|t|,\forall t\in\mathbb{R}$;
- (G3) $\lim_{t\to 0}\frac{G^{-1}(t)}{t}=1$;
- (G4) *if* $\alpha>1$, *then* $\lim_{t\to\infty}\frac{|G^{-1}(t)|^\alpha}{t}=\sqrt{2}$; *if* $\alpha=1$, *then* $\lim_{t\to\infty}\frac{|G^{-1}(t)|}{t}=\sqrt{\frac{2}{3}}$;

(G5) *there exist a positive constant such that*
$$|G^{-1}(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ Ct^{1/\alpha}, & |t| \ge 1; \end{cases}$$
(G6)
$$\frac{G^{-1}(t)t}{g(G^{-1}(t))} \le (G^{-1}(t))^2, \quad \forall t \in \mathbb{R}.$$
(G7) $|G(t)| \le g(t)|t|$, *for any* $t \in \mathbb{R}$*;*

(G8) *for any* $t \in \mathbb{R}$*, we have* $\frac{tg'(t)}{g(t)} \le T(\alpha)$*, where*
$$T(\alpha) = \begin{cases} \alpha - 1, & \alpha \ge \alpha_1 \approx 1.1586, \\ \frac{\alpha^2}{2}\big(\frac{3-\alpha}{2-\alpha}\big)^{\alpha-3}, & 1 \le \alpha < \alpha_1, \end{cases}$$
especially, for the case $1 \le \alpha < \alpha_1$*, for accuracy,* $T(\alpha)$ *can be taken as* $\rho(s_0)$*, where* s_0 *satisfies* $\rho'(s_0) = 0$ *and* $1 \le \alpha < \alpha_1$*,*
$$\rho(s) = \frac{(\alpha-1)\alpha^2 s(1+s)^\alpha + (2-\alpha)\alpha^2 s(1-s)^{\alpha-1}}{2(1+s^2) + \alpha^2 s(1+s)^\alpha}, \quad s \ge 0.$$
(G9) *for each* $\lambda > 1$*, one has*
$$|G^{-1}(\lambda t)|^2 \le \lambda^2 |G^{-1}(t)|^2, \quad \forall t \in \mathbb{R}.$$
(G10) *the function* $(G^{-1}(t))^2$ *is strictly convex, and especially*
$$|G^{-1}(\lambda t)|^2 \le \lambda |G^{-1}(t)|^2, \quad \forall t \in \mathbb{R},\ \lambda \in [0,1].$$
(G11) *there exists a constant* $c > 0$ *such that* $|G^{-1}(t)|^\alpha \le c|t|$ *for all* $t \in \mathbb{R}$.

Proof. The proof of (G2)–(G4) and (G8) can be found in [4]. By the definition of g and direct calculation, (G1) and (G6) hold. In addition, since G^{-1} is an odd function, (G5) and (G11) are consequences of (G3) and (G4), moreover (G7) is also consequence of [4, 5]. To prove (G9), by (G7), we have
$$t = G(G^{-1}(t)) \le g(G^{-1}(t))G^{-1}(t), \quad \text{for all } t \ge 0.$$
Thus
$$\frac{[(G^{-1}(s))^2]'t}{(G^{-1}(t))^2} = \frac{2G^{-1}(t)(G^{-1}(t))'t}{(G^{-1}(t))^2} = \frac{2G^{-1}(t)t}{g(G^{-1}(t))(G^{-1}(t))^2} \le \frac{2(G^{-1}(t))^2}{(G^{-1}(t))^2} = 2,$$
for all $t \ge 0$. Then
$$\ln\Big(\frac{(G^{-1}(\lambda t))^2}{(G^{-1}(t))^2}\Big) = \int_t^{\lambda t} \frac{[(G^{-1}(s))^2]'}{(G^{-1}(s))^2} ds \le 2\ln\lambda = \ln\lambda^2,$$
for all $t \ge 0$ and $\lambda > 1$, which implies that
$$(G^{-1}(\lambda t))^2 \le \lambda^2 (G^{-1}(t))^2, \quad \text{for all } t \ge 0.$$
Since G^{-1} is an odd function, and $(G^{-1})^2$ is a even function, so the above inequality holds for all $t \in \mathbb{R}$.

