Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 107, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MODEL AND ANALYSIS FOR QUASISTATIC FRICTIONAL CONTACT OF A 2D ELASTIC BAR

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Abstract. This article constructs and analyzes a mathematical model that describes the quasistatic evolution of a 2D elastic bar that may come in frictional contact with a deformable foundation. The model and the underlying mechanical assumptions are described in detail and so are the assumptions on the problem data. The variational formulation of the problem is derived and, since friction is taken into account, it is in the form of an evolutionary variational inequality for the displacement field. Existence of solutions for the problem is established by using arguments of evolutionary variational inequalities.

1. Introduction

This work derives a model for a thin long plate that, because of its symmetry and the structure of applied forces, reduces to a model of a 2D bar, which is especially suitable for the description of contact processes. In particular, we use it in this work to describe frictional contact between the bar and a reactive foundation. Whereas it captures both the normal and tangential tractions on the contact surface, it is simpler than the standard 2D elastic plate. Furthermore, the frictional contact problem is reformulated as a variational inequality which allows us to establishes its solvability using arguments from the theory of abstract variational inequalities.

Our main interest lies in the fact that although the body is long and thin, i.e., its lengthwise dimension (x-direction) is much bigger than its thickness (y-direction), the model has two dependent variables, the displacements; one that depends only on x and t and describes the motion of the central axis of the bar, and the other one depends on x, y and t and allows to take tangential shear into account. In this way this 2D bar model may be considered as a '1.5-dimensional.' This simpler structure allows for better analysis and much faster simulations, while fully capturing the essential processes involved in contact.

We first derive the 2D bar model from the 3D problem under appropriate hypothesis on external forces, we derive a two-dimensional (actually 1.5-dimensional) version of the model. The interest in the model lies in the fact that one can easily prescribe tangential, in addition to the usual normal, contact conditions. Indeed,

2010 *Mathematics Subject Classification.* 74M10, 74M15, 74K20, 58E35, 35Q74.
Key words and phrases. 2D bar; frictional contact; normal compliance; weak solution; variational inequality.

Submitted January 30, 2017. Published May 8, 2018.

the setting allows us to study the friction process between the bar and the foundation, which cannot be done in the usual models of plates. We analyze the problem of quasistatic frictional contact. We assume that the foundation is deformable and describe contact with the normal compliance contact condition and we use the associated Coulomb's law of dry friction. Then, we derive a variational formulation of the model and since friction is taken into account, it is in the form of an evolutionary variational inequality for the displacement field. The existence of solutions for the problem is established by using arguments of evolutionary variational inequalities.

It is a new contribution to the Mathematical Theory of Contact Mechanics, MTCM, which has seen considerable progress, especially since the beginning of this century, in modeling, mathematical analysis, numerical analysis and simulations of various contact processes and, as a result, MTCM is currently reaching a state of maturity. The theory is concerned with mathematical structures that underly general contact processes with different constitutive laws, i.e., different materials, different possible geometries and different contact conditions, see for instance [5, 8, 15, 18, 19, 21] and the many references therein. MTCM aims to provide a sound, clear and rigorous framework for models of processes involved in contact, and the necessary tools and ideas to prove the existence, uniqueness and regularity results for the solutions of these models. Moreover, the theory assigns precise meaning to the solutions. In addition, the variational formulation of the models leads directly and naturally to sophisticated numerical methods with proven convergence for the computer approximations of the solutions. The MTCM has been using, and extending, various mathematical concepts which include variational and hemivariational inequalities and differential inclusions. This 2D bar model extends the theory a bit further.

The interest in contact problems involving thin structures such as beams and plates lies, on the one hand, in the fact that their mathematical analysis avoids some of the complications arising in 3D settings and often provides insight into the possible types of behavior of the solutions. On the other hand, such structures abound in all branches of engineering and so there is intrinsic interest in these models, too. Furthermore, these models allow for faster and more comprehensive computer simulations. Finally, one may use such models as tests and benchmarks for computer schemes meant for simulation of complicated multidimensional contact problems. Models, analysis and computer simulations of various contact problems for beams can be found in [1, 2, 3, 4, 7, 13, 20] and the references therein. A model similar to the one in this work has been described in [6], but with *ad hoc* derivation and no analysis was done there.

Following the Introduction, the rest of paper is structured as follows. In Section 2 we describe a general model for frictional contact between a 3D elastic body and a reactive foundation. Then, using the system symmetries and the fact that it is long and thin, we derive the 2D bar model, Problem $\mathcal{P}_{2D}$. In Section 3 we list the assumptions on the problem data and derive the corresponding variational formulation $\mathcal{P}_{2D}^V$. Then, in Section 4 we state and prove our main existence result, Theorem 4.1. Finally, in Section 5 we provide a few short comments and concluding remarks. The paper ends with an Appendix in which we recall a general existence results for evolutionary variational inequalities used in the proof of Theorem 4.1.

2. The model

In this section we first describe a 3D contact problem with friction. Then, under the assumptions that the problem has certain symmetry and the solid is long in the x-direction and thin in the other two directions, we obtain the 2D elastic bar model. Then, we pose the problem for the process of frictional contact of this 2D bar. It is seen that the framework is especially well suited for the mathematical description of the problem.

We consider an elastic 3D rectangular solid that occupies, in a fixed and undeformed reference configuration, the region $\mathcal{B}$ in $\mathbb{R}^3$. We denote by x, y, z the spatial variables and assume that $\mathcal{B}$ is sufficiently long in the direction Oz so that the end effects in this direction are negligible. Thus, $\mathcal{B} = (0, L) \times (-h, h) \times (-\infty, +\infty)$. Since $\mathcal{B}$ is a 3D rectangular region, which is infinite in the direction of the Oz, we refer to $\mathcal{B}$ as a plate. Moreover, L and $2h$ represent its length and its thickness, respectively. We denote in what follows by $\Omega = (0, L) \times (-h, h)$ the cross section of the plate and, therefore, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. Moreover, when $h << L$ we refer to Ω as a 2D bar.

The plate is clamped on on $\Gamma_D = \{0\} \times (-h, h) \times (-\infty, +\infty)$ and so the displacement field vanishes there. It is free on $\Gamma_F = \{L\} \times (-h, h) \times (-\infty, +\infty)$ and, on the top $\Gamma_N = \{0, L\} \times \{h\} \times (-\infty, +\infty)$, is subjected to a distributed surface tractions of density $\mathbf{p}$. On the bottom $\Gamma_C = \{0, L\} \times \{-h\} \times (-\infty, +\infty)$ the plate may come in frictional contact with a cylindrical foundation described by a function $y = \Psi(x) - h$ which, for the sake of simplicity, is assumed to be time independent. The cross section of the plate is depicted in Fig. 1. Contact (in the vertical direction) is modeled with the normal compliance condition and friction (in the horizontal direction) with the Coulomb law of dry friction. It is assumed that the forces and tractions vary sufficiently slowly so that the quasistatic approximation is valid. In addition, for the sake of simplicity, body forces are neglected.

We denote by $\boldsymbol{\nu}$ the normal vector to $\mathcal{B}$ and we use the index ν and τ to represent the normal and tangential components of vectors and tensors, respectively. The time interval of interest is $[0, T]$, with $T > 0$, and a dot above a variable represents its partial time derivative. We denote by $\mathcal{S}^3$ the linear space of second order symmetric tensors in $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, while "$\cdot$" and $\|\cdot\|$ represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$.

