Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 113, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF WEAK SOLUTIONS FOR QUASILINEAR PARABOLIC SYSTEMS IN DIVERGENCE FORM WITH VARIABLE GROWTH

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Communicated by Vicentiu D. Radulescu

Abstract. In this article we study the existence of weak solutions for quasilinear parabolic system in divergence form with variable growth. By means of Young measures, Galerkin's approximation method and the theory of variable exponents spaces, we obtain the existence of weak solutions.

1. Introduction and statement of main result

The spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ were first discussed by Kováčik and Rákosník in [24]. Lately, a lot of attention has been paid to the study of various mathematical problems with variable exponent growth conditions; see [9, 11, 12, 14, 16] for the properties of such spaces, and [6, 15, 20] for applications of variable exponent spaces on partial differential equations. The theory of variable exponent spaces has been driven by various problems in nonlinear elastic mechanics, imaging processing, electrorheological fluids and other physics phenomena; see for example [1, 2, 3, 7, 27, 39].

When $p(x)$ is a constant function, Norbert Hungerbühler studied the following problem in [22]:

$$\begin{gathered} -\operatorname{div}\sigma\big(x,u(x),Du(x)\big)=f, \quad x\in\Omega \\ u(x)=0, \quad x\in\partial\Omega \end{gathered} \tag{1.1}$$

The classical monotone operator methods developed by [5, 25, 29, 36] cannot be applied here. Norbert Hungerbühler obtain the existence of weak solutions for (1.1) by Young measures which were proposed by Young in [38]. Many applications and developments of Young measures to the calculus of variations, optimal control theory and nonlinear partial differential equations are presented by MacShane, Gamkrelidze and Tarter in [21, 26, 35, 37]. Inspired by the works mentioned above, results from [22] were extended in [18] to the case that σ satisfies variable growth conditions, by Young measures generated by sequences in variable exponent spaces; see [18, 19] for the basic theorems and properties.

2010 *Mathematics Subject Classification.* 35D30, 35K59, 46E35.
Key words and phrases. Variable exponent; Young measures; weak solutions; quasilinear parabolic system; Galerkin's approximation.

Submitted November 16, 2017. Published May 10, 2018.

In this article, we consider the initial and boundary value problem for the quasilinear parabolic system:

$$
\begin{gathered}
\frac{\partial u}{\partial t} - \operatorname{div} \sigma\big(x, t, Du(x,t)\big) = -\operatorname{div} f, \quad (x,t) \in Q \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\
u(x,0) = u_0(x), \quad x \in \Omega\,,
\end{gathered} \tag{1.2}
$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open domain, $u : \Omega \times (0,T) \to \mathbb{R}^m$, $0 < T < \infty$, $Q = \Omega \times (0,T)$, $p(x)$ is Lipschitz continuous and $1 < p_- := \inf_{x\in\bar\Omega} p(x) \leq p(x) \leq p_+ := \sup_{x\in\bar\Omega} p(x) < \infty$, $f \in L^{p'(x)}(Q; \mathbb{M}^{m\times N})$, $u_0 \in L^2(\Omega; \mathbb{R}^m)$, σ satisfies the conditions (H1)–(H3) below. Inspired by [8], we consider that $p(x)$ only depends x in this problem. The related definition and properties will be given in section 2. In our paper, we denote by $\mathbb{M}^{m\times n}$ the real vector space of $m \times n$ matrices equipped with the inner product

$$
M \circ N := \sum_{1\leq i\leq m,\, 1\leq j\leq n} M_{ij} N_{ij}.
$$

Now we give conditions required for σ in (1.2).

(H1) (Continuity) $\sigma : \Omega \times (0,T) \times \mathbb{M}^{m\times N} \to \mathbb{M}^{m\times N}$ is a Carathéodory function, i.e. $(x,t) \mapsto \sigma(x,t,\xi)$ is measurable for every $\xi \in \mathbb{M}^{m\times N}$ and $\xi \mapsto \sigma(x,t,\xi)$ is continuous for almost every $(x,t) \in Q$.

(H2) (Growth and coercivity) There exist $c_1 \geq 0, c_2 > 0, 0 < a \in L^{p'(x)}(Q), b \in L^1(Q)$, such that

$$
\begin{gathered}
|\sigma(x,t,\xi)| \leq a(x,t) + c_1 |\xi|^{p(x)-1}, \\
\sigma(x,t,\xi) \circ \xi \geq -b(x,t) + c_2 |\xi|^{p(x)}.
\end{gathered}
$$

(H3) (Monotonicity) σ satisfies one of the following conditions:
(i) For all $(x,t) \in Q$, $\xi \mapsto \sigma(x,t,\xi)$ is a C^1-function and is monotone, i.e. for all $(x,t) \in Q$ and $\xi, \eta \in \mathbb{M}^{m\times N}$, we have

$$
\big(\sigma(x,t,\xi) - \sigma(x,t,\eta)\big) \circ (\xi - \eta) \geq 0.
$$

(ii) There exists a function $W : \Omega \times (0,T) \times \mathbb{M}^{m\times N} \to \mathbb{R}$ such that $\sigma(x,t,\xi) = D_\xi W(x,t,\xi)$, and $\xi \to W(x,t,\xi)$ is convex and C^1 for all $(x,t) \in Q$.
(iii) σ is strictly monotone, i.e. σ is monotone and $\big(\sigma(x,t,\xi) - \sigma(x,t,\eta)\big) \circ (\xi - \eta) = 0$ implies $\xi = \eta$.
(iv)

$$
\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,\lambda) - \sigma(x,t,\overline{\lambda})\big) \circ (\lambda - \overline{\lambda})\, \mathrm{d}\nu_{(x,t)}(\lambda)\, \mathrm{d}x\, \mathrm{d}t > 0
$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, I \rangle$, $\nu = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a bounded sequence in $L^{p(x)}(Q)$ and not a Dirac measure for a.e. $(x,t) \in Q$.

Our main result is as follows:

Theorem 1.1. *If σ satisfies conditions* (H1)–(H3)*, then problem* (1.2) *has a weak solution for every $f \in L^{p(x)}(Q; \mathbb{M}^{m\times N})$ and every $u_0 \in L^2(\Omega; \mathbb{R}^m)$.*

Condition (H2) states the variable growth and coercivity condition. (H3)(iv) is weaker than typical strictly monotone condition, even than the p-quasimonotone condition introduced by Norbert Hungerbühler in [22] when $p(x)$ is a constant.

This article is organized as the following: In Section 2, several important properties on variable exponent spaces and the theory of Young measures will be presented. In Section 3, we will give the Galerkin approximation and necessary priori estimates. In section 4, the existence of weak solutions for problem (1.2) will be proved; the conclusions will be given in section 5.

2. Preliminaries

In this section, we recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$.

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,+\infty)$, where $\Omega\subset\mathbb{R}^n(n\geq 2)$ is a nonempty open subset. Denote

$$\rho_{p(x)}(u)=\int_\Omega |u(x)|^{p(x)}dx, \tag{2.1}$$

$$\|u\|_{p(x)}=\inf\{t>0:\rho_{p(x)}(\frac{u}{t})\leq 1\}. \tag{2.2}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(t_0u)<\infty$ for some $t_0>0$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.2). (2.1) is called the modular of u in $L^{p(x)}(\Omega)$.

