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PLACEMENT OF A SOURCE OR A WELL FOR OPTIMIZING THE ENERGY INTEGRAL

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Abstract. This article concerns the maximization and minimization of the energy integral associated with solutions to partial differential equations with coefficients depending on a suitable source or well. Under suitable geometrical conditions on the domain we find the optimal configurations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$, and let $f = f(x, y)$ and $g = g(x, y)$ be nonnegative bounded functions. We assume f to be positive in a subset with a positive measure. Consider the boundary value problem

$$-\Delta u + gu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

This Dirichlet problem has a unique solution $u \in H_0^1(\Omega)$. By standard regularity results, $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ and is positive on Ω.

The corresponding energy integral is the following

$$I = \int_\Omega (|\nabla u|^2 + gu^2)\, dx\, dy = \int_\Omega fu\, dx\, dy.$$

Let $\mathcal{F}$ be the class of rearrangements of a given function f_0. A typical problem is the investigation of the maximum or the minimum of I for $f \in \mathcal{F}$. Again, let $\mathcal{G}$ be the class of rearrangements of a given function g_0. One can investigate the maximum or the minimum of I for $g \in \mathcal{G}$. These problems have been discussed in many papers, we refer to [1, 2, 3] and references therein.

In this article we consider subclasses of $\mathcal{F}$ and $\mathcal{G}$. More precisely, let $f_0 = \chi_{D_0}$, where D_0 is a given subset of Ω. We shall investigate the maximum and the minimum of $I(D)$ for g fixed and $f = \chi_D$ where $D \subset \Omega$ is any translation or rotation of D_0. Furthermore, let $g_0 = \chi_{D_0}$, where D_0 is a given subset of Ω. We shall investigate the maximum and the minimum of $I(D)$ for f fixed and $g = \chi_D$. These problems are inspired by the paper [6], where the case of eigenvalues is discussed. However the situation here is different for the simultaneous presence of f and g.

We shall consider only the case D is a disc. In this case it is easy to prove existence for a maximizer or a minimizer. Our main effort will be the localization

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of a maximizer or a minimizer. This will be possible under suitable symmetry assumptions on Ω.

The equation $-\Delta u + gu = f$ models the temperature u in Ω in case of a steady state situation. The term gu corresponds to the density of heat absorbed, while f corresponds to the density of the heat produced. The energy integral $\int_\Omega fu\,dx\,dy$ is related with the average temperature in Ω. One may be interested in the maximization or minimization of the average temperature acting either on the data g or the data f. At the end of Section 2 and Section 3, some precise situation is described.

The paper is organized as follows. In Section 2 we study the case g fixed. In Section 3 we discuss the case f fixed. In Section 4 we assume Ω to be a ball in $\mathbb{R}^N$ and discuss the optimization of the energy integral in general classes of rearrangements.

2. First problem

Let Ω be a bounded plane domain, and let D be a disc contained in Ω. Consider the Dirichlet problem

$$-\Delta u + gu = \chi_D \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where $g = g(x, y)$ is a non negative bounded function. The corresponding energy integral is

$$I(D) = \int_\Omega (|\nabla u|^2 + gu^2)\,dx\,dy = \int_\Omega \chi_D u\,dx\,dy.$$

When the location of D (that we call source) changes through Ω, $I(D)$ may change. We are interested in finding the locations of D which realize the maximum or the minimum value of $I(D)$.

We first discuss the effect of a translation of the source. If D_ϵ is the domain D shifted at a distance ϵ in the x direction (assuming this is allowed), the new problem reads as

$$-\Delta u_\epsilon + gu_\epsilon = \chi_{D_\epsilon} \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega, \tag{2.2}$$

and the corresponding energy integral is

$$I(D_\epsilon) = \int_\Omega (|\nabla u_\epsilon|^2 + gu_\epsilon^2)\,dx\,dy = \int_\Omega \chi_{D_\epsilon} u_\epsilon\,dx\,dy.$$

So, we are interested in finding the sign of

$$I(D_\epsilon) - I(D) = \int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u\big]\,dx\,dy$$

for a small $\epsilon > 0$. From the equations in (2.1) and (2.2) we find

$$-\Delta(u_\epsilon - u) + g(u_\epsilon - u) = \chi_{D_\epsilon} - \chi_D. \tag{2.3}$$

