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REGULARITY CRITERIA FOR WEAK SOLUTIONS TO 3D INCOMPRESSIBLE MHD EQUATIONS WITH HALL TERM

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ABSTRACT. We study the regularity conditions for a weak solution to the incompressible 3D magnetohydrodynamic equations with Hall term in the whole space $\mathbb{R}^3$. In particular, we show the regularity criteria in view of gradient vectors in various spaces.

1. INTRODUCTION

We consider the incompressible 3D magneto hydro dynamic (MHD) equations with Hall term

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b, \tag{1.1}$$

$$\partial_t b - \Delta b + u \cdot \nabla b - b \cdot \nabla u + \nabla \times ((\nabla \times b) \times b) = 0, \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.3}$$

Here $u : Q_T := \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ is the flow velocity vector, $b : Q_T \to \mathbb{R}^3$ is the magnetic vector, $\pi = p + \frac{|b|^2}{2} : Q_T \to \mathbb{R}$ is the total pressure. We consider the initial value problem of (1.1)–(1.3), which requires initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad b(x, 0) = b_0(x) \quad x \in \mathbb{R}^3 \tag{1.4}$$

The initial conditions satisfy the compatibility condition, i.e.

$$\operatorname{div} u_0(x) = 0, \quad \text{and} \quad \operatorname{div} b_0(x) = 0.$$

Definition 1.1. A weak solution pair (u, b) of the incompressible 3D MHD equations with the Hall term (1.1)–(1.4) is regular in Q_T provided that $\|u\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)} < \infty$.

For a long time, the effects of Hall current on fluids has been a subject of great interest to researchers. A current induced in a direction normal to the electric and magnetic fields is commonly called Hall current [22]. In particular, the effects of Hall current are very important if the strong magnetic field is applied

The mathematical derivations of the incompressible 3D MHD equations with the Hall term could be given in [1] from either two-fluids or kinetic models. It is well-known that the global existence of weak solutions, local existence and uniqueness

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of smooth solutions to the system (1.1)–(1.4) were established in [5, 6]. Recently, various results for this equation were proved in view of partial regularity, temporary decay and regularity or blow-up conditions (see [5, 6, 7, 8, 9, 12, 11, 21, 26, 27] and references therein.)

We list only some results relevant to our concerns. In view of the regularity conditions in Lorentz space, He and Wang [13] proved that a weak solution pair (u, b) becomes regular in the presence of a certain type of the integral conditions, typically referred to as Serrin's condition, namely,

$$u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3)) \quad \text{with } 3/p + 2/q \le 1,\ 3 < p \le \infty,$$

or

$$\nabla u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3)) \quad \text{with } 3/p + 2/q \le 2,\ \frac{3}{2} < p \le \infty,$$

(also see [3, 4, 17]). Also, Wang proved in [25] that a weak solution pair (u, b) become regular if u satisfies

$$u \in L^2(0, T; BMO(\mathbb{R}^3).$$

On the other hand, recently, Zhang [27] obtained the regularity criterion

$$u \in L^{\frac{2}{1-\alpha}}(0, T; \dot{B}^{-\alpha}_{\infty,\infty}), \quad \nabla b \in L^{\frac{2}{1-\beta}}(0, T; \dot{B}^{-\beta}_{\infty,\infty}) \tag{1.5}$$

with $-1 < \alpha < 1$ and $0 < \beta < 1$. Our study is motivated by these viewpoints, we obtain the regularity conditions for a weak solution to the incompressible 3D MHD equations with the Hall term (1.1)–(1.4) in a whole space. Our proof of main results is based on a priori estimate for the gradient of the velocity field.

Our main results reads as follows.

Theorem 1.2. *Suppose that (u, b) is a weak solution of (1.1)–(1.4) with initial condition $u_0, b_0 \in H^2(\mathbb{R}^3)$. If (u, b) satisfies one of the following cases:*

$$\int_0^T \Big(\|\nabla u\|^q_{L^{p,\infty}} + \|\nabla b\|^m_{L^{l,\infty}}\Big) dt < \infty \tag{1.6}$$

with the relations $\frac{3}{p} + \frac{2}{q} = 2$, $\frac{3}{2} < p \le \infty$ and $\frac{3}{l} + \frac{2}{m} = 1$, $3 < l \le \infty$. or

$$\int_0^T \Big(\|\nabla u\|^q_{L^{p,\infty}} + \|\nabla b\|^{\frac{2}{1-\beta}}_{\dot{B}^{-\beta}_{\infty,\infty}}\Big) dt < \infty \tag{1.7}$$

with the relations $\frac{3}{p} + \frac{2}{q} = 2$, $\frac{3}{2} < p \le \infty$ and $0 < \beta < 1$, then (u, b) is regular in Q_T.