In the end, we prove (G10). In fact, for $1 \le \alpha < \alpha_1$, we have that $\phi(\alpha) = \frac{\alpha^2}{2}\big(\frac{3-\alpha}{2-\alpha}\big)^{\alpha-3}$ is increasing and $0 < \phi(\alpha) < 1$, thus by (g_8), for any $1 \le \alpha < 2$ and $s \in \mathbb{R}$, we have
$$\frac{sg'(s)}{g(s)} \le T(\alpha) < 1,$$

which yields

$$[(G^{-1}(s))^2]'' = \frac{2}{g^2(G^{-1}(t))} - \frac{2G^{-1}(t)g'(G^{-1}(t))}{g^3(G^{-1}(t))} > 0.$$

And then, from the convexity of $(G^{-1}(t))^2$, for all $\lambda \in [0,1]$, one gets

$$|G^{-1}(\lambda t)|^2 \le \lambda |G^{-1}(t)|^2, \quad \forall t \in \mathbb{R}.$$

$\square$

Lemma 2.4. *Assume that $\{v_n\} \subset E$ is a (PS)-sequence of I. Then $\{v_n\}$ is bounded in E.*

Proof. Suppose $\{v_n\} \subset E$ is a (PS)-sequence of I, that is

$$I(v_n) \to c, \quad (1+\|v_n\|)I'(v_n) \to 0, \quad \text{as } n \to \infty. \tag{2.3}$$

By using (2.3), (A1), (G6) and (A6), we get

$$\begin{aligned}
c + o(1) &= I(v_n) - \frac{1}{\mu}\langle I'(v_n), v_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}dx + \frac{1}{\mu}\int_{\mathbb{R}^N} \frac{f(x,G^{-1}(v_n))v_n}{g(G^{-1}(v_n))}dx \\
&\quad - \int_{\mathbb{R}^N} F(x, G^{-1}(v_n))dx \qquad (2.4)\\
&\ge \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} \Big(\frac{f(x,G^{-1}(v_n))v_n}{g(G^{-1}(v_n))} - \mu F(x, G^{-1}(v_n))\Big)dx \\
&\ge \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx,
\end{aligned}$$

which implies that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \le C_1. \tag{2.5}$$

Obviously, from (2.5), if there exists a constant $C_2 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \ge C_2\|v_n\|^2, \tag{2.6}$$

then $\{v_n\}$ is bounded in E. To do this, let

$$\|v_n\|_0^2 =: \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx, \tag{2.7}$$

and $v_n \ne 0$ (if $v_n = 0$, the conclusion obviously holds). Suppose (2.6) is not true, then passing to a subsequence, one has

$$\lim_{n\to+\infty} \frac{\|v_n\|_0^2}{\|v_n\|^2} = 0.$$

Set

$$u_n = \frac{v_n}{\|v_n\|}, \quad k_n = \frac{(G^{-1}(v_n))^2}{\|v_n\|^2},$$

then we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(x) k_n(x) dx \to 0, \quad n \to \infty. \tag{2.8}$$

Thus

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to 0, \quad \int_{\mathbb{R}^N} V(x) k_n(x) dx \to 0, \quad \int_{\mathbb{R}^N} V(x) u_n^2 dx \to 1. \tag{2.9}$$

Now according to the strategy in [51, 63], we claim that for each $\varepsilon > 0$, there exists a constant $C_2 > 0$ such that $\operatorname{meas}(B_n) \leq \varepsilon$, where $\operatorname{meas}(\cdot)$ denotes the standard Lebesgue measure and

$$B_n = \{x \in \mathbb{R}^N : |v_n| \geq C_2\}.$$

Otherwise, there exists $\varepsilon_0 > 0$ and a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that for any positive integer n

$$\operatorname{meas}(A_n) \geq \varepsilon_0,$$

where $A_n = \{x \in \mathbb{R}^N : |v_n| \geq n\}$. By (G5) and (V_0), we have

$$\|v_n\|_0^2 \geq \int_{\mathbb{R}^N} V(x) |G^{-1}(v_n)|^2 dx \geq \int_{A_n} V(x) |G^{-1}(v_n)|^2 dx \geq C_3 n^{1/\alpha} \varepsilon_0 \to +\infty,$$

as $n \to \infty$, which contradicts with (2.5), thus our claim is true.