For a given $p(x)\in\mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x)=\begin{cases}\infty, & \text{if } x\in\Omega_1=\{x\in\Omega:p(x)=1\};\\ \frac{p(x)}{p(x)-1}, & \text{for other } x\in\Omega.\end{cases}$$

Lemma 2.1 ([9]). *Let $p\in\mathbf{P}(\Omega)$, then*

$$\int_\Omega |u(x)\cdot v(x)|dx\leq 2\|u\|_{p(x)}\|v\|_{p'(x)}$$

for every $u\in L^{p(x)}(\Omega)$ and every $v\in L^{p'(x)}(\Omega)$.

In the rest of this section, for every $p\in\mathbf{P}(\Omega)$, we assume that $1\leq p_-\leq p(x)\leq p_+<\infty$.

Lemma 2.2 ([16]). *For every $u\in L^{p(x)}(\Omega)$, we have:*

1. *If $\|u\|_{p(x)}\geq 1$, then $\|u\|_{p(x)}^{p_-}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p_+}$.*
2. *If $\|u\|_{p(x)}<1$, then $\|u\|_{p(x)}^{p_+}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p_-}$.*

Lemma 2.3 ([16]). *If $p_->1$, $L^{p(x)}(\Omega)$ is reflexive, and the dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$.*

Lemma 2.4 ([24]). *Let $|\Omega|<\infty$, where $|\Omega|$ denotes the Lebesgue measure of Ω, $p_1(x),p_2(x)\in\mathbf{P}(\Omega)$, then a necessary and sufficient condition for $L^{p_2(x)}(\Omega)\subset L^{p_1(x)}(\Omega)$ is that $p_1(x)\leq p_2(x)$ for almost every $x\in\Omega$, and in this case the embedding is continuous.*

We assume that $\Omega\subset\mathbb{R}^d$ is a bounded domain, $(0,T)\subset\mathbb{R}$, T is a fixed real number, $Q=\Omega\times(0,T)$, p is a Lipschitz function. We define

$$X(Q):=\Big\{u\in L^2(Q)^d:\nabla u\in L^{p(\cdot,\cdot)}(Q)^{d\times d}, u(\tau,\cdot)\in V_\tau(\Omega)\text{ a.e. }\tau\in(0,T)\Big\}$$

The norm on $X(Q)$ is given by

$$\|u\|_{X(Q)} := \|u\|_{L^2(Q)^d} + \|\nabla u\|_{L^{p(\cdot,\cdot)}(Q)^{d\times d}}$$

where for all $\tau \in (0,T)$, we have

$$V_\tau(\Omega) := \{u \in L^2(\Omega)^d \cap W_0^{1,1}(\Omega)^d : \nabla u \in L^{p(\tau,\cdot)}(\Omega)^{d\times d}\}$$

The norm on $V_\tau(\Omega)$ is defined by

$$\|u\|_{V_\tau(\Omega)} := \|u\|_{L^2(\Omega)^d} + \|\nabla u\|_{L^{p(\tau,\cdot)}(\Omega)^{d\times d}}$$

Lemma 2.5 ([10]). *The space $X(Q)$ is a Banach space under the norm $\|\cdot\|_{X(Q)}$; and $C_0^\infty(Q)$ is density in $X(Q)$.*

Lemma 2.6 ([10]). *The space $X(Q)$ is reflexible.*

Lemma 2.7 ([10]). *The dual space $X'(Q)$ is isomorphic to the subspace of $\mathcal{D}'(Q)$ consisting of distributions T of the form*

$$\|T\| := \inf\Big\{\|g\|_{L^2(Q)^d} + \|G\|_{L^{p(\cdot,\cdot)}(Q)^{d\times d}} : T = g - \operatorname{div} G\Big\},$$

where $g \in L^2(Q)^d$.

Lemma 2.8 ([4]). *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $z_j : \Omega \to \mathbb{R}^m$, $j = 1,2,\dots$, be a sequence of Lebesgue measurable functions. Then there exists a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of nonnegative Radon measures on $\mathbb{R}^n$, such that*

(i) *$\|\nu_x\| := \int d\nu_x \le 1$ for almost every $x \in \Omega$.*
(ii) *$\varphi(z_k) \rightharpoonup^* \bar\varphi$ weakly* in $L^\infty(\Omega)$ for any $\varphi \in C_0(\mathbb{R}^m)$, where $\bar\varphi(x) = \langle \nu_x, \varphi\rangle$ and $C_0(\mathbb{R}^m) = \{\varphi \in C(\mathbb{R}^m) : \lim_{|z|\to\infty} |\varphi(z)| = 0\}$.*
(iii) *If for any $R > 0$*

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \operatorname{meas}\{x \in \Omega \cap B(0,R) : |z_k(x)| \ge L\} = 0,$$

then $\|\nu_x\| = 1$ for almost every $x \in \Omega$, and for any measurable $A \subset \Omega$ there holds $\varphi(z_k) \rightharpoonup \bar\varphi = \langle \nu_x, \varphi\rangle$ weakly in $L^1(A)$ for continuous φ provided the sequence $\varphi(z_k)$ is weakly precompact in $L^1(A)$.

Lemma 2.9 ([4]). *If $\operatorname{meas}\Omega < \infty$ and ν_x is a Young measure generated by the sequence $\{u_j\}$, then u_j converges by measures to u if and only if for a.e. $x \in \Omega$ we have $\nu_x = \delta_{u(x)}$.*

Lemma 2.10 ([5]). *Let $\{f_j\}$ be a uniformly boundedness in $L^1(\Omega)$,*

$$\sup_j \|f_j\|_{L^1(\Omega)} = C < \infty.$$

There exists a subsequence, not relabled, a nonincreasing sequence of measurable sets Ω_n, $\Omega_{n+1} \subset \Omega_n$, and $f \in L^1(\Omega)$ such that

$$f_j \rightharpoonup f \quad in\ L^1(\Omega \setminus \Omega_n)$$

for all n.

Lemma 2.11 ([5]). *If $\{z_j\}$ is a sequence of measurable functions with associated Young measure $\nu = \{\nu_x\}_{x\in\Omega}$,*

$$\liminf_{j\to\infty} \int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x \ge \int_E \int_{\mathbb{R}^m} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x,$$

for every nonnegative, Carathéodory function ψ and every measurable subset $E \subset \Omega$.

The above theorem is obtained in [31] by proving a complicated lemma. We will give a much easier proof by using the contradiction method.