Theorem 1.3. *Suppose that (u, b) is a weak solution of (1.1)–(1.4) with initial condition $u_0, b_0 \in H^2(\mathbb{R}^3)$. If (u, b) satisfies one of the following two conditions:*

$$\int_0^T \|\nabla u\|_{BMO(\mathbb{R}^3)} + \|\nabla b\|^2_{BMO(\mathbb{R}^3)}\ dt < \infty, \tag{1.8}$$

or

$$\int_0^T \|\nabla u\|^2_{BMO^{-1}(\mathbb{R}^3)} + \|\nabla^2 b\|^2_{BMO^{-1}(\mathbb{R}^3)}\ dt < \infty. \tag{1.9}$$

then (u, b) is regular in Q_T.

Theorem 1.2 extends the result by He and Wang [13] with respect to the gradient of the velocity field. Moreover, using the estimate in [24, Lemma A.5], we obtain $BMO^{-1}(\mathbb{R}^3)$-regularity condition.

This article is organized as follows: In Section 2 we recall the notion of weak solutions and review some known results. In Section 3, we present the proofs of the Theorem 1.2 and 1.3.

2. Preliminaries

In this section we introduce the notation and definitions to be used in this paper. We also recall the well-known results for our analysis. For $1 \le q \le \infty$, $W^{k,q}(\mathbb{R}^3)$ indicates the usual Sobolev space with standard norm $\|\cdot\|_{k,q}$, i.e.

$$W^{k,q}(\mathbb{R}^3) = \{u \in L^q(\mathbb{R}^3) : D^\alpha u \in L^q(\mathbb{R}^3), 0 \le |\alpha| \le k\}.$$

When $q = 2$, we denote $W^{k,q}(\mathbb{R}^3)$ by H^k. All generic constants will be denoted by C, which may vary from line to line.

2.1. **BMO and Lorentz spaces.** The John-Nirenberg space or the Bounded Mean Oscillation space (in short BMO space) [14] consists of all functions f which are integrable on every ball $B_R(x) \subset \mathbb{R}^3$ and satisfy:

$$\|f\|_{BMO}^2 = \sup_{x\in\mathbb{R}^3} \sup_{R>0} \frac{1}{B(x,R)} \int_{B(x,R)} |f(y) - f_{B_R}(y)| dy < \infty.$$

Here, f_{B_R} is the average of f over all ball $B_R(x)$ in $\mathbb{R}^3$. It will be convenient to define BMO in terms of its dual space, $\mathcal{H}^1$. On the other hand, following [16] let w be the solution to the heat equation $w_t - \Delta w = 0$ with initial data v. Then

$$\|v\|_{BMO}^2 = \sup_{x\in\mathbb{R}^3} \sup_{R>0} \frac{1}{B(x,R)} \int_{B(x,R)} \int_0^{R^2} |w|^2 \, dt \, dy.$$

and define the BMO^{-1}-norm by

$$\|v\|_{BMO^{-1}}^2 = \sup_{x\in\mathbb{R}^3} \sup_{R>0} \frac{1}{B(x,R)} \int_{B(x,R)} \int_0^{R^2} |\nabla w|^2 \, dt \, dy.$$

We note that if u is a tempered distribution. Then $u \in BMO^{-1}$ if and only if there exist $f^i \in BMO$ with $u = \sum \partial_i f^i$ in [16, Theorem 1].

Let $m(\varphi, t)$ be the Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}$, i.e.

$$m(\varphi, t) := m\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}.$$

We denote by the Lorentz space $L^{p,q}(\mathbb{R}^3)$ with $1 \le p,\ q \le \infty$ with the norm [23]

$$\|\varphi\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \Big(\int_0^\infty t^q (m(\varphi,t))^{q/p} \frac{dt}{t} \Big)^{1/q} < \infty, & \text{for } 1 \le q, \\ \sup_{t\ge 0}\{t (m(\varphi,t))^{1/p}\}, & \text{for } q = \infty\,. \end{cases} \tag{2.1}$$

Followed in [23], Lorentz space $L^{p,q}(\mathbb{R}^3)$ may be defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3),\, L^{p_2}(\mathbb{R}^3))_{\alpha,q}, \tag{2.2}$$

with $\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}$, $1 \le p_1 < p < p_2 \le \infty$. From the interpolation method above, we note that

$$L^{\frac{2p}{p-1},2}(\mathbb{R}^3) = \Big(L^2(\mathbb{R}^3), L^6(\mathbb{R}^3) \Big)_{\frac{3}{2p},2}. \tag{2.3}$$

We also need the Hölder inequality in Lorentz spaces (see [20]) for our proof.