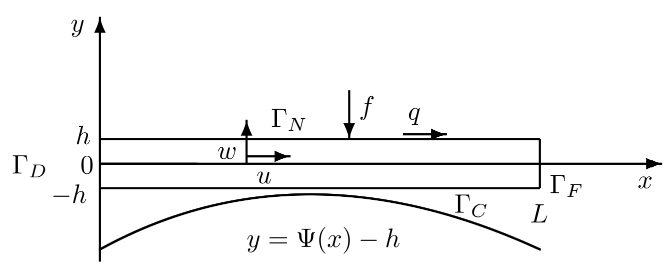

Figure 1. The cross section of the plate; Γ_C is the potential contact surface and Ψ describes the obstacle or foundation.

The mathematical model that describes the quasistatic process of frictional contact of the elastic plate under the above assumptions is as follows.

Problem $\mathcal{P}_{3D}$. *Find a displacement field* $\mathbf{u} : \mathcal{B} \times (0,T) \to \mathbb{R}^3$ *and a stress field* $\boldsymbol{\sigma} : \mathcal{B} \times (0,T) \to \mathcal{S}^3$ *such that*

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\ \boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}_3 + 2\delta\, \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \mathcal{B} \times (0,T) \tag{2.1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \mathcal{B} \times (0,T) \tag{2.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T) \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \mathbf{0} \quad \text{on } \Gamma_F \times (0,T) \tag{2.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{p} \quad \text{on } \Gamma_N \times (0,T) \tag{2.5}$$

$$-\sigma_\nu = \lambda_{nc}(u_\nu - g)_+ \quad \text{on} \quad \Gamma_C \times (0,T) \tag{2.6}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \le \mu|\sigma_\nu| \quad \text{on } \Gamma_C \times (0,T) \\ &-\boldsymbol{\sigma}_\tau = \mu|\sigma_\nu| \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \quad \text{on } \Gamma_C \times (0,T) \end{aligned}\right\} \tag{2.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \mathcal{B}. \tag{2.8}$$

A short description of the model, the equations and conditions (2.1)–(2.8), follows. Equation (2.1) represents the linear elastic constitutive law of the solid material in which λ and δ denote the Lamé coefficients, both positive constants, $\boldsymbol{\varepsilon}(\mathbf{u})$ the linearized strain deformation tensor associated to the displacement field $\mathbf{u}$, $\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})$ denotes its trace and $\mathbf{I}_3$ is the identity tensor in $\mathcal{S}^3$. The tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} u_x & \frac{1}{2}\big(u_y + w_x\big) & \frac{1}{2}\big(u_z + v_x\big) \\ \frac{1}{2}\big(u_y + w_x\big) & w_y & \frac{1}{2}\big(v_y + w_z\big) \\ \frac{1}{2}\big(u_z + v_x\big) & \frac{1}{2}\big(v_y + w_z\big) & v_z \end{pmatrix} \tag{2.9}$$

where u, w and v represent the components of the displacement field, i.e. $\mathbf{u} = (u, w, v)$. Here and below, the indices x, y, z denote the partial derivatives with respect to the corresponding spatial variables.

Equation (2.2) represents the internal forces balance since we assume that the process is quasistatic and we neglect body forces. Moreover, $\mathrm{Div}\, \boldsymbol{\sigma}$ is the divergence of the stress field $\boldsymbol{\sigma}$. Condition (2.3) is the Dirichlet condition and (2.4) and (2.5) are the traction conditions, described above. Next, (2.6) represents the so-called normal compliance condition in which g denotes the gap between the body's bottom surface and the obstacle, measured in the direction of the outward normal; λ_{nc} is the normal compliance stiffness coefficient of the foundation and $r_+ = \max\{0, r\}$. The normal compliance condition was introduced in [14] and was studied extensively, see, e.g., [8, 11, 9, 10, 19] and the many references therein, and more general normal compliance conditions can be found there, as well. Condition (2.7) represents Coulomb's law of dry friction in which μ is the coefficient of friction, assumed to be a positive constant. References to this condition include [5, 8, 19, 21], among a host of others. Finally, condition (2.7) is the initial condition, in which $\mathbf{u}_0$ is the given initial displacement.

We note that although (2.2) is the equilibrium balance of the forces, the problem is quasistatic, thus time dependent, because of the dependence of friction on the velocity.

Next, following [6, 7], we introduce additional assumptions on the size and symmetry of the setting that allow us to derive a simplified two-dimensional model associated with Problem $\mathcal{P}_{3D}$. To that end, we assume that

$$\mathbf{p} = (q, f, 0) \quad \text{with} \quad f = f(x,t) \quad \text{and} \quad q = q(x,t), \tag{2.10}$$

i.e., the plate is subjected on the top $y = h$ to a distributed vertical load f and tangential traction q, which do not depend on z. Such a load gives rise to deformations of the plate with displacement field $\mathbf{u}$ that is independent of z of the form

$$\mathbf{u} = (u, w, 0) \text{ with } u = u(x, y, t) \quad \text{and} \quad w = w(x, t). \tag{2.11}$$

Here, u is the horizontal displacement and w is the vertical one. Since $h << L$, in (2.11) and below we neglect the dependence of the vertical displacement w on y, which means that w describes the vertical displacement of the central line. Nevertheless, due to the action of the tangential traction that act on $y = h$, it is reasonable to assume that the horizontal displacement u does depend on both x and y, as is show in (2.11). Then, using (2.9) and (2.11) it is straightforward to deduce that the strain tensor is given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} u_x & \frac{1}{2}(u_y + w_x) & 0 \\ \frac{1}{2}(u_y + w_x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) = u_x$ and using the elastic constitutive law (2.1) shows that the stress tensor is given by

$$\boldsymbol{\sigma} = \begin{pmatrix} (\lambda + 2\delta)u_x & \delta(u_y + w_x) & 0 \\ \delta(u_y + w_x) & \lambda u_x & 0 \\ 0 & 0 & \lambda u_x \end{pmatrix}. \tag{2.12}$$

We note that the strain is two-dimensional while the stress is three-dimensional, which is the so-called plane-strain case.

We now introduce the Young modulus $E = \lambda + 2\delta$ and the shear modulus $G = \delta$. It follows that $\lambda = E - 2G$ and, therefore, (2.12) becomes

$$\boldsymbol{\sigma} = \begin{pmatrix} Eu_x & G(u_y + w_x) & 0 \\ G(u_y + w_x) & (E - 2G)u_x & 0 \\ 0 & 0 & (E - 2G)u_x \end{pmatrix}. \tag{2.13}$$

Moreover, we note that the components of the stress field do not depend on the variable z, therefore, taking into account (2.13) and (2.11) it follows that the balance equation (2.1) reduces to the following two scalar equations:

$$Eu_{xx}(x, y, t) + Gu_{yy}(x, y, t) = 0, \quad (x, y) \in \Omega,\ t \in [0, T], \tag{2.14}$$

$$Gw_{xx}(x, t) + (E - G)u_{xy}(x, y, t) = 0, \quad (x, y) \in \Omega,\ t \in [0, T]. \tag{2.15}$$

We turn to the boundary conditions. First, we combine (2.3) and (2.11) to deduce that

$$u(0, y, t) = w(0, t) = 0, \tag{2.16}$$

for all $y \in [-h, h]$, $t \in [0, T]$. Next, the outward unit normal at the boundary Γ_F $(x = L)$ is given by $\boldsymbol{\nu} = (1, 0, 0)$. Therefore, using (2.13), we conclude that $\boldsymbol{\sigma\nu} = (Eu_x, G(u_y + u_x), 0)$ on Γ_F. Thus, the boundary condition (2.4) can be written

$$u_x(L, y, t) = 0, \tag{2.17}$$

$$u_y(L, y, t) + w_x(L, y, t) = 0, \tag{2.18}$$

for all $y \in [-h, h]$ and $t \in [0, T]$.