Next notice that if $v_n \in \mathbb{R}^N \setminus B_n$, it follows from (G5), (G9) and (G10) that

$$\frac{C}{C_2^2} v_n^2 \leq \Big(G^{-1}\big(\frac{v_n}{C_2}\big) \Big)^2 \leq C_3 \Big(G^{-1}(v_n) \Big)^2,$$

which implies

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_n} V(x) u_n^2 dx &\leq C_4 \int_{\mathbb{R}^N \setminus B_n} V(x) \frac{\big(G^{-1}(v_n)\big)^2}{\|v_n\|} dx \leq C_4 \int_{\mathbb{R}^N} V(x) k_n(x) dx \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.10}$$

On the other hand, by the absolute equicontinuity of integral [33], there exists $\varepsilon > 0$ such that whenever $\Omega \subset \mathbb{R}^N$ and $\operatorname{meas}(\Omega) < \varepsilon$

$$\int_\Omega V(x) u_n^2 dx \leq \frac{1}{2}. \tag{2.11}$$

Thus it follows from (2.10) and (2.11) that

$$\int_{\mathbb{R}^N} V(x) u_n^2 dx = \int_{B_n} V(x) u_n^2 dx + \int_{\mathbb{R}^N \setminus B_n} V(x) u_n^2 dx \leq \frac{1}{2} + o(1),$$

which implies that $1 \leq \frac{1}{2}$, a contradiction. Thus (2.6) is indeed true, and then $\{v_n\}$ is bounded in E. $\square$

Lemma 2.5. *Assume that $\{v_n\}$ is bounded in E, then for any $v \in E$, there exists a constant $C_5 > 0$ such that*

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla (v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x) \Big[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v))}{g\,(G^{-1}(v))}\Big](v_n - v) dx \\ &\geq C_5 \|v_n - v\|^2. \end{aligned} \tag{2.12}$$

Proof. Let $v_n \neq v$, otherwise, the conclusion is trivial. Set

$$w_n = \frac{v_n - v}{\|v_n - v\|}, \quad h_n(x) = \Big(\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))} - \frac{G^{-1}(v(x)))}{g\left(G^{-1}(v(x))\right)}\Big)/(v_n(x) - v(x)). \tag{2.13}$$

To obtain (2.12), it suffices to prove that there exists a constant $C_5 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x) h_n(x) w_n^2 dx \geq C_5. \tag{2.14}$$

To do this, we argue it by contradiction. Assume that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x) h_n(x) w_n^2 dx \to 0. \tag{2.15}$$

By (G8),

$$\frac{d}{dt}\Big[\frac{G^{-1}(t)}{g(G^{-1}(t))}\Big] = \frac{1}{g^2(G^{-1}(t))} - \frac{G^{-1}(t) g'(G^{-1}(t))}{g^3(G^{-1}(t))} > 0, \tag{2.16}$$

which implies that $\frac{G^{-1}(t)}{g(G^{-1}(t))}$ is strictly increasing and for each $C_6 > 0$ there exists a constant $\delta_1 > 0$ such that

$$\frac{d}{dt}\Big[\frac{G^{-1}(t)}{g(G^{-1}(t))}\Big] \geq \delta_1 \tag{2.17}$$

as $|t| \leq C_6$. Moreover, by the Mean Value Theorem and (2.16), the second equality of (2.13) becomes

$$\begin{aligned} h_n(x) &= \Big(\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))} - \frac{G^{-1}(v(x))}{g\left(G^{-1}(v(x))\right)}\Big)/(v_n(x) - v(x)) \\ &= \frac{d}{dt}\Big[\frac{G^{-1}(t)}{g(G^{-1}(t))}\Big]\Big|_{t = v_n(x) + \theta(v_n(x) + v(x))} \geq 0. \end{aligned} \tag{2.18}$$