Lemma 2.1. *Assume* $1 \le p_1,\ p_2 \le \infty,\ 1 \le q_1,\ q_2 \le \infty$ *and* $u \in L^{p_1,q_1}(\mathbb{R}^3)$, $v \in L^{p_2,q_2}(\mathbb{R}^3)$. *Then* $uv \in L^{p_3,q_3}(\mathbb{R}^3)$ *with* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q_3} \le \frac{1}{q_1} + \frac{1}{q_2}$, *and*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3)} \le C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|v\|_{L^{p_2,q_2}(\mathbb{R}^3)}\,. \tag{2.4}$$

2.2. **Besov space.** Following [23], let $\mathcal{B} = \{\xi \in \mathbb{R}^d,\ |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$. Choose two nonnegative smooth radial function χ, φ supported, respectively, in $\mathcal{B}$ and $\mathcal{C}$ such that

$$\chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$
$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)f(x-y)dy,$$
$$S_j f = \sum_{k\le j-1}\Delta_k f = \chi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d}\tilde{h}(2^j y)f(x-y)dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{C_1 2^j \le |\xi| \le C_2 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \le C2^j\}$. One can easily verify that with our choice of φ,

$$\Delta_j\Delta_k f = 0 \text{ if } |j-k| \ge 2 \quad \text{and} \quad \Delta_j(S_{k-1}f\Delta_k f) = 0 \text{ if } |j-k| \ge 5.$$

With the introduction of Δ_j and S_j, let us recall the definition of the Besov space.

Let $s \in \mathbb{R}$, $p, q \in [1,\infty]$, the homogeneous space is defined as

$$\dot{B}^s_{p,q} = \{f \in \mathcal{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{sjq}\|\Delta_j f\|^q_{L^p}\right)^{1/q}, & \text{for } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^p}, & \text{for } q = \infty, \end{cases}$$

In particular, when $p = q = 2$, the Besov space and Sobolev space are equivalence; that is

$$\dot{H}^s \approx \dot{B}^s_{2,2}, \quad H^s \approx B^s_{2,2}.$$

Now we recall first the definition of weak solutions.

Definition 2.2. Let $u_0, b_0 \in L^2(\mathbb{R}^3)$ with the divergence free conditions. We say that (u,b) is a weak solution of Hall-MHD equations (1.1)–(1.3) with initial condition $u_0, b_0 \in L^2(\mathbb{R}^3)$, if u and b satisfy the following:

(i) $u \in L^\infty([0,T];L^2(\mathbb{R}^3)) \cap L^2([0,T];H^1(\mathbb{R}^3))$, and $b \in L^\infty([0,T];L^2(\mathbb{R}^3)) \cap L^2([0,T];H^1(\mathbb{R}^3))$.

(ii) (u,b) satisfies (1.1)–(1.2) in the sense of distribution; that is

$$\int_0^T\int_{\mathbb{R}^3}\Big(\frac{\partial\phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\Big)u\,dx\,dt + \int_{\mathbb{R}^3}u_0\phi(x,0)\,dx = \int_0^T\int_{\mathbb{R}^3}(b\cdot\nabla)\phi\,b\,dx\,dt$$

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \Big(\frac{\partial \phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\Big) b \,dx\,dt + \int_{\mathbb{R}^3} b_0 \phi(x,0)\,dx \\
&= \int_0^T \int_{\mathbb{R}^3} (b\cdot\nabla)\phi\, u \,dx\,dt + \int_0^T \int_{\mathbb{R}^3} (\nabla\times b)\times b\cdot(\nabla\times\phi)\,dx\,dt,
\end{aligned}
$$

for all $\phi \in C_0^\infty(\mathbb{R}^3\times[0,T))$ with $\operatorname{div}\phi = 0$, and

$$
\int_{\mathbb{R}^3} u\cdot\nabla\psi dx = 0, \quad \int_{\mathbb{R}^3} b\cdot\nabla\psi dx = 0,
$$

for every $\psi \in C_0^\infty(\mathbb{R}^3)$. □

3. Proof of main results

Proof of Theorem 1.2. (L^2-estimate or energy estimate): By the standard energy estimate, we obtain

$$
\frac{1}{2}\frac{d}{dt}\int (|u|^2 + |b|^2)\,dx + \int (|\nabla u|^2 + |\nabla b|^2)\,dx = 0. \tag{3.1}
$$