In a similar way, the outward unit normal at the boundary Γ_N $(y = h)$ is given by $\boldsymbol{\nu} = (0, 1, 0)$. Therefore, using (2.13), we deduce that the tractions on this surface are given by $\boldsymbol{\sigma\nu} = (G(u_y + w_x), (E - 2G)u_x, 0)$. As a result, the boundary condition (2.5) combined with assumption (2.10) imply that

$$G(u_y(x, h, t) + w_x(x, t)) = q(x, t), \tag{2.19}$$

$$(E - 2G)u_x(x, h, t) = f(x, t), \tag{2.20}$$

for all $x \in [0, L]$, and $t \in [0, T]$.

We turn to the contact conditions. We recall that the normal and tangential components of the displacement field are given by

$$u_\nu = \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{and} \quad \mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}, \tag{2.21}$$

respectively. Also, the normal and tangential components of the stress field are given by

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{2.22}$$

Next, on the contact surface Γ_C $(y = -h)$ the outward unit normal is given by $\boldsymbol{\nu} = (0, -1, 0)$. Therefore, using assumption (2.11) and (2.21) we deduce that

$$u_\nu = -w \quad \text{and} \quad \mathbf{u}_\tau = (u, 0, 0) \quad \text{on } \Gamma_C \times (0, T). \tag{2.23}$$

A similar argument based on (2.13) and (2.22) yields

$$\sigma_\nu = (E - 2G)u_x \text{ and } \boldsymbol{\sigma}_\tau = (-G(u_y + w_x), 0, 0) \quad \text{on } \Gamma_C \times (0, T). \tag{2.24}$$

Moreover, since Ψ is a negative function, we conclude that the gap between the bottom $y = -h$ and the obstacle is given by

$$g = -\Psi \quad \text{on } \Gamma_C \times (0, T). \tag{2.25}$$

Using (2.23)–(2.25) shows that the contact condition (2.6) becomes

$$(E - 2G)u_x(x, -h, t) = -\lambda_{nc}(\Psi(x) - w(x, t))_+, \tag{2.26}$$

for all $x \in [0, L]$ and $t \in [0, T]$. Also, using again (2.23)–(2.26), it follows that the friction law (2.7) can be written as follows:

$$\begin{gathered} G|u_y(x, -h, t) + w_x(x, t)| \le \mu\lambda_{nc}(\Psi(x) - w(x, t))_+, \\ G(u_y(x, -h, t) + w_x(x, t)) = \mu\lambda_{nc}(\Psi(x) - w(x, t))_+ \frac{\dot{u}}{|\dot{u}|} \quad \text{if } \dot{u} \ne 0, \end{gathered} \tag{2.27}$$

for all $x \in [0, L]$ and $t \in [0, T]$. We note in passing that in this problem friction is controlled by the combination $\mu\lambda_{nc}/G$.

Finally, we assume that the initial displacement $\mathbf{u}_0$ is of the form

$$\mathbf{u}_0 = (u_0, w_0, 0) \quad \text{with } u_0 = u_0(x, y) \quad \text{and} \quad w_0 = w_0(x). \tag{2.28}$$

Then, using (2.8), (2.11) and (2.28) we deduce that

$$u(x, y, 0) = u_0(x, y) \quad \text{and} \quad w(x, 0) = w_0(x), \tag{2.29}$$

for all $x \in [0, L]$ and $y \in [-h, h]$.

Collecting the equations and conditions above leads to the following 'classical' two-dimensional mathematical model which describes the quasistatic frictional contact problem of the 2D bar.

Problem $\mathcal{P}_{2D}$. *Find the horizontal displacement field* $u = u(x, y, t) : [0, L] \times [-h, h] \times [0, T] \to \mathbb{R}$ *and the vertical displacement* $w = (x, t) : [0, L] \times [0, T] \to \mathbb{R}$ *such that* (2.14)–(2.20), (2.26), (2.27) and (2.29) *hold.*

To analyze the problem we set it in a variational form in the following section. The existence of a weak solution is provided in Section 4.

We remark that Problem $\mathcal{P}_{2D}$ is formulated in terms of the displacements. Once the unknowns u and w are found, then the stress field is obtained by using (2.13). We also note that the frictionless problems is obtained by simply setting $\mu = 0$.

3. Variational formulation

In this section we list the assumption on the problem data and derive the variational formulation of problem $\mathcal{P}_{2D}$. To that end, everywhere below we use the standard notation for the Lebesgue and Sobolev spaces of real-valued or vector-valued functions. Next, recalling that $\Omega = (0, L) \times (-h, h)$, we introduce the spaces

$$V = \{u \in H^1(\Omega) : u(0, \cdot) = 0\}, \quad W = \{w \in H^1(0, L) : w(0) = 0\}. \tag{3.1}$$

Note that equalities $u(0, \cdot) = 0$ and $w(0) = 0$ in the definitions of the spaces V and W are understood in the sense of traces. The spaces V and W are real Hilbert spaces with the canonical inner products defined by

$$(u, \psi)_V = \iint_\Omega (u\psi + u_x\psi_x + u_y\psi_y)\, dx\, dy, \quad \forall u,\, \psi \in V, \tag{3.2}$$

$$(w, \varphi)_W = \int_0^L (w\psi + w_x\psi_x)\, dx, \quad \forall w,\, \varphi \in W. \tag{3.3}$$

The corresponding norms are denoted by $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. In addition, we denote by $X = V \times W$ the product space, endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_X = (u, \psi)_V + (w, \varphi)_W, \quad \forall \mathbf{u} = (u, w),\ \mathbf{v} = (\psi, \varphi) \in X. \tag{3.4}$$

It follows from (3.4) that the norm on X satisfies

$$\|\mathbf{u}\|_X^2 = \|u\|_V^2 + \|w\|_V^2, \quad \forall \mathbf{u} = (u, w) \in X. \tag{3.5}$$

Moreover, the following inequalities hold:

$$\|u\|_V \le \|\mathbf{u}\|_X, \quad \|w\|_V \le \|\mathbf{u}\|_X, \quad \forall \mathbf{u} = (u, w) \in X. \tag{3.6}$$

For an element $\mathbf{u} = (u, w) \in X$, we have that the projection operator $\mathbf{u} \mapsto w : X \to L^2(0, L)$ is a linear compact operator. Therefore, there exist a positive constant c_B, which depends on L and h, such that

$$\|w\|_{L^2(0,L)} \le c_B \|\mathbf{u}\|_X, \quad \forall \mathbf{u} = (u, w) \in X, \tag{3.7}$$

and, moreover,

$$\mathbf{u}_n = (u_n, w_n) \rightharpoonup \mathbf{u} = (u, w) \quad \text{in } X \implies w_n \to w \quad \text{in } L^2(0, L). \tag{3.8}$$

Inequalities (3.6), (3.7) and the weak-strong convergence result (3.8) are used in various places in Section 4 below.

We assume in what follows that:

$$E > 0, \quad G > 0, \quad \lambda_{nc} \ge 0, \quad \mu \ge 0, \tag{3.9}$$

$$f \in W^{1,\infty}(0, T; L^2(0, L)), \quad q \in W^{1,\infty}(0, T; L^2(0, L)), \tag{3.10}$$

$$\Psi \in L^2(0, L), \quad \Psi(x) \le 0 \quad \text{a.e.} x \in (0, L), \tag{3.11}$$

$$u_0 \in V, \quad w_0 \in W. \tag{3.12}$$

Then, we define the bilinear form $a : X \times X \to \mathbb{R}$, the functional $j : X \times X \to \mathbb{R}$ and the function $\mathbf{f} : [0, T] \to X$ as follows:

$$a(\mathbf{u}, \mathbf{v}) = E \iint_\Omega u_x\psi_x dx\, dy + G \iint_\Omega (u_y + w_x)(\psi_y + \varphi_x) dx\, dy, \tag{3.13}$$

$$
\begin{aligned}
j(\mathbf{u},\mathbf{v}) = -\lambda_{nc}\int_0^L (\Psi(x)-w(x))_+\varphi(x)\,dx,\\
+\mu\lambda_{nc}\int_0^L (\Psi(x)-w(x))_+|\psi(x,-h)|\,dx,
\end{aligned} \tag{3.14}
$$

$$
(\mathbf{f}(t),\mathbf{v})_X = \int_0^L q(x,t)\psi(x,h)dx + \int_0^L f(x,t)\varphi(x)dx \tag{3.15}
$$

for all $\mathbf{u}=(u,w)$, $\mathbf{v}=(\psi,\varphi)\in X$ and $t\in[0,T]$. Note that under conditions (3.10), (3.11) the integrals in (3.13)–(3.15) are well-defined. Moreover, the definition of the element $\mathbf{f}$ is based on Riesz's representation theorem.