The Alexandrov-Bakelman-Pucci maximum principle (see [4, Theorem 9.1]) applied to (2.3) yields

$$\sup_\Omega |u_\epsilon - u| \le C\|\chi_{D_\epsilon} - \chi_D\|_{L^2(\Omega)}, \tag{2.4}$$

where the constant C depends on Ω, but it is independent of ϵ and g (see [7]). Note that

$$|\chi_{D_\epsilon} - \chi_D|^2 = \begin{cases} 1 & x \in (D \setminus D_\epsilon) \cup (D_\epsilon \setminus D) \\ 0 & \text{elsewhere in } \Omega. \end{cases}$$

Moreover, if r is the radius of D we have $|D \setminus D_\epsilon| = |D_\epsilon \setminus D| < \pi r \epsilon$. Therefore, we can find a constant C such that

$$\|\chi_{D_\epsilon} - \chi_D\|_{L^2(\Omega)} \le C\epsilon^{\frac{1}{2}}. \tag{2.5}$$

Here and in what follows we denote by C constants (sometimes different from line to line) independent of ϵ. From (2.4) and (2.5) we find

$$\sup_\Omega |u_\epsilon - u| \le C\epsilon^{\frac{1}{2}}. \tag{2.6}$$

Note that (2.5) is equivalent to

$$\int_\Omega |\chi_{D_\epsilon} - \chi_D| \, dx \, dy \le C^2 \epsilon.$$

From the latter estimate and (2.6) we find

$$\int_\Omega (\chi_{D_\epsilon} - \chi_D)(u_\epsilon - u) \, dx \, dy = o(\epsilon). \tag{2.7}$$

Moreover, multiplying (2.1) by u_ϵ, (2.2) by u and integrating over Ω we find

$$\int_\Omega \chi_D u_\epsilon \, dx \, dy = \int_\Omega \chi_{D_\epsilon} u \, dx \, dy. \tag{2.8}$$

Hence,

$$\begin{aligned}
\frac{1}{\epsilon}\big[I(D_\epsilon) - I(D)\big] &= \frac{1}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u\big] \, dx \, dy \quad \text{by using (2.8)} \\
&= \frac{1}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u_\epsilon + \chi_{D_\epsilon} u - \chi_D u\big] \, dx \, dy \quad \text{by using (2.7)} \\
&= \frac{2}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u - \chi_D u\big] \, dx \, dy + \frac{o(\epsilon)}{\epsilon} \\
&= 2\int_D \frac{u(x+\epsilon, y) - u(x,y)}{\epsilon} \, dx \, dy + \frac{o(\epsilon)}{\epsilon}.
\end{aligned}$$

As $\epsilon \to 0$, we find that

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = 2\int_D \frac{\partial u}{\partial x}(x,y) \, dx \, dy.$$

Recall the Green formula

$$\int_D \frac{\partial u}{\partial x}(x,y) \, dx \, dy = \int_{\partial D} u(x,y) \cos(n,x) ds,$$

where n is the exterior normal to ∂D. Hence,

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = 2\int_{\partial D} u(x,y) \cos(n,x) ds. \tag{2.9}$$

To find the sign of the derivative in (2.9) we need some conditions on Ω and on the function $g(x,y)$.

Theorem 2.1. *Assume Ω to be Steiner symmetric with respect to the line $\{x = 0\}$. The function $g(x,y)$ is assumed to satisfy $g(x,y) = g(-x,y)$ and to be non-increasing with respect to x for $x < 0$. Then, if (x,y) is the center of D, the energy integral $I(D)$ associated with problem (2.1) increases as (x,y) approaches the point $(0,y)$.*