• (H^1-estimate): Testing $-\Delta u$ and $-\Delta b$ to the fluid equation and by the magnetic equation of (1.1) and (1.2), respectively, using the integrating by parts, integrating on domain, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\tau)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla b(\tau)\|_{L^2(\mathbb{R}^3)}^2) + \int_{\mathbb{R}^3} (|\Delta u|^2 + |\Delta b|^2) dx \\
&\le -\int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)u] : \nabla u dx + \int_{\mathbb{R}^3} \nabla[(b\cdot\nabla)b] : \nabla u\,dx \\
&\quad -\int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)b]\cdot\nabla b\,dx + \int_{\mathbb{R}^3} \nabla[(b\cdot\nabla)u] : \nabla b dx \\
&\quad + \int \nabla((\nabla\times b)\times b)\nabla\nabla\times b\,dx \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.
\end{aligned} \tag{3.2}
$$

We estimate separately the terms in the right hand side of (3.2). The first term $\mathcal{I}_1$ is computed as follows:

$$
|\mathcal{I}_1| \le \|\nabla u\|_{L^3}^3, \tag{3.3}
$$

where the divergence free condition of u is used.

On the other hand, we observe that

$$
\mathcal{I}_2 + \mathcal{I}_4 \le \int_{\mathbb{R}^3} |\nabla u||\nabla b|^2.
$$

since

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (b\cdot\nabla)\nabla b\cdot\nabla u\,dx + \int_{\mathbb{R}^3} (b\cdot\nabla)\nabla u\cdot\nabla b\,dx \\
&= \sum_{j=1}^3 \int_{\mathbb{R}^3} b_j\Big(\frac{\partial\nabla b}{\partial x_j}\nabla u\,dx + \frac{\partial\nabla u}{\partial x_j}\nabla b\Big)dx \\
&= -\sum_{j=1}^3 \int_{\mathbb{R}^3} b_j\Big(\frac{\partial(\nabla b\nabla u)}{\partial x_j}\Big)dx = 0,
\end{aligned}
$$

where we use the product rule and $\operatorname{div} b = 0$.

Note that

$$\|\nabla f\|_{L^4}^2 \le C\|\nabla f\|_{\dot{B}^{-\beta}_{\infty,\infty}}\|f\|_{\dot{H}^{1+\beta}} \quad \text{with } 0<\beta<1, \tag{3.4}$$

$$\|f\|_{\dot{H}^{1+\beta}} \le C\|\nabla f\|_{L^2}^{1-\beta}\|\Delta f\|_{L^2}^{\beta} \quad \text{with } 0<\beta<1, \tag{3.5}$$

(e.g. see [18, 19] and [2, Theorem 2.42]).

First of all, using the interpolation (2.2), Lemma 2.1, Hölder and Young's inequalities, we estimate $\mathcal{I}_3$ as follows:

$$\begin{aligned}
|\mathcal{I}_3| &\le \int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|dx \le \|\nabla u\|_{L^{p,\infty}}\||\nabla b|^2\|_{L^{\frac{p}{p-1},1}} \\
&= \|\nabla u\|_{L^{p,\infty}}\|\nabla b\|^2_{L^{\frac{2p}{p-1},1}} \\
&\le C\|\nabla u\|_{L^{p,\infty}}\|\nabla b\|_{L^2}^{2\theta}\|\nabla^2 b\|_{L^2}^{2(1-\theta)} \\
&\le C\|\nabla u\|_{L^{p,\infty}}^{\frac{2p}{2p-3}}\|\nabla b\|_{L^2}^2 + \frac{1}{16}\|\nabla^2 b\|_{L^2}^2,
\end{aligned} \tag{3.6}$$

where $\theta = 1-\frac{3}{2p}$. Similarly, from (3.3), we have

$$|\mathcal{I}_1| \le \|\nabla u\|_{L^3}^3 \le C\|\nabla u\|_{L^{p,\infty}}^{\frac{2p}{2p-3}}\|\nabla u\|_{L^2}^2 + \frac{1}{16}\|\nabla^2 u\|_{L^2}^2.$$