Proof. Assume that

$$\liminf_{j\to\infty}\int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x < \infty.$$

Then $\psi(x, z_j(x))$ is a bounded sequence in $L^1(E)$. Let

$$\bar{\psi}(x) = \int_{\mathbb{R}^m} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda).$$

By Lemmas 2.8 and 2.10, there exists $E_n \subset \Omega$, $E_{n+1} \subset E_n$, $\mathrm{meas}E_n \to 0$ as $n \to \infty$, such that

$$\int_{E\setminus E_n} \psi\Big(x, z_j(x)\Big)\,\mathrm{d}x \to \int_{E\setminus E_n} \bar{\psi}\,\mathrm{d}x \tag{2.3}$$

as $j \to \infty$ for all n. On the other hand, it is apparent that

$$\int_{E\setminus E_n} \bar{\psi}\,\mathrm{d}x \to \int_E \bar{\psi}\,\mathrm{d}x$$

as $n \to \infty$. Now we show the proof of our conclusion by contradiction. Assume that

$$\liminf_{j\to\infty}\int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x < \int_E\int_{\mathbb{R}^m} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x.$$

Let

$$a := \int_E\int_{\mathbb{R}^m} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x - \liminf_{j\to\infty}\int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x > 0.$$

Since $\lim_{n\to\infty}\int_{E\setminus E_n}\bar{\psi}\,\mathrm{d}x = \int_E \bar{\psi}\,\mathrm{d}x$, for $a > 0$, there exist n_0 which is large enough, such that

$$\int_E \bar{\psi}\,\mathrm{d}x - \int_{E\setminus E_{n_0}} \bar{\psi}\,\mathrm{d}x < a = \int_E\int_{\mathbb{R}^m} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x - \liminf_{j\to\infty}\int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x.$$

Therefore

$$\int_{E\setminus E_{n_0}} \bar{\psi}\,\mathrm{d}x > \liminf_{j\to\infty}\int_E \psi\big(x, z_j(x)\big)\,\mathrm{d}x.$$

Combining this with (2.3) leads to a contradiction. ☐

Lemma 2.12 ([18])**.** *If $\{u_j\}$ is bounded in $L^{p(x)}(\Omega,\mathbb{R}^m)$, then $\{u_j\}$ can generate Young measure ν_x satisfied that $\|\nu_x\| = 1$ and there is a subsequence of $\{u_j\}$ weakly convergent to $\int_{\mathbb{R}^m}\lambda\,\mathrm{d}\nu_x(\lambda)$ in $L^1(\Omega,\mathbb{R}^m)$.*

3. Galerkin approximation and a priori estimates

Let

$$X := \Big\{u \in L^2(Q;\mathbb{R}^m) : Du \in L^{p(x)}(Q;\mathbb{M}^{m\times N}), u(\cdot,\tau) \in V_\tau(\Omega) \text{ a.e. } \tau \in (0,T)\Big\},$$

where for $\tau \in (0,T)$,

$$V_\tau(\Omega) := \Big\{u \in L^2(\Omega;\mathbb{R}^m) \cap W_0^{1,1}(\Omega;\mathbb{R}^m) : Du(\cdot,\tau) \in L^{p(x)}(\Omega;\mathbb{M}^{m\times N})\Big\}.$$

The norm on X is defined by

$$\|u\|_X := \|u\|_{L^2(Q;\mathbb{R}^m)} + \|Du\|_{L^{p(x)}(Q;\mathbb{M}^{m\times N})}.$$

According to Lemmas 2.5–2.7, it is easy to show that X is a Banach space and $C_0^\infty(Q;\mathbb{R}^m)$ is dense in X. X' denotes the dual space of X. For all $g \in X'$, $u \in X$, there exists $g_0 \in L^2(Q;\mathbb{R}^m), g_1 \in L^{p'(x)}(Q;\mathbb{M}^{m\times N})$, such that

$$\langle g, u\rangle = \int_Q g_0 \cdot u \,\mathrm{d}x\,\mathrm{d}t + \int_Q g_1 \circ Du \,\mathrm{d}x\,\mathrm{d}t.$$

Based on the above notes, we will show the definition of weak solutions for (1.2).

Definition 3.1. A function $u \in L^\infty(0,T;L^2(\Omega)) \bigcap X$ is called as the weak solution of problem (1.2), if

$$-\int_Q u\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u(x,t)\varphi(x,t)\,\mathrm{d}x\Big|_0^T + \int_Q \sigma(x,t,Du)\circ D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q f\circ D\varphi\,\mathrm{d}x\,\mathrm{d}t$$

holds for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$.

We choose an $L^2(\Omega;\mathbb{R}^m)$-orthonormal base $\{\omega_j\}_{j=1}^\infty$, such that

$$\{\omega_j\}_{j=1}^\infty \subset C_0^\infty(\Omega;\mathbb{R}^m), \quad C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\cup_{n=1}^\infty V_n}^{C^1(\bar\Omega;\mathbb{R}^m)}.$$

Here $V_n = \mathrm{span}\{\omega_1,\omega_2,\dots,\omega_n\}$.

Since $f \in L^{p'(x)}(Q;\mathbb{M}^{m\times N})$ and $C_0^\infty(Q;\mathbb{M}^{m\times N})$ is identity in $L^{p'(x)}(Q;\mathbb{M}^{m\times N})$, there exists a sequence $\{f_n\} \subset C_0^\infty(Q;\mathbb{M}^{m\times N})$ such that

$$f_n \to f \quad \text{in } L^{p(x)}(Q;\mathbb{M}^{m\times N})$$

as $n \to \infty$. For every $u_0 \in L^2(\Omega;\mathbb{R}^m)$, there is a sequence $\{\psi_n\}_{n=1}^\infty$, such that $\psi_n \in \cup_{n=1}^\infty V_n$ and

$$\psi_n \to u_0 \quad \text{in } L^2(\Omega;\mathbb{R}^m)$$

as $n \to \infty$.

Definition 3.2. $u_n \in C^1(0,T;V_n)$ is called by the Galerkin solution of problem (1.2), if

$$-\int_{Q_\tau} \frac{\partial u_n}{\partial t}\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} \sigma(x,t,Du_n)\circ D\phi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\tau} f_n \circ D\phi\,\mathrm{d}x\,\mathrm{d}t$$

holds for all $\tau \in (0,T]$ and $\phi \in C^1(0,T;V_k)(k \le n)$, where $Q_\tau = \Omega \times (0,\tau)$.

Now we construct the Galerkin solution of problem (1.2). Define $P_n(t,\eta) : [0,T]\times \mathbb{R}^n \to \mathbb{R}^n$

$$(P_n(t,\eta))_i = \int_\Omega \sigma(x,t,\sum_{j=1}^n \eta_j D\omega_j)\circ D\omega_i\,\mathrm{d}x$$

where $\eta = (\eta_1,\cdot\cdot\cdot,\eta_n)$. Since σ is a Carathéodory function, $P_n(t,\eta)$ is continuous in t,η.