Proof. Suppose the center of D is located on $\{x < 0\}$. The line $\{x = a\}$ passing through the center of D divides Ω into Ω_1 (the small part) and Ω_2. Since Ω is Steiner symmetric, the reflection of Ω_1 with respect to the line $\{x = a\}$ is strictly contained in Ω_2. In Ω_1, define $w = u(2a - x, y) - u(x, y)$. Of course, we have $w = 0$ for $x = a$. Furthermore, recalling that $u(x, y) > 0$ in Ω and $u(x, y) = 0$ on $\partial\Omega$ we have $w(x, y) > 0$ on the part of the boundary $\partial\Omega_1$ with $x < a$. Since D is symmetric with respect to $\{x = a\}$, by (2.1) we find

$$-\Delta u(2a - x, y) + g(2a - x, y)u(2a - x, y) = \chi_D(2a - x, y) \quad \text{in} \quad \Omega_1.$$

Subtracting (2.1) from the latter equation we find

$$-\Delta w + g(2a - x, y)u(2a - x, y) - g(x, y)u(x, y) = \chi_D(2a - x, y) - \chi_D(x, y).$$

It is easy to check that $(x, y) \in D$ if and only if $(2a - x, y) \in D$. Therefore,

$$-\Delta w + g(2a - x, y)u(2a - x, y) - g(x, y)u(x, y) = 0 \quad \text{in} \quad \Omega_1. \tag{2.10}$$

We claim that $g(2a-x, y) \le g(x, y)$ for $x < a$. Indeed, if $2a-x \le 0$ (since $2a-x \ge x$ in Ω_1) this follows from the assumption that $g(x, y)$ is non-increasing with respect to x for $x < 0$. If $2a - x > 0$, using the assumption $g(x, y) = g(-x, y)$ we rewrite the inequality as $g(x - 2a, y) \le g(-x, y)$ and apply the condition that $g(x, y)$ is non-decreasing with respect to x for $x > 0$. The claim follows. Hence, from (2.10) we find that

$$-\Delta w + g(x, y)w(x, y) \ge 0 \quad \text{in } \Omega_1. \tag{2.11}$$

The strong maximum principle (see [4, Theorem 8.19]), yields $w(x, y) > 0$; that is, $u(2a - x, y) > u(x, y)$ in Ω_1. Hence, $u(x, y)$ computed at the right hand side of ∂D is larger than $u(x', y)$ computed at the left hand side of ∂D.

If we denote $(\partial D)^r$ the part of ∂D located at the right hand side with respect to the line $\{x = a\}$, and $(\partial D)^l$ the part of ∂D located at the left hand side with respect to the same line we have

$$\int_{\partial D} u(x, y)\cos(n, x)ds = \int_{(\partial D)^r} u(x, y)\cos(n, x)ds + \int_{(\partial D)^l} u(x, y)\cos(nx)ds.$$

Now, it is easy to note that $\cos(n, x)$ is positive at each point $(x, y) \in (\partial D)^r$. Moreover, if we take $(x, y) \in (\partial D)^r$ and $(x', y) \in (\partial D)^l$, the values of the corresponding $\cos(n, x)$ are opposite each other. This fact, coupled with the information that the value of $u(x, y)$ computed at $(x, y) \in (\partial D)^r$ is larger than the value of $u(x', y)$ computed at $(x', y) \in (\partial D)^l$ (for the same value of y) yields

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = 2\int_{\partial D} u(x, y)\cos(nx)ds > 0. \tag{2.12}$$

Therefore, the energy integral increases moving D in the x direction until the center of D belongs to the y axes.
By symmetry, if the center of D is located on $\{x > 0\}$, the energy integral increases moving D in the opposite direction of x until the center of D belongs to the y axes. The proof is complete. □

Theorem 2.1 allows us to solve the optimization problem in some simple cases.

- Let Ω be a disc (larger than D) centered at the origin $(0, 0)$. If $g(x, y) \ge 0$ is radially symmetric and non-decreasing with respect to the distance from (x, y) to $(0, 0)$, the maximum of $I(D)$ is attained when D is concentric with Ω, and the minimum is attained when D is (internally) tangent to $\partial\Omega$.

• Let Ω be the rectangle $(-a_1, a_1) \times (-a_2, a_2)$. Let $g(x, y) \geq 0$ satisfy $g(x, y) = g(-x, y) = g(x, -y)$, and let $g(x, y)$ be non-increasing with respect to x for $x < 0$ and non-increasing with respect to y for $y < 0$. Then the maximum of $I(D)$ occurs when D is centered at $(0, 0)$, and the minimum occurs when D is located at one corner of the rectangle (there are four locations for the minimum).