Case 1: Again, using the interpolation (2.2), Lemma 2.1, Hölder and Young's inequalities, we bound $\mathcal{I}_5$ as follows.

$$\begin{aligned}
|\mathcal{I}_5| &\le C\|\nabla b\|_{L^{l,\infty}}\|\nabla b\|_{L^{\frac{2l}{l-2},2}}\|\Delta b\|_{L^{2,2}} \\
&\le C\|\nabla b\|_{L^{l,\infty}}\|\nabla b\|_{L^2}^{\frac{l-3}{l}}\|\Delta b\|_{L^2}^{\frac{l+3}{l}} \\
&\le C\|\nabla b\|_{L^{l,\infty}}^{\frac{2l}{l-3}}\|\nabla b\|_{L^2}^2 + \frac{1}{16}\|\Delta b\|_{L^2}^2.
\end{aligned}$$

Summing the terms $\mathcal{I}_1$–$\mathcal{I}_5$, inequality (3.2) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx \\
&\le C\Big(\|\nabla u\|_{L^{p,\infty}}^{\frac{2p}{2p-3}} + \|\nabla b\|_{L^{l,\infty}}^{\frac{2l}{l-3}}\Big)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
\end{aligned} \tag{3.7}$$

Case 2: Using (3.4) and (3.5), we bound $\mathcal{I}_5$ as follows:

$$\begin{aligned}
\mathcal{I}_5 &= -\sum_i \int (\nabla\times b\times\partial_i b)\partial_i\nabla\times b dx \le C\|\nabla b\|_{L^4}^2\|\Delta b\|_{L^2} \\
&\le C\|\nabla b\|_{\dot{B}^{-\beta}_{\infty,\infty}}\|b\|_{\dot{H}^{1+\beta}}\|\Delta b\|_{L^2} \le C\|\nabla b\|_{\dot{B}^{-\beta}_{\infty,\infty}}\|\nabla b\|_{L^2}^{1-\beta}\|\Delta b\|_{L^2}^{1+\beta} \\
&\le \frac{1}{16}\|\Delta b\|_{L^2}^2 + C\|\nabla b\|_{\dot{B}^{-\beta}_{\infty,\infty}}^{\frac{2}{1-\beta}}\|\nabla b\|_{L^2}^2.
\end{aligned}$$

Summing the terms $\mathcal{I}_1$–$\mathcal{I}_5$, the (3.2) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx \\
&\le C\Big(\|\nabla u\|_{L^{p,\infty}}^{\frac{2p}{2p-3}} + \|\nabla b\|_{\dot{B}^{-\beta}_{\infty,\infty}}^{\frac{2}{1-\beta}}\Big)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
\end{aligned} \tag{3.8}$$

For Cases 1 and 2 with the given conditions, we apply the Grownwall's inequality to estimates (3.7) and (3.8), respectively, to find

$$\begin{aligned}&\sup_{0<\tau\le T}(\|\nabla u(\tau)\|_{L^2}^2+\|\nabla b(\tau)\|_{L^2}^2)+\int_0^T\int_{\mathbb{R}^3}(|\nabla^2 u|^2+|\nabla^2 b|^2)\,dx\,dt\\&\le C(\|\nabla u(0)\|_{L^2}^2+\|\nabla b(0)\|_{L^2}^2).\end{aligned}$$

• (H^2-estimate) Applying the operator Δ to (1.1)–(1.2), then multiplying it by Δu and Δb, respectively, and integrating on domain, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Delta u\|^2+\|\Delta b\|^2)dx+(\|\nabla\Delta u|_{L^2}^2+\|\nabla\Delta b|_{L^2}^2)\\&=-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u\,dx+\int_{\mathbb{R}^3}\Delta(b\cdot\nabla b)\cdot\Delta u\,dx\\&\quad-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla b)\cdot\Delta b\,dx\\&\quad+\int_{\mathbb{R}^3}\Delta(b\cdot\nabla u)\cdot\Delta b\,dx-\int_{\mathbb{R}^3}\Delta((\nabla\times b)\times b)\cdot\Delta\nabla\times b\,dx\\&:=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5\end{aligned}$$