Consider the ordinary differential equation

$$\begin{gathered} \eta'(t) + P_n\big(t,\eta(t)\big) = F_n \\ \eta(0) = U_n(0) \end{gathered} \tag{3.1}$$

where

$$(F_n)_i = \int_\Omega f_n \circ D\omega_i\,\mathrm{d}x, \quad (U_n(0))_i = \int_\Omega \psi_n(x)\omega_i\,\mathrm{d}x.$$

From (3.1) we have $\eta'\eta + P_n(t,\eta)\eta = F_n\eta$. Furthermore,

$$\begin{aligned} P_n(t,\eta)\eta &= \int_Q \sigma(x,t,\sum_{j=1}^n \eta_j D\omega_j)\circ \sum_{i=1}^n \eta_i D\omega_i \,\mathrm{d}x\,\mathrm{d}t \\ &\geq -\int_Q b(x,t)\,\mathrm{d}x\,\mathrm{d}t + c_2\int_Q \Big|\sum_{i=1}^n \eta_i D\omega_i\Big|^{p(x)}\mathrm{d}x\,\mathrm{d}t \geq C. \end{aligned}$$

It is apparent that

$$\eta'\eta + C \leq F_n\eta \leq \frac{1}{2}|F_n|^2 + \frac{1}{2}|\eta(t)|^2.$$

Consequently,

$$\frac{1}{2}\frac{\partial|\eta(t)|^2}{\partial t} \leq \frac{1}{2}|F_n|^2 + \frac{1}{2}|\eta(t)|^2 + C$$

After integrating the both sides of this inequality , we obtain

$$|\eta(t)|^2 \leq C_n + \int_0^T |\eta(s)|^2\,\mathrm{d}s$$

Then by Gronwall's inequality, $|\eta(t)| \leq C_n(T)$. Let

$$M_n = \max_{(t,\eta)\in[0,T]\times B(\eta(0),2C_n(T))} |F_n - P_n(t,\eta)|,$$

$$T_n = \min\big\{T, \frac{2C_n(T)}{M_n}\big\}$$

where $B\Big(\eta(0), 2C_n(T)\Big)$ is a ball of radius $2C_n(T)$ with the center at the point $\eta(0)$ in $\mathbb{R}^n$.

By Peano's theorem, (3.1) has a C^1 solution on $[0,T_n]$. Let $t_1 = T_n$ and $\eta(t_1)$ be a initial value, then we can repeat the above process and get a C^1 solution on $[t_1,t_2]$, where $t_2 = t_1 + T_n$. Thus there is a interval $[t_{i-1}, t_{i-2}] \subset [0,T]$, such that (3.1) admits a solution on $[t_{i-1}, t_{i-2}]$, where $t_i = t_{i-1} + T_n, i = 1,2,\dots,l-1, t_l = T$. Moreover we can get a solution $\eta_n(t) \in C^1([0,T])$.

From the definition of P_n, it is easy to know that $u_n(x,t) = \sum_{j=1}^n \big(\eta_n(t)\big)_j\omega_j(x)$ is the Galerkin solution of (1.2).

Now we study the boundedness and convergence of some function sequences.

Lemma 3.3. *The sequence $\{u_n\}$ is bounded in X, and $\{\sigma(x,t,Du_n)\}$ is bounded in $L^{p'(x)}(Q;\mathbb{M}^{m\times N})$.*

Proof. Let $\phi = u_n$. By Definition 3.2, for every $\tau \in [0,T]$, one has

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} u_n \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} \sigma(x,t,Du_n)\circ Du_n\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\tau} f_n \circ Du_n\,\mathrm{d}x\,\mathrm{d}t,$$

which is denoted as $I + II = III$. By integration and (H2),

$$I = \frac{1}{2}\|u_n(\cdot,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(\cdot,0)\|^2_{L^2(\Omega)}$$

and

$$II \geq -\int_{Q_\tau} b(x,t)\,\mathrm{d}x\,\mathrm{d}t + c_2\int_{Q_\tau} |Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t$$

Since $f_n \in L^{p'(x)}(Q;\mathbb{M}^{m\times N})$, we have

$$III \leq C\|f_n\|_{L^{p(x)}(Q_\tau;\mathbb{M}^{m\times N})}\|Du_n\|_{L^{p(x)}(Q_\tau;\mathbb{M}^{m\times N})}$$

We know that $u_n(x,0)=\psi_n(x)\to u_0$ in $L^2(\Omega)$. As a result,
$$\int_\Omega u_n^2(x,0)\,\mathrm{d}x=\int_\Omega|\psi_n(x)|^2\,\mathrm{d}x\le C\quad\text{forall } n.$$
Consequently,
$$\begin{aligned}&\frac12\|u_n(\cdot,\tau)\|^2_{L^2(\Omega)}+c_2\int_{Q_\tau}|Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\\&\le\frac12\|u_n(\cdot,0)\|^2_{L^2(\Omega)}+\|b\|_{L^1(Q_\tau)}\\&\quad+C\|f_n\|_{L^{p'(x)}(Q_\tau;\mathbb{M}^{m\times N})}\|Du_n\|_{L^{p(x)}(Q_\tau;\mathbb{M}^{m\times N})}\\&\le C+C\|Du_n\|_{L^{p(x)}(Q_\tau;\mathbb{M}^{m\times N})}\end{aligned}\tag{3.2}$$
By Lemma 2.2, it follows that
$$\|Du_n\|_{L^{p(x)}(Q_\tau)}\le\max\Big\{\Big(\int_{Q_\tau}|Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p_-},\Big(\int_{Q_\tau}|Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p_+}\Big\}.$$
If $\|Du_n\|_{L^{p(x)}(Q_\tau;\mathbb{M}^{m\times N})}$ is unbounded, then $\int_{Q_\tau}|Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t$ is unbounded. This contradict (3.2). Thus
$$\|Du_n\|_{L^{p(x)}(Q;\mathbb{M}^{m\times N})}\le C.$$
Moreover
$$\|u_n(\cdot,\tau)\|^2_{L^2(\Omega)}\le C\tag{3.3}$$
Then we can get the conclusion that $\{u_n\}$ is bounded in X. By Lemma 2.6, there is a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) satisfying $u_n\rightharpoonup u$ in X, as $n\to\infty$.

Owing to (H2), we obtain
$$\begin{aligned}&\int_Q|\sigma(x,t,Du_n)|^{p'(x)}\,\mathrm{d}x\,\mathrm{d}t\\&\le C\Big(\int_Q|a(x,t)|^{p'(x)}\,\mathrm{d}x\,\mathrm{d}t+c_1\int_Q|Du_n|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t\Big).\end{aligned}$$
Since $a\in L^{p'(x)}(Q)$ and $\|Du_n\|_{L^{p(x)}(Q;\mathbb{M}^{m\times N})}\le C$, it follows that
$$\int_Q|\sigma(x,t,Du_n)|^{p'(x)}\,\mathrm{d}x\,\mathrm{d}t\le C.$$
From Lemma 2.2 we have
$$\|\sigma(x,t,Du_n)\|_{L^{p'(x)}(Q;\mathbb{M}^{m\times N})}\le C.\tag{3.4}$$
Then $\sigma(x,t,Du_n)\rightharpoonup\chi$ in $L^{p'(x)}(Q;\mathbb{M}^{m\times N})$ as $n\to\infty$ (we can choose a proper subsequence if necessary). □

Lemma 3.4. *For function sequences $\{u_n\}$ constructed above, we have*
$$u_n(\cdot,T)\rightharpoonup u(\cdot,T)\quad\text{in} L^2(\Omega),$$
$$u(\cdot,0)=u_0.$$

Proof. Thanks to (3.3), the sequence $\{u_n\}$ is bounded in $L^\infty\big(0,T;L^2(\Omega)\big)$. Thus there exists a subsequence (also denoted by $\{u_n\}$) such that
$$u_n(\cdot,T)\rightharpoonup z\quad\text{in } L^2(\Omega)$$
as $n\to\infty$. We will prove that $z=u(\cdot,T)$, and $u(\cdot,0)=u_0$. We denote $u(\cdot,T)$ as $u(T)$, and denote $u(\cdot,0)$ as $u(0)$.