• Let Ω be a regular polygon of n sides centered at the origin. If $g(x, y) \geq 0$ is radially symmetric and non-decreasing with respect to the distance from (x, y) to $(0, 0)$, the maximum of $I(D)$ is attained when D is concentric with Ω, and the minimum occurs when D is located at one corner (there are n locations for the minimum).

• By the proof of Theorem 2.1 we can find indications for the maximum even if Ω is not Steiner symmetric as in the following example. Let Ω be the union of the half disc $\{x^2 + y^2 < 1,\ x \leq 0\}$ and the square $\{0 < x < 2,\ -1 < y < 1\}$. Note that Ω is Steiner symmetric with respect to the x axis, but it is not symmetric with respect to any line $\{x = a\}$. However, the reflection of the half disc $\{x^2 + y^2 < 1,\ x \leq 0\}$ with respect to $\{x = 0\}$ is contained in Ω, and the reflection of the rectangle $\{1 \leq x < 2,\ -1 < y < 1\}$ with respect to the line $\{x = 1\}$ is contained in Ω. If $g(x, y)$ satisfy suitable conditions (for example if $g(x, y)$ is a constant) one can conclude that if $I(D)$ is maximum then the center of D is located on $(a, 0)$, where a has a suitable value such that $0 < a < 1$.

Theorem 2.1 also holds for $N = 1$. Let us show that without any assumption on $g(x)$ the result may not hold. Consider the following example (corresponding to the case the segment D is centered at the origin)

$$-u'' + \lambda^2 \chi_{[-1,1]} u = \chi_{[-1,1]}, \quad u(-2) = u(2) = 0.$$

Here λ is any real positive number. Note that the function $g(x) = \lambda^2 \chi_{[-1,1]}(x)$ does not satisfy the monotonicity condition required by Theorem 2.1. The solution of this problem can be written as

$$u(x) = \begin{cases} A(e^{\lambda x} + e^{-\lambda x}) + \lambda^{-2} & |x| \leq 1, \\ B(|x| - 2) & 1 < |x| < 2, \end{cases}$$

where A and B satisfy

$$A(e^{\lambda} + e^{-\lambda}) + \lambda^{-2} = -B, \quad \lambda A(e^{\lambda} - e^{-\lambda}) = B.$$

We find that

$$A\big(e^{\lambda} + e^{-\lambda} + \lambda(e^{\lambda} - e^{-\lambda})\big) + \lambda^{-2} = 0.$$

It is clear that $A < 0$. Hence,

$$I = \int_{-1}^{1} u(x)\, dx = \int_{-1}^{1} \big[A(e^{\lambda x} + e^{-\lambda x}) + \lambda^{-2}\big] dx < \frac{2}{\lambda^2}.$$

Now we consider the following problem (note that D is not centered at the origin)

$$-u'' + \lambda^2 \chi_{[-1,1]} u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

Let us show that for $1 < x < 2$ we have $u(x) > v(x)$, where

$$-v'' = 1, \quad v(1) = v(2) = 0.$$

This fact follows from the familiar comparison principle in that, for $1 < x < 2$ we have

$$-u'' = 1, \quad u(1) > 0, \quad u(2) = 0.$$

Therefore,

$$J = \int_0^2 u(x)\,dx > \int_1^2 v(x)\,dx = \int_1^2 \Big(-\frac{x^2}{2} + \frac{3}{2}x - 1\Big)dx = \frac{1}{12}.$$

As a consequence, for λ such that $\frac{2}{\lambda^2} \le \frac{1}{12}$ the energy integral I is less than the energy integral J. Hence, the maximum of the energy integral does not correspond to the case D is centered at the origin.