It follows from (2.13), (2.14) and (2.18) that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx \to 0, \quad \int_{\mathbb{R}^N} V(x) h_n(x) w_n^2 dx \to 0, \quad \int_{\mathbb{R}^N} V(x) w_n^2 dx \to 1. \tag{2.19}$$

Thus similar to the argument of (2.10) and (2.11), one can get a contradiction. So (2.12) holds. $\square$

Now let $\{e_i\}$ be an orthonormal basis of E and define $X_i = \mathbb{R} e_i$, then $E = \oplus_{i=1}^{\infty} X_i$. Let

$$V_n = \oplus_{i=1}^{n} X_i, \quad W_n = \oplus_{i=n}^{\infty} X_i, \quad n \in \mathbb{Z},$$

then V_n is finite dimensional. By [50, Lemma 3.8], we have the following conclusion.

Lemma 2.6. *Assume* (V_0) *and* $2 \leq s < 2^*$, *then* $\sup_{v \in W_n, \|v = 1\|} \|v\|_s \to 0$ *as* $n \to \infty$.

Lemma 2.7. *Assume* (A1), (A3) *and* (A4) *hold. Then there exist constants* $\rho, \delta > 0$ *and positive integer* $k \geq 1$ *such that* $I|_{\partial S_\rho \cap W_k} \geq \delta$ *and* $I(v)$ *satisfies the (PS)-condition.*

Proof. Firstly, we prove that, for any $v \in S_\rho$, there exists a positive constant C_7 such that

$$\|v\|_0^2 =: \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx \geq C_7 \|v\|^2. \tag{2.20}$$

Otherwise, there exists a sequence $\{v_n\} \subset S_\rho$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \le \frac{1}{n}\|v\|^2,$$

which yields $\|v\|_0^2/\|v\|^2 \to 0$ as $n \to +\infty$. Similar to (2.6), one gets a contraction. Thus (2.20) holds.

On the other hand, for any $\epsilon > 0$, form (A3) and (A4) there exists $C_\epsilon > 0$ such that

$$|f(x,t)| \le \epsilon|t| + C_\epsilon|t|^{r-1}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.21}$$

Moreover it follows from Lemma 2.6 and $2\alpha < r < 2^*\alpha$ that there exists an integer $k \ge 1$ such that

$$\|v\|_2^2 \le C_8\|v\|^2, \quad \|v\|_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} \le C_9\|v\|^{\frac{r}{\alpha}}, \quad \forall v \in W_k. \tag{2.22}$$

Thus for any $v \in W_k$ and $v \in S_\rho$, by (2.20)-(2.22), (G2) and (G11), we have

$$\begin{aligned}
I(v) &\ge \frac{C_7}{2}\|v\|^2 - \epsilon\int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx - C_\epsilon \int_{\mathbb{R}^N} |G^{-1}(v)|^r dx \\
&\ge \frac{C_7}{2}\|v\|^2 - \epsilon\int_{\mathbb{R}^N} |v|^2 dx - \widetilde{C}_\epsilon \int_{\mathbb{R}^N} |v|^{\frac{r}{\alpha}} dx \\
&\ge \frac{C_7}{2}\rho^2 - C_8\epsilon\rho^2 - \widetilde{C}_\epsilon C_9 \rho^{\frac{r}{\alpha}} = \delta > 0,
\end{aligned} \tag{2.23}$$

for small $\epsilon > 0$ and $\rho > 0$, that is $I|_{\partial S_\rho \cap W_k} \ge \delta$.