For every $\psi \in C^\infty([0,T])$, $v \in V_k$, $k \le n$, we have

$$\int_0^T\int_\Omega \partial_t u_n v\psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sigma(x,t,Du_n)\circ Dv\psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_n \circ Dv\psi \,\mathrm{d}x\,\mathrm{d}t.$$

After integrating, one gets

$$\begin{aligned}
&\int_\Omega u_n(T)\psi(T)v\,\mathrm{d}x - \int_\Omega u_n(0)\psi(0)v\,\mathrm{d}x\\
&= -\int_0^T\int_\Omega \sigma(x,t,Du_n)\circ Dv\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega f_n\circ Dv\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T\int_\Omega u_n v\psi'\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

If $n \to \infty$, then

$$\begin{aligned}
&\int_\Omega z\psi(T)v\,\mathrm{d}x - \int_\Omega u_0\psi(0)v\,\mathrm{d}x\\
&= \int_0^T\int_\Omega f\circ Dv\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \chi\circ D\psi v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi' vu\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.5}$$

Let $\psi(0) = \psi(T) = 0$. Then

$$\int_0^T\int_\Omega f\circ Dv\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \chi\circ D\psi v\,\mathrm{d}x = -\int_0^T\int_\Omega \psi' vu\,\mathrm{d}x = \int_0^T\int_\Omega \psi vu'\,\mathrm{d}x.$$

Thus by (3.5), we can obtain

$$\begin{aligned}
\int_\Omega z\psi(T)v\,\mathrm{d}x - \int_\Omega u_0\psi(0)v\,\mathrm{d}x &= \int_0^T\int_\Omega \psi vu'\,\mathrm{d}x + \int_0^T\int_\Omega \psi' vu\,\mathrm{d}x\\
&= \int_\Omega u\psi v\,\mathrm{d}x\Big|_0^T\\
&= \int_\Omega u(T)\psi(T)v\,\mathrm{d}x - \int_\Omega u(0)\psi(0)v\,\mathrm{d}x
\end{aligned}$$

Let $k \to \infty$, if we take $\psi(T) = 0$ and $\psi(0) = 1$, then we have $u(0) = u_0$; if we take $\psi(T) = 1$ and $\psi(0) = 0$, then we have $u(T) = z$. $\square$

4. Existence of weak solutions

The proof of Lemma 3.3 implies that $\{Du_n\}$ is bounded in $L^{p(x)}(Q;\mathbb{M}^{m\times N})$. By Lemma 2.12, $\{Du_n\}$ can generate a family of Young measures $\nu_{(x,t)}$, and $\langle \nu_{(x,t)}, I\rangle = Du(x,t)$. By Lemmas 2.3 and 2.6, we can choose a proper subsequence if necessary such that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } X, \quad n \to \infty,\\
Du_n \rightharpoonup Du \quad \text{in } L^{p(x)}(Q;\mathbb{M}^{m\times N}).
\end{gathered}$$

Lemma 4.1. *Suppose that σ satisfies* (H1)–(H3)*, then the Young measures $\nu_{(x,t)}$ generated by $\{Du_n\}$, which is the gradient of Galerkin sequence $\{u_n\}$ constructed before, satisfy*

$$\begin{aligned}
&\int_Q\int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ\lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_Q\int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ Du\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.1}$$

Proof. Consider the sequence

$$\begin{aligned}
I_n &:= \big(\sigma(x,t,Du_n) - \sigma(x,t,Du)\big) \circ (Du_n - Du) \\
&= \sigma(x,t,Du_n) \circ (Du_n - Du) - \sigma(x,t,Du) \circ (Du_n - Du) \\
&=: I_{n,1} + I_{n,2}
\end{aligned}$$

Assumption (H2) implies

$$\begin{aligned}
&\int_Q |\sigma(x,t,Du)|^{p'(x)} \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\Big(\int_Q |a(x,t)|^{p'(x)} \,\mathrm{d}x\,\mathrm{d}t + c_1 \int_Q |Du|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t \Big).
\end{aligned}$$

Since $Du \in L^{p(x)}(Q;\mathbb{M}^{m\times N})$, it follow that $\sigma \in L^{p'(x)}(Q;\mathbb{M}^{m\times N})$. Because of the weak convergence of $\{Du_n\}$, we obtain $I_{n,2} \to 0$ as $n \to \infty$. It follows from Lemma 2.11 that

$$\begin{aligned}
I &:= \liminf_{n\to\infty} \int_Q I_n \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \int_Q I_{n,1} \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \int_Q \sigma(x,t,Du_n) \circ (Du_n - Du) \,\mathrm{d}x\,\mathrm{d}t \\
&\ge \int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda) \circ (\lambda - Du) d\nu_{(x,t)}(\lambda) \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.2}$$

Now we prove that $I \le 0$. Using

$$\int_Q \frac{\partial u_n}{\partial t} u_n \,\mathrm{d}x\,\mathrm{d}t + \int_Q \sigma(x,t,Du_n) \circ Du_n \,\mathrm{d}x\,\mathrm{d}t = \int_Q f_n \circ Du_n \,\mathrm{d}x\,\mathrm{d}t,$$

we find that

$$\begin{aligned}
I &= \liminf_{n\to\infty} \int_Q \sigma(x,t,Du_n) \circ (Du_n - Du) \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \Big(\int_Q \sigma(x,t,Du_n) \circ Du_n \,\mathrm{d}x - \int_Q \sigma(x,t,Du_n) \circ Du \,\mathrm{d}x\,\mathrm{d}t \Big) \\
&= \liminf_{n\to\infty} \Big(\int_Q f_n \circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q u_n \partial_t u_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q \sigma(x,t,Du_n) \circ Du \,\mathrm{d}x\,\mathrm{d}t \Big).
\end{aligned}$$

Obviously,

$$\begin{aligned}
\int_Q f_n \circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q f \circ Du \,\mathrm{d}x\,\mathrm{d}t &= \int_Q f_n \circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q f \circ Du_n \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q f \circ Du_n \,\mathrm{d}x\,\mathrm{d}t + \int_Q f \circ Du \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since

$$\|f_n - f\|_{L^{p'(x)}(Q;\mathbb{M}^{m\times N})} \to 0, \quad \text{as } n \to \infty,$$

it is easy to see that

$$\begin{aligned}
&\int_Q f_n \circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q f \circ Du_n \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\|f_n - f\|_{L^{p'(x)}(Q;\mathbb{M}^{m\times N})} \|Du_n\|_{L^{p(x)}(Q;\mathbb{M}^{m\times N})} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Because of the weak convergence of Du_n,

$$\int_Q f\circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q f\circ Du \,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \text{as } n\to\infty.$$

Consequently,

$$\int_Q f_n\circ Du_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q f\circ Du \,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \text{as } n\to\infty.$$

From the weak convergence of $\sigma(x,t,Du_n)$,

$$\int_Q \sigma(x,t,Du_n)\circ Du \,\mathrm{d}x\,\mathrm{d}t \to \int_Q \chi\circ Du \,\mathrm{d}x\,\mathrm{d}t, \quad \text{as } n\to\infty.$$

For every $\psi\in C^1(0,T;V_k), k\le n$,

$$\int_Q \psi\partial_t u_n \,\mathrm{d}x\,\mathrm{d}t - \int_Q \sigma(x,t,Du_n)\circ D\psi \,\mathrm{d}x\,\mathrm{d}t = \int_Q f_n\circ D\psi \,\mathrm{d}x\,\mathrm{d}t.$$