With these notation in place, we are in a position to derive the variational formulation of the Problems $\mathcal{P}_{2D}$. We proceed formally and assume in what follows that $\mathbf{u}=(u(x,y,t),w(x,t))$ represents a solution to the Problem $\mathcal{P}_{2D}$ and let $t\in[0,T]$, $\mathbf{v}=(\psi(x,y),\varphi(x))\in X$ be fixed. Then, multiplying (2.14) by $\psi-\dot u(t)$ and integrating over Ω, we obtain

$$
\begin{aligned}
&\iint_\Omega Eu_{xx}(x,y,t)(\psi(x,y)-\dot u(x,y,t))\,dx\,dy\\
&+\iint_\Omega Gu_{yy}(x,y,t)(\psi(x,y)-\dot u(x,y,t)\,dx\,dy = 0.
\end{aligned} \tag{3.16}
$$

Next, we write

$$
u_{xx}(\psi-\dot u) = (u_x(\psi-\dot u))_x - u_x(\psi_x-\dot u_x),
$$

and use Green's formula to see that

$$
\begin{aligned}
&\iint_\Omega Eu_{xx}(x,y,t)(\psi(x,y)-\dot u(x,y,t))\,dx\,dy\\
&= E\int_{-h}^h u_x(L,y,t)(\psi(L,y)-\dot u(L,y,t))\,dy\\
&\quad - E\int_{-h}^h u_x(0,y,t)(\psi(0,y)-\dot u(0,y,t))\,dy\\
&\quad - E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dx\,dy = 0.
\end{aligned} \tag{3.17}
$$

Similar arguments show that

$$
\begin{aligned}
&\iint_\Omega Gu_{yy}(x,y,t)(\psi(x,y)-\dot u(x,y,t))\,dx\,dy\\
&= -G\int_0^L u_y(x,-h,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\
&\quad + G\int_0^L u_y(x,h,t)(\psi(x,h)-\dot u(x,h,t))\,dx\\
&\quad - G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy = 0.
\end{aligned} \tag{3.18}
$$

We now add the equalities (3.17) and (3.18), then we use equality (3.16), the boundary conditions (2.16), (2.17) and the definition (3.1) of the space V to deduce that

$$\begin{aligned} &E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dx\,dy\\ &+G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy\\ &=-G\int_0^L u_y(x,-h,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\ &\quad+G\int_0^L u_y(x,h,t)(\psi(x,h)-\dot u(x,h,t))\,dy. \end{aligned} \tag{3.19}$$

Next, it is straightforward to show that the frictional condition (2.27) implies that

$$\begin{aligned} &-G(u_y(x,-h,t)+w_x(x,t))(\psi(x,-h)-\dot u(x,-h,t))\\ &\ge \mu\lambda_{nc}(\Psi(x)-w(x,t))(|\dot u(x,-h,t)|-|\psi(x,-h)|) \end{aligned}$$

for $x\in[0,L]$, and, therefore,

$$\begin{aligned} &-G\int_0^L u_y(x,-h,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\ &\ge G\int_0^L w_x(x,-h,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\ &\quad+\mu\lambda_{nc}\int_0^L(\Psi(x)-w(x,t))(|\dot u(x,-h,t)|-|\psi(x,-h)|)\,dx. \end{aligned} \tag{3.20}$$

We now use the boundary condition (2.19) to see that

$$\begin{aligned} &G\int_0^L u_y(x,h,t)(\psi(x,h)-\dot u(x,h,t))\,dx\\ &=\int_0^L(q(x,t)-Gw_x(x,t))(\psi(x,h)-\dot u(x,h,t))\,dx. \end{aligned} \tag{3.21}$$

Finally, we combine relations (3.19)–(3.21) to deduce that

$$\begin{aligned} &E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dx\,dy\\ &+G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy\\ &\ge G\int_0^L w_x(x,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\ &\quad+\mu\lambda_{nc}\int_0^L(\Psi(x)-w(x,t))(|\dot u(x,-h,t)|-|\psi(x,-h)|)\,dx\\ &\quad+\int_0^L(q(x,t)-Gw_x(x,t))(\psi(x,h)-\dot u(x,h,t))\,dx. \end{aligned} \tag{3.22}$$

We now keep $x\in[0,L]$ fixed and integrate equation (2.15) with respect to y on $[-h,h]$, thus,

$$2hGw_{xx}(x,t)+(E-G)\int_{-h}^h u_{xy}(x,y,t)\,dy=0. \tag{3.23}$$

Then, we write

$$\int_{-h}^{h} u_{xy}(x,y,t)\,dy = u_x(x,h,t) - u_x(x,-h,t)$$

and use the boundary conditions (2.20) and (2.26) to see that

$$\begin{aligned}(E-G)&\int_{-h}^{h} u_{xy}(x,y,t)\,dy\\ &= f(x,t) + \lambda_{nc}(\Psi(x) - w(x,t))_+ + G(u_x(x,h,t) - u_x(x,-h,t).\end{aligned} \tag{3.24}$$

Next, we subtract equalities (3.23) and (3.24) and deduce that

$$\begin{aligned}&-2hGw_{xx}(x,t)\\ &= f(x,t) + \lambda_{nc}(\Psi(x) - w(x,t))_+ + G(u_x(x,h,t) - u_x(x,-h,t)).\end{aligned} \tag{3.25}$$

We multiply (3.25) with $\xi = \xi(x) \in W$, integrate over $[0,L]$ and obtain

$$\begin{aligned}&-2hG\int_0^L w_{xx}(x,t)\xi(x)\,dx\\ &= \int_0^L f(x,t)\xi(x)\,dx + \lambda_{nc}\int_0^L (\Psi(x) - w(x,t))_+\xi(x)\,dx\\ &\quad + G\int_0^L (u_x(x,h,t) - u_x(x,-h,t)\xi(x)\,dx.\end{aligned} \tag{3.26}$$

Next, we perform an integration by parts and note that $\xi(0) = 0$, thus

$$\begin{aligned}&-2hG\int_0^L w_{xx}(x,t)\xi(x)\,dx\\ &= -2hGw_x(L,t)\xi(L) + 2hG\int_0^L w_x(x,t)\xi_x(x)\,dx,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&G\int_0^L (u_x(x,h,t) - u_x(x,-h,t))\xi(x)\,dx.\\ &= G(u(L,h,t) - u(L,-h,t))\xi(L)\\ &\quad - G\int_0^L u(x,h,t) - u(x,-h,t))\xi_x(x)\,dx.\end{aligned} \tag{3.28}$$

We now substitute equalities (3.27), (3.28) in (3.26) and obtain

$$\begin{aligned}&2hG\int_0^L w_x(x,t)\xi_x(x)\,dx\\ &= \int_0^L f(x,t)\xi(x)\,dx + \lambda_{nc}\int_0^L (\Psi(x) - w(x,t))_+\xi(x)\,dx\\ &\quad - G\int_0^L u(x,h,t) - u(x,-h,t))\xi_x(x)\,dx\\ &\quad + G(u(L,h,t) - u(L,-h,t) + 2hw_x(L,t))\xi(L).\end{aligned} \tag{3.29}$$