Let us give a simple model described by problem (2.1) in case of $g(x,y) \equiv 0$. Let Ω be a plane thermal conductor subject to a source of heat with density $\chi_D(x,y)$. In other words, we have a stove which occupies D and produces heat with density one (in D only). Assume the temperature is equal to zero on the boundary $\partial\Omega$. Then, in a steady state situation, the solution $u(x,y)$ to problem (2.1) yields the temperature in Ω. The energy integral

$$I(D) = \int_D u_D(x,y)\,dx\,dy$$

is related with the average of the temperature in D, and it depends on the location of our stove in Ω. By our previous results, for particular domains we can find the exact location of D which maximizes $I(D)$. This result agrees with the physical intuition in that, for maximizing the energy integral, D has to be located as far as possible from the boundary $\partial\Omega$.

3. Second problem

Now we discuss a new problem which is, in some sense, complementary to the previous one. Using the previous notation, consider the boundary-value problem

$$-\Delta u + \chi_D u = f, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where $f = f(x,y) \ge 0$ is a bounded function, positive in a subset of positive measure. Physically $f(x,y)$ represents the density of the heat produced (source) and $\chi_D u$ represents the density of the heat absorbed (well). The associated energy integral is

$$I(D) = \int_\Omega f(x,y) u_D\,dx\,dy.$$

Recall that D is a disc with a fixed radius. We shall investigate the minimum and the maximum of $I(D)$ for $D \subset \Omega$.

As in the previous case, we investigate the effect of a translation of D. If D_ϵ is the domain D shifted at a distance ϵ in the x direction (assuming this is allowed), the new problem reads as

$$-\Delta u_\epsilon + \chi_{D_\epsilon} u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega, \tag{3.2}$$

and the corresponding energy integral is

$$I(D_\epsilon) = \int_\Omega f(x,y) u_\epsilon\,dx\,dy.$$

So, we are interested in finding the sign of

$$I(D_\epsilon) - I(D) = \int_\Omega f(x,y)(u_\epsilon - u)\,dx\,dy$$

for a small $\epsilon > 0$. Here we have denoted with u_ϵ the solution to problem (3.2), and with u the solution to problem (3.1). From equation (3.1) we find

$$\int_\Omega \big(\nabla u \cdot \nabla u_\epsilon + \chi_D u u_\epsilon\big)\, dx\, dy = \int_\Omega f(x,y) u_\epsilon\, dx\, dy.$$

Similarly, from (3.2) we find

$$\int_\Omega \big(\nabla u_\epsilon \cdot \nabla u + \chi_{D_\epsilon} u_\epsilon u\big)\, dx\, dy = \int_\Omega f(x,y) u\, dx\, dy.$$

It follows that

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u\, dx\, dy = \int_\Omega f(x,y)(u_\epsilon - u)\, dx\, dy = I(D_\epsilon) - I(D). \tag{3.3}$$

From equations (3.1) and (3.2) we also find

$$-\Delta(u_\epsilon - u) + \chi_D(u_\epsilon - u) = (\chi_D - \chi_{D_\epsilon}) u_\epsilon.$$

By Alexandrov-Bakelman-Pucci maximum principle applied to the latter equation we have

$$\sup_\Omega |u_\epsilon - u| \le C \|(\chi_D - \chi_{D_\epsilon}) u_\epsilon\|_{L^2(\Omega)} \le C \|\chi_D - \chi_{D_\epsilon}\|_{L^2(\Omega)} \sup_\Omega u_\epsilon. \tag{3.4}$$

The Alexandrov-Bakelman-Pucci maximum principle applied to (3.1) and (3.2) also yields

$$\sup_\Omega u \le C\|f\|_{L^2(\Omega)} \quad \text{and} \quad \sup_\Omega u_\epsilon \le C\|f\|_{L^2(\Omega)}. \tag{3.5}$$

Moreover, as already observed, it is easy to find a constant C such that

$$\int_\Omega |\chi_{D_\epsilon} - \chi_D|\, dx\, dy \le C\epsilon \quad \text{and} \quad \|\chi_D - \chi_{D_\epsilon}\|_{L^2(\Omega)} \le C\epsilon^{\frac{1}{2}}. \tag{3.6}$$

Hence, (3.4) implies

$$\sup_\Omega |u_\epsilon - u| \le C\epsilon^{\frac{1}{2}}. \tag{3.7}$$

Since

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u\, dx\, dy = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2\, dx\, dy + \int_\Omega (\chi_D - \chi_{D_\epsilon})(u_\epsilon - u) u\, dx\, dy,$$

using (3.6) and (3.7) we find

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u\, dx\, dy = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2\, dx\, dy + o(\epsilon).$$