After integrating, we have

$$\begin{aligned}
&\int_\Omega u_n(\cdot,T)\psi(T)v \,\mathrm{d}x - \int_\Omega u_n(\cdot,0)\psi(0)v \,\mathrm{d}x - \int_Q u_n\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \sigma(x,t,Du_n)\circ D\psi \,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q f_n\circ D\psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Letting $n\to\infty$, we have

$$\begin{aligned}
&\int_\Omega u(\cdot,T)\psi(T)v \,\mathrm{d}x - \int_\Omega u(\cdot,0)\psi(0)v \,\mathrm{d}x - \int_Q u\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t + \int_Q \chi\circ D\psi \,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q f\circ D\psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Let $k\to\infty$, for all $\psi\in C^1\big(0,T;C^1(\bar\Omega)\big)$. The the above equality is valid. Then for all $\psi\in C_0^\infty(Q)$, the above equality also holds. Thus

$$-\int_Q u\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t = -\int_Q \chi\circ D\psi \,\mathrm{d}x\,\mathrm{d}t + \int_Q f\circ D\psi \,\mathrm{d}x\,\mathrm{d}t = \langle \operatorname{div}(\chi - f),\psi\rangle.$$

Obviously $\partial_t u = \operatorname{div}(\chi - f)$. For $u\in X$, we can derive that

$$\int_Q u\partial_t u \,\mathrm{d}x\,\mathrm{d}t = -\int_Q \chi\circ Du \,\mathrm{d}x\,\mathrm{d}t + \int_Q f\circ Du \,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand,

$$\begin{gathered}
\int_Q u\partial_t u \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|u(\cdot,T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u(\cdot,0)\|_{L^2(\Omega)}^2,\\
\int_Q u_n\partial_t u_n \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|u_n(\cdot,T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_n(\cdot,0)\|_{L^2(\Omega)}^2.
\end{gathered}$$

From the structure of u_n, we obtain

$$\|u_n(\cdot,0)\|_{L^2(\Omega)} \to \|u(\cdot,0)\|_{L^2(\Omega)}.$$

Using Lemma 3.4, we have $u_n(\cdot,T) \rightharpoonup u(\cdot,T)$ in $L^2(\Omega)$. Owing to the weakly lower semicontinuity of the norm,

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le \liminf_{n\to\infty} \|u_n(\cdot,T)\|_{L^2(\Omega)}.$$

Clearly,

$$\liminf_{n\to\infty}\Big(-\int_Q u_n\partial_t u_n \,\mathrm{d}x\,\mathrm{d}t\Big) \le -\frac{1}{2}\|u(\cdot,T)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(\cdot,0)\|^2_{L^2(\Omega)}.$$

Thus we arrived at the conclusion that $I \le 0$. $\square$

Lemma 4.2. *For a.e. $(x,t) \in Q$, we have*

$$\big(\sigma(x,t,\lambda) - \sigma(x,t,Du)\big) \circ (\lambda - Du) = 0 \quad \text{on} \quad \operatorname{supp}\nu_{(x,t)}.$$

Proof. Since

$$\int_{\mathbb{M}^{m\times N}} \lambda \,\mathrm{d}\nu_{(x,t)}(\lambda) = \langle \nu_{(x,t)}, I\rangle = Du(x,t),$$

and $\nu_{(x,t)}$ is a family of probability measures, $\int_{\mathbb{M}^{m\times N}} 1\mathrm{d}\nu_{(x,t)} = 1$. Consequently

$$\begin{aligned}
&\int_Q\int_{\mathbb{M}^{m\times N}} \sigma(x,t,Du)\circ(\lambda - Du)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q\int_{\mathbb{M}^{m\times N}} \sigma(x,t,Du)\circ\lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_\Omega\int_{\mathbb{M}^{m\times N}} \sigma(x,t,Du)\circ Du\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q \sigma(x,t,Du)\circ\int_{\mathbb{M}^{m\times N}} \lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_\Omega \sigma(x,t,Du)\circ Du\int_{\mathbb{M}^{m\times N}} 1\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q \sigma(x,t,Du)\circ Du\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \sigma(x,t,Du)\circ Du\int_{\mathbb{M}^{m\times N}} 1\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

From Lemma 4.1, we obtain

$$\int_\Omega\int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ(\lambda - Du)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Thus

$$\int_\Omega\int_{\mathbb{M}^{m\times N}} (\sigma(x,t,\lambda) - \sigma(x,t,Du))\circ(\lambda - Du)d\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

By the monotonicity of σ, the integrand in the above inequality is nonnegative. Then for a.e. $(x,t) \in Q$, we can obtain that

$$(\sigma(x,t,\lambda) - \sigma(x,t,Du))\circ(\lambda - Du) = 0 \quad \text{in } \operatorname{supp}\nu_x.$$

$\square$

We are now in a position to show the existence of solutions of (1.2).

Proof of Theorem 1.1. We consider 4 cases which correspond to the 4 cases in (H3).

Case (i). We prove that for a.e. $(x,t) \in Q$ and every $\mu \in \mathbb{M}^{m\times N}$ the following equation holds on $\operatorname{supp}\nu_x$,

$$\sigma(x,t,\lambda)\circ\mu = \sigma(x,t,Du)\circ\mu + \Big(\nabla\sigma(x,t,Du)\mu\Big)\circ(Du-\lambda), \tag{4.3}$$

where ∇ is the derivative with respect to the third variable of σ. Actually, by the monotonicity of σ, for all $\alpha \in \mathbb{R}$, we have

$$\Big(\sigma(x,t,\lambda)-\sigma(x,t,Du+\alpha\mu)\Big)\circ(\lambda-Du-\alpha\mu)\geq 0.$$

From Lemma 4.2 on $\operatorname{supp}\nu_x$ we obtain

$$\begin{aligned}
&\Big(\sigma(x,t,\lambda)-\sigma(x,t,Du+\alpha\mu)\Big)\circ(\lambda-Du-\alpha\mu)\\
&=\sigma(x,t,\lambda)\circ(\lambda-Du)-\sigma(x,t,\lambda)\circ\alpha\mu-\sigma(x,t,Du+\alpha\mu)\circ(\lambda-Du-\alpha\mu)\\
&=\sigma(x,t,Du)\circ(\lambda-Du)-\sigma(x,t,\lambda)\circ\alpha\mu-\sigma(x,t,Du+\alpha\mu)\circ(\lambda-Du-\alpha\mu).
\end{aligned}$$

It can be easily seen that

$$-\sigma(x,t,\lambda)\circ\alpha\mu \geq -\sigma(x,t,Du)\circ(\lambda-Du)+\sigma(x,t,Du+\alpha\mu)\circ(\lambda-Du-\alpha\mu),$$

and

$$\sigma(x,t,Du+\alpha\mu)=\sigma(x,t,Du)+\nabla\sigma(x,t,Du)\alpha\mu+o(\alpha).$$