On the other hand, elementary manipulations yield

$$u(L,h,t) - u(L,-h,t) + 2hw_x(L,t) = \int_{-h}^{h} \big(u_y(L,y,t) + w_x(L,t)\big)\,dy$$

and, therefore, condition (2.18) implies that

$$u(L,h,t)-u(L,-h,t)+2hw_x(L,t)=0. \tag{3.30}$$

We now substitute (3.30) in (3.29) and choose $\xi(x)=\varphi(x)-\dot w(x,t)$ in the resulting equality to obtain

$$\begin{aligned}
&2hG\int_0^L w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\\
&=\int_0^L f(x,t)(\varphi(x)-\dot w(x,t))\,dx\\
&\quad+\lambda_{nc}\int_0^L(\Psi(x)-w(x,t))_+(\varphi(x)-\dot w(x,t))\,dx\\
&\quad-G\int_0^L(u(x,h,t)-u(x,-h,t))(\varphi_x(x)-\dot w_x(x,t))\,dx.
\end{aligned} \tag{3.31}$$

Next, we add (3.31) and (3.22) and use the definitions (3.14) and (3.15) to find that

$$\begin{aligned}
&E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dx\,dy\\
&+G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy\\
&+2hG\int_0^L w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\\
&+G\int_0^L(u(x,h,t)-u(x,-h,t))(\varphi_x(x)-\dot w_x(x,t))\,dx\\
&+G\int_0^L w_x(x,t)(\psi(x,h)-\dot u(x,h,t))\,dx\\
&-G\int_0^L w_x(x,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\\
&\ge(\mathbf{f},\mathbf{v}-\dot{\mathbf{u}}(t))_X.
\end{aligned} \tag{3.32}$$

We now use the identities

$$2hG\int_0^L w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dx=G\iint_\Omega w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\,dy,$$

$$\begin{aligned}
&G\int_0^L(u(x,h,t)-u(x,-h,t))(\varphi_x(x)-\dot w_x(x,t))\,dx\\
&=G\iint_\Omega u_y(x,y,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\,dy,
\end{aligned}$$

$$\begin{aligned}
&G\int_0^L w_x(x,t)(\psi(x,h)-\dot u(x,h,t))\,dx\\
&-G\int_0^L w_x(x,t)(\psi(x,-h)-\dot u(x,-h,t))\,dx\\
&=G\iint_\Omega w_x(x,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy
\end{aligned}$$

and inequality (3.32) to deduce that

$$\begin{aligned}
&E\iint_\Omega u_x(x,y,t)(\psi_x(x,y)-\dot u_x(x,y,t))\,dx\,dy\\
&+G\iint_\Omega u_y(x,y,t)(\psi_y(x,y)-\dot u_y(x,y,t))\,dx\,dy\\
&+G\iint_\Omega w_x(x,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\,dy\\
&+G\iint_\Omega u_y(x,y,t)(\varphi_x(x)-\dot w_x(x,t))\,dx\,dy\\
&+G\iint_\Omega w_x(x,t)(\psi_y(x,t)-\dot u_y(x,y,t))\,dx\,dy+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\\
&\ge(\mathbf{f},\mathbf{v}-\dot{\mathbf{u}}(t))_X.
\end{aligned}\tag{3.33}$$

Finally, using the definition (3.13) of the bilinear from a in (3.33) yields

$$a(\mathbf{u}(t),\mathbf{v}-\dot{\mathbf{u}}(t))+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\ge(\mathbf{f},\mathbf{v}-\dot{\mathbf{u}}(t))_X.\tag{3.34}$$

Combining now inequality (3.34) with the initial condition (2.29) we obtain the variational problem P_{2D}.

Problem $\mathcal{P}^V_{2D}$. *Given $\mathbf{u}_0=(u_0,v_0)$, find a pair of functions $\mathbf{u}=(u,w):[0,T]\to X$ such that*

$$\begin{gathered}
a(\mathbf{u}(t),\mathbf{v}-\dot{\mathbf{u}}(t))+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\\
\ge(\mathbf{f},\mathbf{v}-\dot{\mathbf{u}}(t))_X\quad\forall\mathbf{v}\in X,\ \text{a.e. } t\in(0,T),
\end{gathered}\tag{3.35}$$

$$\mathbf{u}(0)=\mathbf{u}_0.\tag{3.36}$$

This problem has the structure of an evolutionary variational inequality that appears often in contact problems with friction. However, the choice of the variables is the novelty in the problem. The solvability of Problem P^V_{2D} is shown in the next section. It is based on an abstract existence result for evolutionary variational inequalities that for the convenience of the reader, we recall in Section 6.

4. Existence result

Our existence and uniqueness result in the study of Problem P^V_{2D}, which is the other main result in this work (in addition to the model itself), is the following.

Theorem 4.1. *Assume that* (3.9)–(3.12) *hold and, moreover, assume that*

$$a(\mathbf{u}_0,\mathbf{v})+j(\mathbf{u}_0,\mathbf{v})\ge(\mathbf{f}(0),\mathbf{v})_X\quad\forall\mathbf{v}\in X.\tag{4.1}$$

Then, there exists a constant ω_0 that depends only on E,G,L and h such that Problem P^V_{2D} has at least one solution, provided that $\lambda_{nc}(1+\mu)\le\omega_0$. Moreover, the solution has the regularity $\mathbf{u}\in W^{1,\infty}(0;T;V)$.

The proof of Theorem 4.1 is carried out in several steps and it is based on Theorem 6.1. We assume in the sequel that (3.9)–(3.12) hold and we start by investigating the properties of the form a.

Lemma 4.2. *The bilinear form a defined by* (3.13) *is symmetric, continuous and coercive, i.e., it satisfies condition* (6.3).

Proof. First, we note that a is a bilinear and symmetric from on X. Moreover, an elementary computation shows that

$$a(\mathbf{u},\mathbf{v}) \le (E+2G)\|\mathbf{u}\|_X\|\mathbf{v}\|_X \quad \forall \mathbf{u},\mathbf{v}\in X, \tag{4.2}$$

which implies that a is continuous, i.e., it satisfies condition (6.3)(a). In addition, we claim that a is coercive i.e., there exists a constant $m>0$ such that

$$a(\mathbf{v},\mathbf{v}) \ge m\|\mathbf{v}\|_X^2 \quad \forall \mathbf{v}\in X. \tag{4.3}$$

The inequality is a direct consequence of the Korn's inequality. Nevertheless, for the convenience of the reader, we prove this claim and, to that end, we consider in what follows an arbitrary element $\mathbf{v}=(\psi(x,y),\varphi(x))\in X$. Then, the linearized strain tensor associated with the two-dimensional displacement field $\mathbf{v}$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \begin{pmatrix} \psi_x & \frac{1}{2}(\psi_y+\varphi_x) \\ \frac{1}{2}(\psi_y+\varphi_x) & 0 \end{pmatrix}$$

We denote in what follows by " $\cdot$ " the inner product in the space of the second order symmetric tensors on $\mathbb{R}^2$. We have

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2 = \boldsymbol{\varepsilon}(\mathbf{v})\cdot\boldsymbol{\varepsilon}(\mathbf{v}) = \psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2 \quad \text{a.e. on } \Omega. \tag{4.4}$$

Note also that the function $\mathbf{u}$ vanishes on the part of Γ characterized by $x=0$ which is, obviously, of positive one-dimensional measure. Therefore, we are in a position to use Korn's inequality (for a proof in there-dimensional case see, for instance, [17, p.79]). Therefore, there exists a constant $c_K>0$, which depends on h, such that

$$\iint_\Omega \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2\,dx\,dy \ge c_K\,\|\mathbf{v}\|^2_{H^1(\Omega)^2}. \tag{4.5}$$

We now combine (4.4) and (4.5) to deduce that

$$\iint_\Omega \big(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\big)\,dx\,dy \ge c_K \iint_\Omega \big(\psi^2+\psi_x^2+\psi_y^2+\varphi^2+\varphi_x^2\big)\,dx\,dy$$

and then, using (3.2)–(3.4), we obtain that

$$\iint_\Omega \big(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\big)\,dx\,dy \ge \widetilde{c}_K\|\mathbf{v}\|_X^2 \tag{4.6}$$

where $\widetilde{c}_K>0$ depends on c_K and h.