Now, by (3.3) and the latter equation we have

$$I(D_\epsilon) - I(D) = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2\, dx\, dy + o(\epsilon).$$

Finally, we get

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{I(D_\epsilon) - I(D)}{\epsilon} &= -\lim_{\epsilon \to 0} \int_\Omega \frac{\chi_{D_\epsilon} - \chi_D}{\epsilon} u^2\, dx\, dy \\
&= -\lim_{\epsilon \to 0} \int_D \frac{u^2(x+\epsilon, y) - u^2(x,y)}{\epsilon}\, dx\, dy \\
&= -\int_D \frac{\partial u^2}{\partial x}\, dx\, dy.
\end{aligned}$$

Hence,

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = -\int_D \frac{\partial u^2}{\partial x}\,dx\,dy.$$

By using Green's formula we find

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = -\int_{\partial D} u^2\ \cos(n,x)ds. \tag{3.8}$$

Theorem 3.1. *Assume Ω to be Steiner symmetric with respect to the line $\{x = 0\}$. The function $f(x,y)$ is assumed to satisfy $f(x,y) = f(-x,y)$ and to be non-decreasing with respect to x for $x < 0$. Then the energy integral $I(D)$ associated with problem* (3.1) *decreases as the center (x,y) of the disc D approaches the point $(0,y)$.*

Proof. To find the sign of the derivative (3.8), we proceed as in the previous case. Suppose the center of D is located on $\{x < 0\}$. The line $\{x = a\}$ passing through the center of D divides Ω into Ω_1 and Ω_2, and the reflection of Ω_1 with respect to the line $\{x = a\}$ is strictly contained in Ω_2. In Ω_1, define $w = u(2a-x,y)-u(x,y)$. We have $w = 0$ for $x = a$ and $u(x,y) = 0$ on the part of the boundary $\partial\Omega_1$ with $x < a$.

Since D is symmetric with respect to $\{x = a\}$ we have $\chi_D(x,y) = \chi_D(2a-x,y)$, and equation (3.1) at the point $(2a-x,y)$ reads as

$$-\Delta u(2a-x,y) + \chi_D u(2a-x,y) = f(2a-x,y).$$

Subtracting the equation (3.1) from the latter equation we find

$$-\Delta w + \chi_D w = f(2a-x,y) - f(x,y). \tag{3.9}$$

We claim that for $x < a$ we have

$$f(2a-x,y) - f(x,y) \geq 0.$$

Indeed, if $2a-x \leq 0$ (recall that $2a-x > x$ in Ω_1) the inequality holds since $f(x,y)$ is non-decreasing for $x < 0$. If $2a-x > 0$, recalling that $f(-x,y) = f(x,y)$ we can rewrite the inequality as

$$f(x-2a,y) - f(-x,y) \geq 0,$$

and this holds since $x-2a < -x$ and $f(x,y)$ is non-increasing for $x > 0$. The claim follows. Hence, (3.9) yields

$$-\Delta w + \chi_D w \geq 0 \quad \text{in } \Omega_1. \tag{3.10}$$

By the strong maximum principle we have $w(x,y) > 0$, and, equivalently,

$$u(2a-x,y) > u(x,y) \quad \text{in } \Omega_1.$$

Hence, $u(x,y)$ computed at the right hand side of ∂D is larger than $u(x',y)$ computed at the left hand side of ∂D (for the same value of y). Arguing as in the previous case we find

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = -\int_{\partial D} u^2(x,y)\ \cos(n,x)ds < 0.$$

Therefore, the energy integral decreases moving D in the x direction until the center of D belongs to the y axes. By symmetry, if the center of D is located on $\{x > 0\}$, the energy integral decreases moving D in the opposite direction of x until the center of D belongs to the y axes. The proof is complete. □

We underline that in this second problem, $f(x,y)$ is non-decreasing with respect to x for $x < 0$, whereas, in the first problem, $g(x,y)$ was non-increasing. The conclusions concerning the location of the maximum and the minimum are reversed. As in the previous case, Theorem 3.1 allows one to solve the optimization problems for special domains Ω.