Then we infer that

$$\begin{aligned}
&\sigma(x,t,Du+\alpha\mu)\circ(\lambda-Du-\alpha\mu)\\
&=\sigma(x,t,Du+\alpha\mu)\circ(\lambda-Du)-\sigma(x,t,Du+\alpha\mu)\circ\alpha\mu\\
&=\sigma(x,t,Du)\circ(\lambda-Du)+\nabla\sigma(x,t,Du)\alpha\mu\circ(\lambda-Du)\\
&\quad-\sigma(x,t,Du)\circ\alpha\mu+\nabla\sigma(x,t,Du)\alpha\mu\circ\alpha\mu+o(\alpha)\\
&=\sigma(x,t,Du)\circ(\lambda-Du)+\alpha\Big(\nabla\sigma(x,t,Du)\mu\circ(\lambda-Du)-\sigma(x,t,Du)\circ\mu\Big)+o(\alpha).
\end{aligned}$$

Moreover

$$-\sigma(x,t,\lambda)\circ\alpha\mu\geq\alpha\Big(\big(\nabla\sigma(x,t,Du)\mu\big)\circ(\lambda-Du)-\sigma(x,t,Du)\circ\mu\Big)+o(\alpha)$$

Since the sign of α is arbitrary, the above equation implies (4.2). Set $\mu=E_{ij}$, where E_{ij} is the matrix whose entry in the ith row and jth column is 1 and others are 0. Then by (4.2),

$$\sigma(x,t,\lambda)_{ij}=\sigma(x,t,Du)_{ij}+\Big(\nabla\sigma(x,t,Du)E_{ij}\Big)\circ(Du-\lambda).$$

Furthermore,

$$\begin{aligned}
&\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,\lambda)_{ij}\,\mathrm{d}\nu_{(x,t)}(\lambda)\\
&=\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,Du)_{ij}\,\mathrm{d}\nu_{(x,t)}(\lambda)\\
&\quad+\Big(\nabla\sigma(x,t,Du)E_{ij}\Big)\circ\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda).
\end{aligned}$$

Note that

$$\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)=Du(x,t)-\int_{\operatorname{supp}\nu_{(x,t)}}\lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)=0.$$

Thus we can derived that

$$\begin{aligned}\int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,u,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda) &= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,Du)\,\mathrm{d}\nu_{(x,t)}(\lambda)\\ &= \sigma(x,t,Du)\int_{\operatorname{supp}\nu_{(x,t)}} \mathrm{d}\nu_{(x,t)}(\lambda)\\ &= \sigma(x,t,Du).\end{aligned}$$

Since $\{\sigma(x,t,Du_n)\}$ is weakly convergent in $L^{p'(x)}(Q;\mathbb{M}^{m\times N})$. By Dunford-Pettis criterion and Lemma 2.9, $\{\sigma(x,t,Du_n)\}$ has a L^1-weak limit:

$$\overline{\sigma} := \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda) = \sigma(x,t,Du).$$

Evidently,

$$\sigma(x,t,Du_n) \rightharpoonup \sigma(x,t,Du) \quad \text{in} L^{p'(x)}(Q;\mathbb{M}^{m\times N}).$$

For all $\phi\in C^1(0,T;V_k), k\le n$, one has

$$\int_Q \phi\partial_t u_n\,\mathrm{d}x\,\mathrm{d}t + \int_Q \sigma(x,t,Du_n)\circ D\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f_n\circ D\phi\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\int_Q \phi\partial_t u_n\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega u_n(\cdot,T)\phi(T)\,\mathrm{d}x - \int_\Omega u_n(\cdot,0)\phi(0)v\,\mathrm{d}x - \int_Q u_n\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t.$$

Letting $n\to\infty$ we obtain

$$\begin{aligned}&\int_\Omega u(\cdot,T)\phi(T)\,\mathrm{d}x - \int_\Omega u(\cdot,0)\phi(0)v\,\mathrm{d}x - \int_Q u\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t + \int_Q \sigma(x,t,Du)\circ D\phi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q f\circ D\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Let $k\to\infty$, then for $\phi\in C^1(0,T;C_0^\infty(\Omega))$, we are led to the conclusion that

$$-\int_Q u\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u(x,t)\phi(x,t)\,\mathrm{d}x\Big|_0^T + \int_Q \sigma(x,t,Du)\circ D\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f\circ D\phi\,\mathrm{d}x\,\mathrm{d}t.$$

Case (ii). We prove that for all $(x,t)\in Q$ we have

$$\begin{aligned}&\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}\\ &= \big\{\lambda\in\mathbb{M}^{m\times N} : W(x,t,\lambda) = W(x,t,Du) + \sigma(x,t,Du)\circ(\lambda - Du)\big\}.\end{aligned}$$

If $\lambda\in\operatorname{supp}\nu_{(x,t)}$, by Lemma 4.2, for every $\beta\in[0,1]$,

$$(1-\beta)\Big(\sigma(x,t,\lambda)-\sigma(x,t,Du)\Big)\circ\big(\lambda - Du\big) = 0.$$

By monotonicity, for $\beta\in[0,1]$, we have

$$(1-\beta)\Big(\sigma\big(x,t,Du+\beta(\lambda-Du)\big)-\sigma(x,t,\lambda)\Big)\circ\big(Du-\lambda\big)\ge 0.$$

Thus for all $\beta\in[0,1]$,

$$(1-\beta)\Big(\sigma\big(x,t,Du+t(\lambda-Du)\big)-\sigma(x,t,Du)\Big)\circ\big(Du-\lambda\big)\ge 0.$$

In view of the monotonicity condition,

$$\Big(\sigma\big(x,t,Du+\beta(\lambda-Du)\big)-\sigma(x,t,Du)\Big)\circ\beta\big(\lambda-Du\big)\ge 0.$$

Since $\beta \in [0,1]$, we have

$$\Big(\sigma\big(x,t,Du+\beta(\lambda-Du)\big)-\sigma(x,t,Du)\Big)\circ(1-\beta)\big(\lambda-Du\big)\geq 0.$$

For all $\beta\in[0,1]$, if $\lambda\in\operatorname{supp}\nu_{(x,t)}$, then

$$\Big(\sigma\big(x,t,Du+\beta(\lambda-Du)\big)-\sigma(x,t,Du)\Big)\circ\big(\lambda-Du\big)=0. \tag{4.4}$$

It follows that

$$\begin{aligned}W(x,t,\lambda)&=W(x,t,Du)+\int_0^1\sigma\big(x,t,Du+\beta(\lambda-Du)\big)\circ(\lambda-Du)\,\mathrm{d}\beta\\&=W(x,t,Du)+\sigma(x,t,Du)\circ(\lambda-Du).\end{aligned}$$

So we can get $\lambda\in K_{(x,t)}$, i.e. $\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}$.