On the other hand, using the definition (3.13) and inequality (4.6) yields

$$a(\mathbf{v},\mathbf{v}) \ge \min(E,2G)\iint_\Omega \big(\psi_x^2 + \frac{1}{2}(\psi_y+\varphi_x)^2\big)\,dx\,dy. \tag{4.7}$$

We now combine (4.7), (4.6) and assumption (3.9) to see that inequality (6.3)(b) holds with $m=\widetilde{c}_K\min(E,2G)>0$, which concludes the proof. □

We turn now to the properties of the functional j given in (3.14). First, we note that j satisfies condition (6.4). Moreover, we have the following results.

Lemma 4.3. *The functional j satisfies assumptions* (6.9) *and* (6.10).

Proof. Let $\boldsymbol{\eta}=(\eta,\theta),\mathbf{u}=(u,w)$, $\overline{\mathbf{u}}=(\overline{u},\overline{w})\in X$ and let $\alpha\in(0,1]$. Using (3.14) results in

$$\begin{aligned} &j(\boldsymbol{\eta},\mathbf{u}-\overline{\mathbf{u}}-\lambda\mathbf{u}) - j(\boldsymbol{\eta},\mathbf{u}-\overline{\mathbf{u}}) \\ &= \alpha\lambda_{nc}\int_0^L (\Psi(x)-\theta(x))_+ w(x)\,dx \end{aligned}$$

$$+\mu\lambda_{nc}\int_0^L(\Psi(x)-\theta(x))_+\Big(|u(x)-\overline{u}(x)-\alpha u(x)|-|u(x)-\overline{u}(x)|\Big)\,dx,$$

and, since

$$\begin{aligned}&|u(x)-\overline{u}(x)-\alpha u(x)|-|u(x)-\overline{u}(x)|\\&=|(1-\alpha)(u(x)-\overline{u}(x))-\alpha\overline{u}(x)|-|u-\overline{u}(x)|\\&\le-\alpha|u(x)-\overline{u}(x)|+\alpha|\overline{u}(x)|\\&\le\alpha|\overline{u}(x)|\quad \text{a.e.}x\in[0,L],\end{aligned}$$

we deduce that

$$\begin{aligned}&j(\boldsymbol{\eta},\mathbf{u}-\overline{\mathbf{u}}-\alpha\mathbf{u})-j(\boldsymbol{\eta},\mathbf{u}-\overline{\mathbf{u}})\\&=\alpha\lambda_{nc}\int_0^L(\Psi(x)-\theta(x))_+w(x)\,dx+\alpha\mu\lambda_{nc}\int_0^L(\Psi(x)-\theta(x))_+|\overline{u}(x)|\,dx.\end{aligned}$$

Therefore, using definition (6.8), we obtain

$$\begin{aligned}j_2'(\boldsymbol{\eta},\mathbf{u}-\overline{\mathbf{u}};-\mathbf{u})\le\lambda_{nc}&\int_0^L(\Psi(x)-\theta(x))_+w(x)\,dx\\&+\mu\lambda_{nc}\int_0^L(\Psi(x)-\theta(x))_+|\overline{u}(x)|\,dx.\end{aligned}\tag{4.8}$$

Let us now consider the sequences $\{\mathbf{u}_n\}=\{(u_n,w_n)\}\subset X$, $\{t_n\}\subset[0,1]$ and let $\overline{\mathbf{u}}=(\overline{u},\overline{w})\in X$. Then, since $t_n\in[0,1]$, it is straightforward to see that

$$(\Psi(x)-t_nw_n(x))_+w_n(x)\le(\Psi(x)-t_nw_n(x))_+\Psi(x)\le0\quad\text{a.e. } x\in[0,L],$$

and, therefore, (4.8) yields

$$j_2'(t_n\mathbf{u}_n,\mathbf{u}_n-\overline{\mathbf{u}};-\mathbf{u}_n)\le\mu\lambda_{nc}\int_0^L(\Psi(x)-t_nw_n(x))_+|\overline{u}(x)|\,dx$$

for all $n\in\mathbb{N}$. Thus,

$$j_2'(t_n\mathbf{u}_n,\mathbf{u}_n-\overline{\mathbf{u}};-\mathbf{u}_n)\le\mu\lambda_{nc}\big(\|\Psi\|_{L^2(0,L)}+\|w_n\|_{L^2(0,L)}\big)\|\overline{u}\|_{L^2(0,L)},$$

and using (3.6) and (3.7) it is found that

$$j_2'(t_n\mathbf{u}_n,\mathbf{u}_n-\overline{\mathbf{u}};-\mathbf{u}_n)\le c_B\mu\lambda_{nc}\big(\|\Psi\|_{L^2(0,L)}+c_B\|\mathbf{u}_n\|_X\big)\|\overline{\mathbf{u}}\|_X.$$

It follows that if $\|\mathbf{u}_n\|_X\to\infty$, then

$$\liminf_{n\to\infty}\Big[\frac{1}{\|\mathbf{u}_n\|_V^2}j_2'(t_n\mathbf{u}_n,\mathbf{u}_n-\overline{\mathbf{u}};-\mathbf{u}_n)\Big]\le0,$$

which shows that j satisfies the assumption (6.9).

Let us now consider the sequences $\{\mathbf{u}_n\}=\{(u_n,w_n)\}\subset X$, $\{\boldsymbol{\eta}_n\}=\{(\eta_n,\theta_n)\}\subset X$ such that

$$\|\mathbf{u}_n\|_X\to\infty,\quad n\to\infty,\tag{4.9}$$

$$\|\boldsymbol{\eta}_n\|_X\le C,\quad n\in\mathbb{N},\tag{4.10}$$

where $C>0$. Let $\overline{\mathbf{u}}=(\overline{w},\overline{w})\in X$. Then, using (4.8), we have

$$\begin{aligned}j_2'(\boldsymbol{\eta}_n,\mathbf{u}_n-\overline{\mathbf{u}};-\mathbf{u}_n)\le\lambda_{nc}&\int_0^L(\Psi(x)-\theta_n(x))_+w_n(x)\,dx\\&+\mu\lambda_{nc}\int_0^L(\Psi(x)-\theta_n(x))_+|\overline{u}(x)|\,dx,\end{aligned}$$

which implies that

$$\begin{aligned} j_2'(\boldsymbol{\eta}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) &\le \lambda_{nc}\big(\|\Psi\|_{L^2(0,L)} + \|\theta_n\|_{L^2(0,L)}\big)\|w_n\|_{L^2(0,L)} \\ &\quad + \mu\lambda_{nc}\big(\|\Psi\|_{L^2(0,L)} + \|\theta_n\|_{L^2(0,L)}\big)\|\overline{u}\|_{L^2(0,L)}, \end{aligned}$$

for all $n \in \mathbb{N}$. Using again inequalities (3.6) and (3.7), we find that

$$\begin{aligned} j_2'(\boldsymbol{\eta}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) &\le c_B\lambda_{nc}\big(\|\Psi\|_{L^2(0,L)} + c_B\|\boldsymbol{\eta}_n\|_X\big)\|\mathbf{u}_n\|_X \\ &\quad + c_B\mu\lambda_{nc}\big(\|\Psi\|_{L^2(0,L)} + c_B\|\boldsymbol{\eta}_n\|_X\big)\|\overline{\mathbf{u}}\|_X, \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from (4.9) and (4.10) that

$$\liminf_{n\to\infty} \Big[\frac{1}{\|\mathbf{u}_n\|_V^2} j_2'(\boldsymbol{\eta}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n)\Big] \le 0.$$