Let $\{v_n\} \subset E$ be any (PS)-sequence of $I(v)$, i.e; there exists $c > 0$ such that $|I(v_k)| \le c$ and $I'(v_k) \to 0$ as $k \to \infty$. From Lemma 2.4, we know $\{v_n\}$ is bounded in E. Thus, up to a subsequence, we have $v_n \rightharpoonup v$ in E. Moreover, the compactness of embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ $(s \in [2, 2^*))$ implies that $v_n \to v$ in $L^s(\mathbb{R}^N)$ for any $2 \le s < 2^*$ and $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^N$.

According to (G11) and (G5), we have

$$|G^{-1}(t)|^\alpha \le c|t| \le cg(G^{-1}(t))|G^{-1}(t)|,$$

which yields

$$\frac{|G^{-1}(t)|^{\alpha-1}}{g(G^{-1}(t))} \le c. \tag{2.24}$$

Thus by (G1), (G2), (G11) and (2.24), one has

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} \Big[\frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\Big](v_n - v)dx\Big| \\
&\le \int_{\mathbb{R}^N} \Big[\Big|\frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\Big| + \Big|\frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}\Big|\Big]|v_n - v|dx \\
&\le \int_{\mathbb{R}^N} [\epsilon\Big(|G^{-1}(v_n)| + |G^{-1}(v)|\Big) + C_\epsilon\Big(\frac{|G^{-1}(v_n)|^{r-1}}{g(G^{-1}(v_n))} + \frac{|G^{-1}(v)|^{r-1}}{g(G^{-1}(v))}\Big)]|v_n - v|dx \\
&\le \int_{\mathbb{R}^N} [\epsilon\left(|v_n| + |v|\right) + \widetilde{C}_\epsilon\left(|v_n|^{\frac{r-\alpha}{\alpha}} + |v|^{\frac{r-\alpha}{\alpha}}\right)]|v_n - v|dx \\
&\le C_{10}(\|v_n\|_2 + \|v\|_2)\|v_n - v\|_2 + C_{11}\left(\|v_n\|_{\frac{r}{\alpha}}^{\frac{r-\alpha}{\alpha}} + \|v\|_{\frac{r}{\alpha}}^{\frac{r-\alpha}{\alpha}}\right)\|v_n - v\|_{\frac{r}{\alpha}} \\
&= o_n(1).
\end{aligned} \tag{2.25}$$

Thus Lemma 2.5, (2.25) and $I'(v_n) \to 0$ imply

$$\begin{aligned} o(1) &= \langle I'(v_n) - I'(v), v_n - v\rangle \\ &= \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x)[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v))}{g\left(G^{-1}(v)\right)}](v_n - v)dx \\ &\quad - \int_{\mathbb{R}^N} [\frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}](v_n - v)dx \\ &\geq C_5\|v_n - v\|^2 + o_n(1), \end{aligned}$$

which yields $v_n \to v$ in E. The proof is complete. □

Lemma 2.8. *For each finite-dimensional subspace $E' \subset E$, there is a constant $R > \rho$ such that $I|_{E'\setminus B_R} \leq 0$.*

Proof. Suppose that the conclusion of the lemma is not invalid for some finite-dimensional subspace $E' \subset E$. Then there is a sequence $\{v_n\} \subset E'$ such that $\|v_n\| \to \infty$ and $I(v_n) > 0$, that is

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) dx > \int_{\mathbb{R}^N} F(x, G^{-1}(v_n))dx. \tag{2.26}$$

By (G2), we have

$$\int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))dx}{\|v_n\|^2} < \frac{1}{2}. \tag{2.27}$$