By the commutator estimate in [10, Theorem 2.1 or Corollary 2.1] or [15], we note that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\Delta[(u\cdot\nabla)u],\Delta u\,dx\Big|&\le C\|\nabla u\|_{H^2}\|u\|_{H^2}^2,\\\Big|\int_{\mathbb{R}^3}\Delta[(u\cdot\nabla)b],\Delta b\,dx\Big|&\le C\|\nabla u\|_{H^2}\|b\|_{H^2}^2,\\\Big|\int_{\mathbb{R}^3}\Delta[(b\cdot\nabla)u],\Delta b\,dx\Big|&\le C\|\nabla u\|_{H^2}\|b\|_{H^2}^2.\end{aligned}$$

Also, integrating by parts we obtain the estimate for the remaining convection term follows as:

$$\Big|\int_{\mathbb{R}^3}\Delta[(b\cdot\nabla)b],\Delta u\,dx\Big|\le C|\langle\Delta[b\otimes b],\Delta\nabla u\,dx|\le C\|\nabla u\|_{H^2}\|b\|_{H^2}^2.$$

Thus

$$\begin{aligned}|\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4|&\le C\|\nabla u\|_{H^2}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)\\&\le C(\|u\|_{H^2}^4+\|b\|_{H^2}^4)+\frac{1}{128}\|\nabla u\|_{H^2}^2\\&\le C(\|u\|_{H^2}^2+\|b\|_{H^2}^2)(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\frac{1}{128}\|\nabla u\|_{H^2}^2\end{aligned}\tag{3.9}$$

Case 1: For the term $\mathcal{J}_5$, using the chain rule, we note that

$$\mathcal{J}_5=\int_{\mathbb{R}^3}(\nabla\times b\times\Delta b+2\partial_i(\nabla\times b)\times\partial_i b)\nabla\Delta b\,dx\tag{3.10}$$

And thus, we have

$$|\mathcal{J}_5|\le\|\nabla b\|_{L^{l,\infty}}\|\Delta b\|_{L^{\frac{2l}{l-2},2}}\|\nabla\Delta b\|_{L^2}\le C\|\nabla b\|_{L^{l,\infty}}^{\frac{2l}{l-3}}\|\Delta b\|_{L^2}^2+\frac{1}{128}\|\nabla\Delta b\|_{L^2}^2$$

Summing the estimate of terms $\mathcal{J}_1$–$\mathcal{J}_5$ with the energy estimate and H^1-estimates, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\frac{1}{2}(\|u|_{H^3}^2+\|b|_{H^3}^2)\\ &\le C\Big(\|\nabla u\|_{L^{q,\infty}}^{\frac{2q}{2q-3}}+\|\nabla b\|_{L^{l},\infty}^{\frac{2l}{l-3}}+\|u\|_{H^2}^2+\|b\|_{H^2}^2\Big)(\|u\|_{H^2}^2+\|b\|_{H^2}^2) \end{aligned} \tag{3.11}$$

Case 2: Using (3.4) and (3.5), we bound $\mathcal{J}_5$ as follows:

$$\begin{aligned} |\mathcal{J}_5| &\le C\|\Delta b\|_{\dot B^{-\beta}_{\infty,\infty}}\|\nabla b\|_{\dot H^{1+\beta}}\|\nabla\Delta b\|_{L^2}\\ &\le C\|\nabla b\|_{\dot B^{-\beta}_{\infty,\infty}}\|\nabla^2 b\|_{L^2}^{1-\beta}\|\nabla\Delta b\|_{L^2}^{1+\beta}\\ &\le C\|\nabla b\|_{\dot B^{-\beta}_{\infty,\infty}}^{\frac{2}{1-\beta}}\|b\|_{H^2}^2+\frac{1}{128}\|\nabla\Delta b\|_{L^2}^2. \end{aligned}$$

As in case 1, summing the estimate of terms $\mathcal{J}_1$–$\mathcal{J}_5$ with the energy estimate and H^1-estimates, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\frac{1}{2}(\|u|_{H^3}^2+\|b|_{H^3}^2)\\ &\le C\Big(\|\nabla u\|_{L^{p,\infty}}^{\frac{2p}{2p-3}}+\|\nabla b\|_{\dot B^{-\beta}_{\infty,\infty}}^{\frac{2}{1-\beta}}+\|u\|_{H^2}^2+\|b\|_{H^2}^2\Big)(\|u\|_{H^2}^2+\|b\|_{H^2}^2) \end{aligned} \tag{3.12}$$