This inequality shows that j satisfies assumption (6.10), which completes the proof. □

Lemma 4.4. *The functional j satisfies assumptions* (6.11) *and* (6.14)*.*

Proof. Let $\{\mathbf{u}_n\} = \{(u_n, w_n)\} \subset X$, $\{\boldsymbol{\eta}_n\} = \{(\eta_n, \theta_n)\} \subset X$ be two sequences such that $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} = (\eta, \theta) \in X$ and $\mathbf{u}_n \rightharpoonup \mathbf{u} = (u_n, w_n) \in X$. Using the compactness of the trace map, (3.8), it follows that

$$j(\boldsymbol{\eta}_n, \mathbf{v}) \to j(\boldsymbol{\eta}, \mathbf{v}) \quad \forall \mathbf{v} \in V, \quad j(\boldsymbol{\eta}_n, \mathbf{u}_n) \to j(\boldsymbol{\eta}, \mathbf{u})$$

as $n \to \infty$, which shows that the functional j satisfies the condition (6.11).

Assume now that $\{\mathbf{u}_n\}$ is a bounded, i.e.,

$$\|\mathbf{u}_n\|_X \le C \quad \forall n \in \mathbb{N}, \tag{4.11}$$

where $C > 0$. An elementary calculus based on the definition (3.14) and inequalities (3.6) and (3.7) yields

$$|j(\boldsymbol{\eta}_n, \mathbf{u}_n) - j(\boldsymbol{\eta}, \mathbf{u}_n)| \le c_B\lambda_{nc}(1+\mu)\big(\|\theta_n - \theta\|_{L^2(0,L)}\big)\|\mathbf{u}_n\|_X. \tag{4.12}$$

Note also that the convergence $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in X$ and (3.8) imply the strong convergence $\theta_n \to \theta \quad \text{in } L^2(0, L)$. Therefore,

$$\|\theta_n - \theta\|_{L^2(0,L)} \to 0. \tag{4.13}$$

We now combine inequality (4.12) with (4.11) and (4.13) and find that j satisfies assumption (6.14). □

Lemma 4.5. *The functional j satisfies the following inequalities*

$$j(\mathbf{u}, \mathbf{v} - \mathbf{u}) - j(\mathbf{v}, \mathbf{v} - \mathbf{u}) \le c_B^2\lambda_{nc}(1+\mu)\|\mathbf{u} - \mathbf{v}\|_X^2, \tag{4.14}$$

$$|j(\boldsymbol{\eta}, \mathbf{u})| \le c_B\lambda_{nc}(1+\mu)\big(\frac{1}{2}\|\Psi\|_{L^2(0,L)}^2 + \frac{1}{2}c_B^2\|\boldsymbol{\eta}\|_X^2 + \|\mathbf{u}\|_X^2\big), \tag{4.15}$$

for all $\boldsymbol{\eta}, \mathbf{u}, \mathbf{v} \in X$.

Proof. Let $\mathbf{u} = (u, w) \in X$ and $\mathbf{v} = (\psi, \varphi) \in X$. Then, using (3.14) we find that

$$\begin{aligned} &j(\mathbf{u}, \mathbf{v} - \mathbf{u}) - j(\mathbf{v}, \mathbf{v} - \mathbf{u}) \\ &\le \lambda_{nc}\|w - \varphi\|_{L^2(0,L)}^2 + \mu\lambda_{nc}\|w - \varphi\|_{L^2(0,L)}\|u - \psi\|_{L^2(0,L)}. \end{aligned}$$

Therefore, using the trace inequality (3.7) again, we find that j satisfies the (4.14).

Consider now two elements $\boldsymbol{\eta} = (\eta, \theta), \mathbf{u} = (u, w) \in X$. Then, using (3.14), it follows that

$$|j(\boldsymbol{\eta}, \mathbf{u})| \le c_B\lambda_{nc}(1+\mu)\big(\|\Psi\|_{L^2(0,L)} + c_B\|\theta\|_{L^2(0,L)}\big)\|u\|_{L^2(0,L)}.$$

Next, using (3.7) we obtain that

$$|j(\boldsymbol{\eta},\mathbf{u})| \le c_B\lambda_{nc}(1+\mu)\big(\|\Psi\|_{L^2(0,L)} + c_B\|\boldsymbol{\eta}\|_X\big)\|\mathbf{u}\|_X$$

and, using the inequality $ab \le \frac{1}{2}\,a^2 + \frac{1}{2}\,b^2$, we deduce that j satisfies (4.15). □

We have now all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. We check that the assumptions of Theorem 6.1 are satisfied. First, we recall that Lemma 4.2 guarantees that the form a satisfies condition (6.3) with

$$m = \widetilde{c}_K \min\{E, 2G\}. \tag{4.16}$$

We also recall that j satisfies the condition (6.4) and we note that assumption (3.10) guarantees that the element $\mathbf{f}$ given by (3.15) satisfies condition (6.5). It also follows from (3.12) and (4.1) that conditions (6.6) and (6.7) hold, too. Finally, Lemmas 4.3–4.5 show that conditions (6.9), (6.10), (6.11) and (6.14) hold.

Next, let

$$\omega_0 = \frac{m}{c_B^2 + c_B}, \tag{4.17}$$

which, clearly, depends only on E, G, L and h. Assume now that

$$\lambda_{nc}(1+\mu) < \omega_0. \tag{4.18}$$

Then,

$$c_B^2\lambda_{nc}(1+\mu) < m, \tag{4.19}$$

and, therefore, inequality (4.14) shows that condition (6.12) holds with

$$c_0 = c_B^2\lambda_{nc}(1+\mu). \tag{4.20}$$

Let $a_1; X \to \mathbb{R}$ and $a_2 : X \to \mathbb{R}$ be two functions defined by

$$a_1(\boldsymbol{\eta}) = c_B\lambda_{nc}(1+\mu), \quad a_2(\boldsymbol{\eta}) = c_B\lambda_{nc}(1+\mu)\big(\frac{1}{2}\|\Psi\|^2_{L^2(0,L)} + \frac{1}{2}c_B^2\|\boldsymbol{\eta}\|^2_X\big), \tag{4.21}$$

for all $\boldsymbol{\eta} \in X$. We now use (4.17)–(4.21) and inequality (4.15) to see that j satisfies condition (6.13).

We are now in a position to apply Theorem 6.1 since all the conditions are satisfied. Thus, we deduce that Problem $\mathcal{P}_{2D}^V$ has at least one solution $\mathbf{u} \in W^{1,\infty}(0,T;V)$. □

5. Concluding remarks

We now return to the 3D contact problem $\mathcal{P}_{3D}$. We note that this problem is formulated in an unbounded domain $\mathcal{B}$ and, for this reason, the standard arguments used in the literature for the variational analysis of 3D frictional contact problems cannot be applied in this case. Nevertheless, as shown in Section 2, assumptions (2.10) and (2.28) on the external forces and the initial displacement, respectively, combined with the specific geometry of $\mathcal{B}$ allow us to associate with Problem $\mathcal{P}_{3D}$ the two dimensional contact problem $\mathcal{P}_{2D}$. Theorem 4.1 provides the weak solvability of this simplified contact problem. Therefore, the solutions of Problem $\mathcal{P}_{2D}^V$ can be considered as weak solutions of the fully three dimensional contact problem $\mathcal{P}_{3D}$, as well. We conclude from here that, under assumptions (2.10) and (2.28), we provided the weak solvability of the Problem $\mathcal{P}_{3D}$. Moreover, its weak solutions have a special structure, (2.11), and can be obtained by solving a simplified two-dimensional problem, Problem $\mathcal{P}_{2D}$. These results represent an interesting addition to the MTCM since the theory contains very few results concerning the study of frictional contact problems defined on unbounded domains.