Theorem 3.1 can be easily extended to the case χ_D is replaced by $\lambda^2\chi_D$ for any $\lambda > 0$. However, it fails to hold even in dimension $N = 1$ without appropriate assumptions on $f(x)$. Indeed, let

$$-u'' + \lambda^2\chi_{[-1,1]}u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

As already noticed, the corresponding energy integral J satisfies

$$J(\lambda) = \int_0^2 u(x)\,dx > \frac{1}{12} \quad \text{for any } \lambda > 0.$$

On the other hand, let

$$-u'' + \lambda^2\chi_{[0,2]}u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

One finds

$$u(x) = \begin{cases} A(x+2) & \text{if } -2 < x < 0 \\ Be^{\lambda x} + Ce^{-\lambda x} + \frac{1}{\lambda^2} & \text{if } 0 < x < 2, \end{cases}$$

with

$$2A = B + C + \frac{1}{\lambda^2}, \quad A = \lambda(B - C), \quad Be^{2\lambda} + Ce^{-2\lambda} + \frac{1}{\lambda^2} = 0.$$

By easy computation one finds

$$B = -\frac{(2\lambda+1)e^{2\lambda} - 1}{\lambda^2\big(2\lambda - 1 + e^{4\lambda}(2\lambda+1)\big)},$$
$$C = -\frac{(2\lambda-1)e^{2\lambda} + e^{4\lambda}}{\lambda^2\big(2\lambda - 1 + e^{4\lambda}(2\lambda+1)\big)}.$$

Since B and C have a negative sign, the corresponding energy integral satisfies

$$I(\lambda) = \int_0^2 \Big(Be^{\lambda x} + Ce^{-\lambda x} + \frac{1}{\lambda^2}\Big)dx < \frac{2}{\lambda^2}.$$

As a consequence, for λ such that $\frac{2}{\lambda^2} \le \frac{1}{12}$ we have $I(\lambda) < J(\lambda)$. Therefore, the configuration corresponding to $D = [-1,1]$ cannot be a minimum of the energy integral.

We have a physical interpretation also for problem (3.1). As before, let Ω be a plane thermal conductor subject to a source of heat with density $f(x,y) \equiv 1$. The term $\chi_D u$ in the left hand side of the equation corresponds to a device which absorbs heat (like any fire extinguisher) located in D. Assume the temperature is equal to zero on the boundary $\partial\Omega$ and that we are in a steady state situation. The solution $u(x,y)$ to problem (3.1) yields the temperature in Ω, and the energy integral

$$I(D) = \int_\Omega f(x,y)u_D(x,y)\,dx\,dy$$

is related to the average temperature in Ω, and it depends on the location of D. By our results, for special domains Ω we can find the exact location of D which minimizes $I(D)$. This result agrees with the physical intuition in that, for minimizing the energy integral (when $f(x,y)\equiv 1$), D has to be located as far as possible from the boundary $\partial\Omega$.

4. Rearrangements

In this section we consider $\Omega = B$, a ball in $\mathbb{R}^N$. Define $\mathcal{F}$ as the class of rearrangements of a bounded non-negative function f_0, and $\mathcal{G}$ as the class of rearrangements of a bounded non-negative function g_0. For $f\in\mathcal{F}$ and $g\in\mathcal{G}$, let

$$-\Delta u + gu = f \text{ in } B, \quad u = 0 \text{ on } \partial B. \tag{4.1}$$

We want to discuss the maximization and the minimization of the functionals

$$\int_B f u_f\,dx,\ f\in\mathcal{F}, \quad \text{and} \quad \int_B f u_g\,dx\ g\in\mathcal{G}.$$

In case of $f_0=\chi_D$, the class $\mathcal{F}$ is the family of all functions χ_E with $|E|=|D|$ (so, E is not necessarily a translate of D). Therefore, our optimization problems are more general than those treated in the previous sections.