Under the assumption, we apply Grown's inequality to the estimates (3.11) and (3.12), respectively, we finally obtain

$$\sup_{0\le\tau\le T}(\|u(\tau)\|_{H^2}^2+\|b(\tau)\|_{H^2}^2)+\|u\|_{H^3}^2+\|b\|_{H^3}^2\le(\|u_0\|_{H^2}^2+\|b_0\|_{H^2}^2)$$

Th proof is complete. □

Proof of Theorem 1.3. (H^1-estimate): Testing $-\Delta u$ and $-\Delta b$ to the fluid equation and the magnetic equation of (1.1) and (1.2), respectively, using the integrating by parts, integrating on domain, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla u(\tau)\|_{L^2(\mathbb{R}^3)}^2+\|\nabla b(\tau)\|_{L^2(\mathbb{R}^3)}^2)+\int_{\mathbb{R}^3}(|\Delta u|^2+|\Delta b|^2)dx\\ &\le-\int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)u]:\nabla u dx+\int_{\mathbb{R}^3}\nabla[(b\cdot\nabla)b]:\nabla u dx\\ &\quad-\int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)b]:\nabla b\,dx+\int_{\mathbb{R}^3}\nabla[(b\cdot\nabla)u]:\nabla b dx\\ &\quad+\int\nabla((\nabla\times b)\times b):\nabla\nabla\times b\,dx\\ &:=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5. \end{aligned} \tag{3.13}$$

Case 1: By the Hölder, Young inequalities and the space duality BMO-H^1, we have

$$\begin{aligned} \int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|dx &\le\|\nabla u\|_{BMO}\||\nabla b|^2\|_{\mathcal{H}^1}\le\|\nabla u\|_{BMO}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}\\ &=\|\nabla u\|_{BMO}\|\nabla b\|_{L^2}^2. \end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{R}^3}|\nabla u|^3dx\le C\|\nabla u\|_{BMO}\|\nabla u\|_{L^2}^2.$$

Again, by the vector identity, the Hölder and Young inequalities, we have

$$
\begin{aligned}
\mathcal{I}_5 &= \int_{\mathbb{R}^3} \nabla[(\nabla\times b)\times b]\cdot\nabla(\nabla\times b)dx \\
&= \int_{\mathbb{R}^3} \Big((\nabla\times b)\times\nabla b - \nabla(\nabla\times b)\times b\Big)\cdot\nabla(\nabla\times b)dx \\
&\le C\|\nabla b\|_{BMO}\||\nabla b||\nabla^2 b|\|_{\mathcal{H}^1} \\
&\le C\|\nabla b\|_{BMO}^2\|\nabla b\|_{L^2}^2 + \frac{1}{8}\|\nabla^2 b\|_{L^2}^2.
\end{aligned} \tag{3.14}
$$

Summing the estimates above, the (3.13) becomes

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int (|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx \\
&\le C(\|\nabla u\|_{BMO} + \|\nabla b\|_{BMO}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
\end{aligned} \tag{3.15}
$$

Case 2: Following [24, Lemma A.5], we note that

$$
\|u\|_{L^4}^2 = \|uu\|_{L^2} \le C\|\nabla u\|_{L^2}\|u\|_{BMO^{-1}}.
$$

Using this estimate, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\nabla b|^2|\nabla u|dx &\le \|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2 \le C\|\nabla u\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla b\|_{BMO^{-1}} \\
&\le C\|\nabla u\|_{L^2}^2\|\nabla b\|_{BMO^{-1}}^2 + \frac{1}{8}\|\nabla^2 b\|_{L^2}^2.
\end{aligned}
$$

Similarly, we obtain

$$
\int_{\mathbb{R}^3} |\nabla u|^3 dx \le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO^{-1}}^2 + \frac{1}{8}\|\nabla^2 u\|_{L^2}^2.
$$

By the vector identity, the Hölder and Young inequalities, we have

$$
\begin{aligned}
\mathcal{I}_5 &= \int_{\mathbb{R}^3} \nabla[(\nabla\times b)\times b]\cdot\nabla(\nabla\times b)dx \\
&= \int_{\mathbb{R}^3} \Big((\nabla\times b)\times\nabla b - \nabla(\nabla\times b)\times b\Big)\cdot\nabla(\nabla\times b)dx \\
&\le C\|(\nabla\times b)\times\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2} \\
&\le C\|\nabla^2 b\|_{\mathrm{BMO}^{-1}}^2\|\nabla b\|_{L^2}^2 + \frac{1}{258}\|\nabla^2 b\|_{L^2}^2
\end{aligned} \tag{3.16}
$$