Moreover, we note that condition (4.1) represents a compatibility condition at the initial moment $t = 0$. Also, we recall that Theorem 4.1 provides the weak solvability of the contact problem $\mathcal{P}_{2D}$, under the smallness assumption $\lambda_{nc}(1+\mu) \le \omega_0$, which involves the stiffness and the friction coefficient. The question if this smallness assumption and the compatibility condition (4.1) represent an intrinsic feature of the contact problem or they represent only a limitation of our mathematical tools is an open problem. And so is the question of finding an accurate estimate of the critical value ω_0, as a function of the geometry of the problem and the elastic coefficients. Clearly, these two issues deserve to be studied in the future, as well as numerical algorithms and computer simulations of the problem. We recall that the uniqueness of the solution is left open. Finally, we would like note that such a model may be easily extended to the case of adhesion that was recently studied in [12], since the new 2D bar may replace the somewhat awkward use of a beam and a rod system, because it combines both.

6. Appendix

In this appendix we state an abstract existence result for evolutionary variational inequalities in a Hilbert space. The functional framework is the following.

Let X be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We denote by "$\rightharpoonup$" and "$\to$" the weak and strong convergence, respectively, on X. Below, $\mathbf{0}_X$ represent the zero element of X and a dot above represents the weak derivative with respect to the time variable. Then, the problem under consideration is:

Find $\mathbf{u} : [0,T] \to X$ *such that*

$$\begin{aligned} &a(\mathbf{u}(t), v - \dot{\mathbf{u}}(t)) + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ &\ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_X \quad \forall \mathbf{v} \in X, \quad \text{a.e. } t \in (0,T), \end{aligned} \tag{6.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{6.2}$$

In the study of (6.1)–(6.2) we consider the following assumptions:

$$\left\{\begin{aligned} &a : X \times X \to \mathbb{R} \text{ is a bilinear symmetric form and} \\ &\text{(a) there exists } M > 0 \text{ such that} \\ &\qquad |a(\mathbf{u},\mathbf{v})|_X \le M\|\mathbf{u}\|_X\|\mathbf{v}\|_X \quad \forall \mathbf{u},\mathbf{v} \in X; \\ &\text{(b) there exists } m > 0 \text{ such that } a(\mathbf{v},\mathbf{v}) \ge m\|\mathbf{v}\|_X^2 \text{ for all } \mathbf{v} \in X. \end{aligned}\right. \tag{6.3}$$

$$\left\{\begin{aligned} &j : X \times X \to \mathbb{R} \text{ and for every } \boldsymbol{\eta} \in X,\ j(\boldsymbol{\eta},\cdot) : X \to \mathbb{R} \text{ is a positively} \\ &\text{homogenuous subadditive functional, i.e.} \\ &\text{(a) } j(\boldsymbol{\eta}, \lambda\mathbf{u}) = \lambda j(\boldsymbol{\eta},\mathbf{u}) \text{ for all } \mathbf{u} \in X,\ \lambda \in \mathbb{R}_+; \\ &\text{(b) } j(\boldsymbol{\eta}, \mathbf{u}+\mathbf{v}) \le j(\boldsymbol{\eta},\mathbf{u}) + j(\boldsymbol{\eta},\mathbf{v}) \text{ for all } \mathbf{u},\ \mathbf{v} \in X. \end{aligned}\right. \tag{6.4}$$

$$\mathbf{f} \in W^{1,\infty}(0,T;X), \tag{6.5}$$

$$\mathbf{u}_0 \in X, \tag{6.6}$$

$$a(\mathbf{u}_0,\mathbf{v}) + j(\mathbf{u}_0,\mathbf{v}) \ge (\mathbf{f}(0),\mathbf{v})_X \quad \forall \mathbf{v} \in X. \tag{6.7}$$

Keeping in mind (6.4), it results that for all $\eta \in X$, $j(\boldsymbol{\eta}, \cdot) : X \to \mathbb{R}$ is a convex functional. Therefore, the directional derivative j_2' exists and is given by

$$j_2'(\boldsymbol{\eta}, \mathbf{u}; \mathbf{v}) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \Big[j(\boldsymbol{\eta}, \mathbf{u} + \lambda \mathbf{v}) - j(\boldsymbol{\eta}, \mathbf{u}) \Big] \quad \forall \boldsymbol{\eta}, \mathbf{u}, \mathbf{v} \in X. \tag{6.8}$$

We consider now the following additional assumptions on the functional j.

For every sequence $\{\mathbf{u}_n\} \subset X$ with $\|\mathbf{u}_n\|_X \to \infty$, every sequence $\{t_n\} \subset [0,1]$ and each $\overline{\mathbf{u}} \in X$ one has $\liminf_{n\to\infty} \big[\frac{1}{\|\mathbf{u}_n\|_X^2} j_2'(t_n \mathbf{u}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) \big] < m$. (6.9)

For every sequence $\{\mathbf{u}_n\} \subset X$ with $\|\mathbf{u}_n\|_X \to \infty$, every bounded sequence $\{\boldsymbol{\eta}_n\} \subset X$ and each $\overline{\mathbf{u}} \in X$ one has $\liminf_{n\to\infty} \big[\frac{1}{\|\mathbf{u}_n\|_X^2} j_2'(\boldsymbol{\eta}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) \big] < m$. (6.10)

For all sequences $\{\mathbf{u}_n\} \subset X$ and $\{\boldsymbol{\eta}_n\} \subset X$ such that $\mathbf{u}_n \rightharpoonup u \in X$, $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in X$ and for every $\mathbf{v} \in X$, one has $\limsup_{n\to\infty} [j(\boldsymbol{\eta}_n, \mathbf{v}) - j(\boldsymbol{\eta}_n, \mathbf{u}_n)] \le j(\boldsymbol{\eta}, \mathbf{v}) - j(\boldsymbol{\eta}, \mathbf{u})$. (6.11)

There exists $c_0 \in (0, m)$ such that $j(\mathbf{u}, \mathbf{v} - \mathbf{u}) - j(\mathbf{v}, \mathbf{v} - \mathbf{u}) \le c_0 \|\mathbf{u} - \mathbf{v}\|_X^2 \quad \forall \mathbf{u}, \mathbf{v} \in X$. (6.12)

There exist two functions $a_1 : X \to \mathbb{R}$ and $a_2 : X \to \mathbb{R}$ which map bounded sets in X into bounded sets in $\mathbb{R}$ such that $|j(\boldsymbol{\eta}, \mathbf{u})| \le a_1(\eta) \|\mathbf{u}\|_X^2 + a_2(\boldsymbol{\eta})$ for all $\boldsymbol{\eta}, \mathbf{u} \in X$, and $a_1(\mathbf{0}_X) < m - c_0$. (6.13)

For every sequence $\{\boldsymbol{\eta}_n\} \subset X$ with $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in X$, and every bounded sequence $\{\mathbf{u}_n\} \subset X$, one has $\lim_{n\to\infty} [j(\boldsymbol{\eta}_n, \mathbf{u}_n) - j(\boldsymbol{\eta}, \mathbf{u}_n)] = 0$. (6.14)

Theorem 6.1. *Assume that* (6.3)–(6.7) *and* (6.9)–(6.14) *hold. Then there exists at least one solution* $u \in W^{1,\infty}(0, T; X)$ *that satisfies* (6.1)–(6.2).

The above theorem was proved in [16]. The proof is based on time-discretization combined with monotonicity, compactness and lower semicontinuity arguments.

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