On account of the convexity of W, for all $\xi\in\mathbb{M}^{m\times N}$,

$$W(x,t,\xi)\geq W(x,u,Du)+\sigma(x,t,Du)\circ(\xi-Du).$$

For all $\lambda\in K_{(x,t)}$, put

$$P(\lambda)=W(x,t,\lambda),\quad Q(\lambda)=W(x,t,Du)+\sigma(x,t,Du)\circ(\lambda-Du).$$

As $\lambda\to W(x,u,\lambda)$ is continuous and differentiable, for every $\varphi\in\mathbb{M}^{m\times N}$, $\gamma\in\mathbb{R}$,

$$\begin{gathered}\frac{P(\lambda+\gamma\varphi)-P(\lambda)}{\gamma}\geq\frac{Q(\lambda+\gamma\varphi)-Q(\lambda)}{\gamma}\quad(\gamma>0),\\\frac{P(\lambda+\gamma\varphi)-P(\lambda)}{\gamma}\leq\frac{Q(\lambda+\gamma\varphi)-Q(\lambda)}{\gamma}\quad(\gamma<0).\end{gathered}$$

Thus $DP=DQ$, and

$$\sigma(x,t,\lambda)=\sigma(x,t,Du)\quad\forall\lambda\in K_{(x,t)}\supset\operatorname{supp}\nu_{(x,t)}. \tag{4.5}$$

Consequently,

$$\begin{aligned}\overline{\sigma}(x,t)&:=\int_{\mathbb{M}^{m\times N}}\sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\\&=\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)=\sigma(x,t,Du).\end{aligned} \tag{4.6}$$

Now we consider the Carathéodory function

$$g(x,t,\lambda)=|\sigma(x,t,\lambda)-\overline{\sigma}(x,t)|,\quad\lambda\in\mathbb{M}^{m\times N}.$$

Since $\sigma(x,t,Du_n)$ is weakly convergent in $L^{p'(x)}(Q;\mathbb{M}^{m\times N})$, then $\sigma(x,t,Du_n)$ is equi-integrable. Thus $g_n(x,t)=g(x,t,Du_n)$ is equi-integrable, and

$$g_n\rightharpoonup\overline{g}\ \text{ in } L^1(Q).$$

Taking (4.4) and (4.5) into consideration, we obtain

$$\begin{aligned}\overline{g}(x,t)&=\int_{\mathbb{M}^{m\times N}}\big|\sigma(x,t,\lambda)-\overline{\sigma}(x,t)\big|\,\mathrm{d}\nu_{(x,t)}(\lambda)\\&=\int_{\operatorname{supp}\nu_{(x,t)}}\big|\sigma(x,t,\lambda)-\overline{\sigma}(x,t)\big|\,\mathrm{d}\nu_{(x,t)}(\lambda)\\&=\int_{\operatorname{supp}\nu_{(x,t)}}\big|\sigma(x,t,\lambda)-\sigma(x,t,Du(x,t))\big|\,\mathrm{d}\nu_{(x,t)}(\lambda)=0.\end{aligned}$$

It turns out that

$$\int_Q \big|\sigma(x,t,Du_n) - \sigma(x,t,Du)\big| \,\mathrm{d}x\,\mathrm{d}t \to 0.$$

The remainder of the argument is similar to that in case (i) and so is omitted.

Case (iii). By the strict monotonicity and Lemma 4.2, we have

$$\operatorname{supp}\nu_{(x,t)} = \{Du(x,t)\}.$$

Thus for a.e. $(x,t) \in Q$, $\nu_{(x,t)} = \delta_{Du(x,t)}$. Using Lemma 2.9, we find $Du_n \to Du$ in measure. For a proper subsequence, we assert that $Du_n \to Du$ a.e. in Q. It follows that $\sigma(x,t,Du_n) \to \sigma(x,t,Du)$ a.e. in Q. Moreover $\sigma(x,t,Du_n) \to \sigma(x,t,Du)$ in measure.

From a similar analysis in case (i), we obtain the existence of (1.2) for case (ii).

Case (iv). Suppose that $\nu_{(x,t)}$ is not a Dirac measure, for a.e. $(x,t) \in Q$, then we have

$$\begin{aligned}
0 &< \int_Q \int_{\mathbb{M}^{m\times N}} \Big(\sigma(x,t,\lambda) - \sigma(x,t,\bar\lambda)\Big) \circ (\lambda - \bar\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \int_{\mathbb{M}^{m\times N}} \Big(\sigma(x,t,\lambda)\circ\lambda - \sigma(x,t,\lambda)\circ\bar\lambda \\
&\quad - \sigma(x,t,\bar\lambda)\circ\lambda + \sigma(x,t,\bar\lambda)\circ\bar\lambda\Big)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since

$$\int_{\mathbb{M}^{m\times N}} 1\,\mathrm{d}\nu_{(x,t)}(\lambda) = 1 \quad \text{and} \quad \int_{\mathbb{M}^{m\times N}} \lambda\,\mathrm{d}\nu_{(x,t)}(\lambda) = \bar\lambda = Du(x,t),$$

we obtain

$$\begin{aligned}
&\int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ\lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&> \int_Q \int_{\mathbb{M}^{m\times N}} \Big(\sigma(x,t,\lambda)\circ\bar\lambda + \sigma(x,t,\bar\lambda)\circ\lambda - \sigma(x,t,\bar\lambda)\circ\bar\lambda\Big)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \Big(\int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\circ\bar\lambda + \sigma(x,t,\bar\lambda)\circ\int_{\mathbb{M}^{m\times N}} \lambda\,\mathrm{d}\nu_{(x,t)}(\lambda) \\
&\quad - \sigma(x,t,\bar\lambda)\circ\bar\lambda\cdot\int_{\mathbb{M}^{m\times N}} 1\,\mathrm{d}\nu_{(x,t)}(\lambda)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\circ\bar\lambda\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\circ Du(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
&\int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ Du\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&\ge \int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\circ\lambda\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&> \int_Q \int_{\mathbb{M}^{m\times N}} \sigma(x,t,\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\circ Du\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

This is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure. Assume that $\nu_{(x,t)} = \delta_{h(x,t)}$. Then

$$h(x,t) = \int_{\mathbb{M}^{m\times N}} \lambda \,\mathrm{d}\delta_{h(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times N}} \lambda \,\mathrm{d}\nu_{(x,t)}(\lambda) = Du(x,t)$$

Thus $\nu_{(x,t)} = \delta_{Du(x,t)}$. Lemma 2.10 implies that $Du_n \to Du$ as $n \to \infty$. Moreover $\sigma(x,t,Du_n) \to \sigma(x,t,Du)$ in measure as $n \to \infty$. An argument similar to the one in case (iii) shows the conclusion we want. The proof is complete. $\square$

Conclusions. In this article, we study the existence of weak solutions for quasilinear parabolic system in divergence form with variable growth by means of Young measures generated by sequences in variable exponent spaces. We can conclude that problem (1.2) has a weak solution under four kinds of monotonicity conditions in (H3). We need notice that (H3)(iii) requires σ is strictly monotone. Actually classical monotonicity operator method can get our result under (H3)(iii). We give the other method to obtain the main theorem by Young measures in our paper under (H3)(iii). But conventional method can not prove the main result under the other monotonicity conditions. And in H3(iv), we define a new monotonicity condition. If σ is strictly monotone, then (H3)(iv) holds. Obviously, (H3)(iv) is weaker than typical strictly monotone condition.

Currently, the research on Young measures generated by sequences in variable exponent Lebesgue and Sobolev spaces is still in exploration. Our results enrich and perfect the theory of variable exponent spaces and Young measures.

For related results on nonlinear problems with variable growth we refer to the monograph by Rădulescu and Repovš [33] and the survey paper by Rădulescu [32]. Recent contributions to this field may be found in the papers [28, 29, 30, 34].

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 11771107, No. 11601251), and by the Natural Science Foundation of Shandong Province (ZR2016AM13, ZR2018PA003).

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