Using the estimates above, (3.13) becomes

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int (|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx \\
&\le C(\|\nabla u\|_{BMO^{-1}}^2 + \|\nabla b\|_{BMO^{-1}}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
\end{aligned} \tag{3.17}
$$

• (H^2-estimate) Taking Δ to (1.1)–(1.2), then multiplying it by Δu and Δb, respectively, and integrating on domain, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u\|^2+\|\nabla^2 b\|^2)dx+(\|\nabla\Delta u|_{L^2}^2+\|\nabla\Delta b|_{L^2}^2)\\
&=-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u\,dx+\int_{\mathbb{R}^3}\Delta(b\cdot\nabla b)\cdot\Delta u\,dx-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla b)\cdot\Delta b\,dx\\
&\quad+\int_{\mathbb{R}^3}\Delta(b\cdot\nabla u)\cdot\Delta b\,dx-\int_{\mathbb{R}^3}\Delta((\nabla\times b)\times b):\Delta\nabla\times b\,dx\\
&:=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5
\end{aligned}\tag{3.18}$$

As in the proof of Theorem 1.2, namely (3.9), we note that

$$|\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4|\le C(\|u\|_{H^2}^2+\|b\|_{H^2}^2)(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\frac{1}{128}\|\nabla u\|_{H^2}^2.$$

Case 1. From (3.10) with the space duality BMO-$\mathcal{H}^1$, we have

$$|\mathcal{J}_5|\le\|\nabla b\|_{\rm BMO}\|\nabla^2 b\|_{L^2}\|\nabla\Delta b\|_{L^2}\le C\|\nabla b\|_{\rm BMO}^2\|\nabla^2 b\|_{L^2}^2+\frac{1}{128}\|\nabla\Delta b\|_{L^2}^2.$$

Summing the estimates $\mathcal{J}_1$–$\mathcal{J}_5$ with the energy estimate and H^1-estimates, the (3.18) becomes

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+(\|u\|_{H^3}^2+\|b|_{H^3}^2)\\
&\le C\Big(\|\nabla u\|_{BMO}+\|\nabla b\|_{\rm BMO}^2+\|u\|_{H^2}^2+\|b\|_{H^2}^2\Big)(\|u\|_{H^2}^2+\|b\|_{H^2}^2).
\end{aligned}\tag{3.19}$$

Case 2. From (3.9), we note that

$$|\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4|\le C(\|u\|_{H^2}^2+\|b\|_{H^2}^2)(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\frac{1}{128}\|\nabla u\|_{H^2}^2.$$

Following [24, Lemma A.5], we note that

$$\|uu\|_{L^2}\le C\|\nabla u\|_{L^2}\|u\|_{BMO^{-1}}.$$

As in the previous proof, namely, from (3.10) with the space duality BMO-$\mathcal{H}^1$, we have

$$|\mathcal{J}_5|\le C\|\nabla^2 b\|_{\rm BMO^{-1}}\|\nabla^2 b\|_{L^2}+\frac{1}{128}\|\nabla\Delta b\|_{L^2}^2.$$

Summing $\mathcal{J}_1$–$\mathcal{J}_5$ with the energy estimate and H^1-estimate, the (3.18) becomes

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+(\|u\|_{H^3}^2+\|b|_{H^3}^2)\\
&\le C(\|\nabla u\|_{BMO^{-1}}^2+\|\nabla^2 b\|_{BMO^{-1}}^2+\|u\|_{H^2}^2+\|b\|_{H^2}^2)(\|u\|_{H^2}^2+\|b\|_{H^2}^2).
\end{aligned}\tag{3.20}$$

Under the assumption, we apply Gronwall's inequality to the estimates (3.19) and (3.20), respectively, we finally obtain

$$\sup_{0\le\tau\le T}(\|u(\tau)\|_{H^2}^2+\|b(\tau)\|_{H^2}^2)+\|u\|_{H^3}^2+\|b\|_{H^3}^2\le(\|u_0\|_{H^2}^2+\|b_0\|_{H^2}^2).$$

The proof is complete. □

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