Let f^* be the radially symmetric non-increasing rearrangement of f, and let f_* be the radially symmetric non-decreasing rearrangement of f. We shall use the following classical results.

$$\int_B f_* g^*\,dx \le \int_B fg\,dx \le \int_B f^* g^*\,dx. \tag{4.2}$$

If $u\ge 0$ and $u\in H_0^1(B)$ then $u^*\ge 0$ and $u^*\in H_0^1(B)$. Furthermore,

$$\int_B |\nabla u^*|^2 dx \le \int_B |\nabla u|^2 dx. \tag{4.3}$$

For a proof of (4.2) and (4.3) we refer to [5].

Another tool we shall use is the variational characterization of the solution u to problem (4.1), that is

$$\int_B fu\,dx = \int_B \big(2fu - |\nabla u|^2 - gu^2\big)\,dx = \sup_{v\in H_0^1(B)} \int_B \big(2fv - |\nabla v|^2 - gv^2\big)\,dx. \tag{4.4}$$

One result is the following. If $g=g_*$ then

$$\begin{aligned}
\int_B f u_f\,dx &= \int_B (2fu_f - |\nabla u_f|^2 - gu_f^2)\,dx \quad \text{using (4.2) and (4.3)}\\
&\le \int_B \big(2f^* u_f^* - |\nabla u_f^*|^2 - g(u_f^*)^2\big)\,dx \quad \text{using (4.4)}\\
&\le \int_B \big(2f^* u_{f^*} - |\nabla u_{f^*}|^2 - g u_{f^*}^2\big)\,dx\\
&= \int_B f^* u_{f^*}\,dx.
\end{aligned}$$

Therefore, f^* is a maximizer for $\int_B fu_f\,dx$ in $\mathcal{F}$. If $f_0=\chi_D$ then $f^*=\chi_{\hat{D}}$, where $\hat{D}$ is a ball concentric with B and $|\hat{D}|=|D|$. This result is in accordance with Theorem 2.1 concerning the maximum. The situation is different for the minimum.

In addition to the condition $g=g_*$ we suppose that the solution $u(r)$ to the problem

$$-r^{1-N}(r^{N-1}u')' + gu = f_*, \quad u'(0) = u(R) = 0$$

satisfies $u'(r) \leq 0$ in $(0, R)$. Then $u_{f_*} = u^*_{f_*}$ and

$$\begin{aligned}\int_B fu_f\,dx &= \int_B (2fu_f - |\nabla u_f|^2 - gu_f^2)\,dx \quad \text{using (4.4)}\\ &\geq \int_B \big(2fu_{f_*} - |\nabla u_{f_*}|^2 - gu_{f_*}^2\big)\,dx \quad \text{using (4.2)}\\ &\geq \int_B \big(2f_*u_{f_*} - |\nabla u_{f_*}|^2 - gu_{f_*}^2\big)\,dx = \int_B f_*u_{f_*}\,dx.\end{aligned}$$

Therefore, f_* is a minimizer for $\int_B fu_f\,dx$ in $\mathcal{F}$. If $f_0 = \chi_D$ then $f_* = \chi_{\check{D}}$, where $\check{D} = B \setminus E$. Here E is a ball concentric with B such that $|B \setminus E| = |D|$.

The same method can be used for investigating the functional $g \to \int_B fu_g\,dx$ for $g \in \mathcal{G}$. The results are the following. If $f = f^*$ then one has

$$\int_B fu_g\,dx \leq \int_B fu_{g_*}\,dx.$$

So, g_* corresponds to a maximum. To investigate the minimum, in addition to the condition $f = f^*$ we suppose that the solution $u(r)$ to the problem

$$-r^{1-N}(r^{N-1}u')' + g_*u = f, \quad u'(0) = u(R) = 0 \tag{4.5}$$

satisfies $u'(r) \leq 0$ in $(0, R)$. Then one finds

$$\int_B fu_g\,dx \geq \int_B fu_{g^*}\,dx.$$

So, g^* corresponds to a minimum. This result is in accordance with Theorem 3.1 concerning the minimum, but, we have used the additional condition that the solution u to problem (4.5) is decreasing on $(0, R)$. This fact is not a surprise in that the class of rearrangements now is larger than that used in Theorem 3.1.

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