On there other hand, let $w_n = \frac{v_n}{\|v_n\|}$. Then up to a sequence, we can assume that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s(\mathbb{R}^N), s \in [2, 2^*)$, $w_n \rightharpoonup w$ for a.e. $x \in \mathbb{R}^N$. Let $\Lambda = \{x \in \mathbb{R}^N : w(x) \neq 0\}$ and $\Lambda_1 = \{x \in \mathbb{R}^N : w(x) = 0\}$, we assert $meas\Lambda = 0$. In fact, if not, by (A5), (G4) and the Fatou's Lemma, one has

$$\int_{\Lambda} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx = \int_{\Lambda} \frac{F(x, G^{-1}(v_n))dx}{(G^{-1}(v_n))^{2\alpha}} \frac{(G^{-1}(v_n))^{2\alpha}}{v_n^2} w_n^2 dx \to +\infty. \tag{2.28}$$

On the other hand, by assumptions (A3)–(A5), there exists a constant $C_{12} > 0$ such that

$$F(x, t) \geq -C_{12}t^2, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

which implies that

$$\int_{\Lambda_1} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx \geq -C_{12} \int_{\Lambda_1} \frac{(G^{-1}(v_n))^2}{\|v_n\|^2} dx \geq -C_{12} \int_{\Lambda_1} w_n^2 dx.$$

Since $w_n \to w$ in $L^2(\mathbb{R}^N)$, by [50], there exists a function $h \in L^2(\mathbb{R}^N)$ such that

$$|w_n(x)| \leq h(x), \quad \text{a.e. } x \in \mathbb{R}^N.$$

Thus Lebesgue's dominated convergence theorem guarantees

$$\liminf_{n\to\infty} \int_{\Lambda_1} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx \geq 0. \tag{2.29}$$

Consequently, (2.28) and (2.29) yield

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx = +\infty, \tag{2.30}$$

which contradicts (2.27). So meas $\Lambda = 0$ and $w(x) = 0$ a.e $x \in \mathbb{R}^N$. According the fact of all norms are equivalent on the finite dimensional space and Sobloev embedding theorem, there is a constant $d > 0$ such that

$$0 = \lim_{n\to\infty} \|w_n\|_2^2 \geq \lim_{n\to\infty} d\|w_n\|^2 = d,$$

a contradiction, and the desired conclusion is obtained. □

Proof of Theorem 2.2. Let $V = V_n$, $X = W_n$, then $E = V \oplus X$, V is a finite-dimensional space. Obviously, by (f_1), we know the functional I is even and $I(0) = 0$. Lemma 2.7 implies that I satisfies (PS)-condition, and by Lemmas 2.7 and 2.8, (i) and (ii) of Lemma 1.1 are also satisfied. Thus, by Lemma 1.1, I possesses a sequence of critical points $\{v_n\} \subset E$ such that $I(v_n) \to \infty$ as $n \to \infty$, i.e., the problem (1.4) has a sequence of solutions $\{u_n\} \subset E$ such that and $\|u_n\| \to \infty$ and $J(u_n) \to \infty$ as $n \to \infty$, where $u_n = G^{-1}(v_n)$. □

3. Further results

In this section, we obtain infinitely many high energy solutions for (1.4) by using some easily verifiable assumptions:

(A7) there exists a constant $M > 0$ such that

$$M_0 = \inf_{(x,t)\in\mathbb{R}^N\times[G^{-1}(M),+\infty)} F(x,t) > 0.$$

Theorem 3.1. *Suppose that* (A2)–(A4), (A6), (A7) *are satisfied. Then* (1.4) *admits a sequence of weak solutions* $\{u_n\} \subset E$ *such that* $\|u_n\| \to \infty$ *and* $J(u_n) \to \infty$ *as* $n \to \infty$.

Proof. We only need to prove that the assumption (A7) is stronger than (A5). To do this, for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, let

$$\varphi(s) = F\Big(x, G^{-1}\big(\frac{t}{s}\big)\Big)s^{\mu}, \quad s \geq 1.$$

It follows from (A6) and (A7) that

$$\begin{aligned}
\varphi'(s) &= f\Big(x, G^{-1}\big(\frac{t}{s}\big)\Big)\big(-\frac{t}{s^2}\big)\big[G^{-1}\big(\frac{t}{s}\big)\big]'s^{\mu} + \mu F\Big(x, G^{-1}\big(\frac{t}{s}\big)\Big)s^{\mu-1} \\
&= \frac{s^{\mu-1}}{g(G^{-1}\big(\frac{t}{s}\big))}\Big[-\frac{t}{s}f\Big(x, G^{-1}\big(\frac{t}{s}\big)\Big) + \mu F\Big(x, G^{-1}\big(\frac{t}{s}\big)\Big)g\Big(G^{-1}\big(\frac{t}{s}\big)\Big)\Big] \\
&\leq 0,
\end{aligned} \tag{3.1}$$

which implies that $\varphi(s)$ is decreasing on $[1, +\infty)$. Thus for $|t| > M$, notice that $\frac{M|t|}{t}$ is an odd function and F is a even function, we have

$$\varphi(1) = F(x, G^{-1}(t)) \geq \varphi\big(\frac{|t|}{M}\big) = F\Big(x, G^{-1}\big(\frac{M|t|}{t}\big)\Big)\big(\frac{|t|}{M}\big)^{\mu} \geq \frac{M_0}{M}|t|^{\mu},$$

for $|t| > M$. Consequently, from (G2), we have

$$\frac{F(x, G^{-1}(t))}{|G^{-1}(t)|^{2\alpha}} \geq \frac{M_0}{M}|t|^{\mu-2\alpha}, \quad |t| > M.$$

Notice $\mu > 2\alpha$, we get

$$\lim_{|t|\to\infty} \frac{F(x, G^{-1}(t))}{|G^{-1}(t)|^{2\alpha}} = +\infty,$$

uniformly for $x \in \mathbb{R}^N$. Further, it follows from (G5) that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{2\alpha}} = +\infty,$$

uniformly for $x \in \mathbb{R}^N$. Consequently, the assumption (A7) implies (A5). The proof is complete. $\square$

In the next theorem, we use the assumption

(A8) $F(x,1) > 0$ for any $x \in \mathbb{R}^N$, and there exists a constant $\sigma > 2\alpha$ such that any $c > 1$,

$$F(x,ct) \ge c^\sigma F(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Theorem 3.2. *Suppose that* (A1)–(A4), (A6), (A8) *are satisfied. Then* (1.4) *admits a sequence of weak solutions* $\{u_n\} \subset E$ *such that* $\|u_n\| \to \infty$ *and* $J(u_n) \to \infty$ *as* $n \to \infty$.

Proof. We consider that (A8) implies (A5). In fact, for any $x \in \mathbb{R}^N$ and $|s| > 1$, we have

$$F(x,|s|) \ge |s|^\sigma F(x,1).$$

Consequently,

$$\frac{F(x,s)}{|s|^{2\alpha}} \ge |s|^{\sigma-2\alpha} F(x,1).$$

It follows from $\sigma > 2\alpha$ and $F(x,1) > 0$ that

$$\lim_{|s|\to\infty} \frac{F(x,s)}{|s|^{2\alpha}} = +\infty,$$

uniformly for $x \in \mathbb{R}^N$. Consequently, the assumption (A8) implies (A5). The proof is complete. $\square$

In the next theorem, we use the assumption

(A9) Assume $\widetilde{F}(x,t) = \frac{1}{4} f(x,t)t - F(x,t) \ge 0$, and there exist constants $c > 0$ and $\frac{2^*}{2^*-1} < \sigma < 2$ such that

$$\widetilde{F}(x,t) \ge c\Big|\frac{F(x,t)}{t}\Big|^\sigma, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R} \quad \text{with } t \text{ large enough.}$$

Theorem 3.3. *Suppose that* (A1)–(A3), (A5), (A9) *are satisfied. Then* (1.4) *admits a sequence of weak solutions* $\{u_n\} \subset E$ *such that* $\|u_n\| \to \infty$ *and* $J(u_n) \to \infty$ *as* $n \to \infty$.

Following the method in [51, 63], the theorem above can be obtained directly.

Acknowledgments. The authors were supported financially by the National Natural Science Foundation of China (11571296).

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