Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 311, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO AN EVOLUTION p-LAPLACIAN EQUATION WITH A NONLINEAR GRADIENT TERM

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Abstract. We study the evolution p-Laplacian equation with the nonlinear gradient term

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) - B(x)|\nabla u|^q,$$

where $a(x), B(x) \in C^1(\overline{\Omega})$, $p > 1$ and $p > q > 0$. When $a(x) > 0$ and $B(x) > 0$, the uniqueness of weak solution to this equation may not be true. In this study, under the assumptions that the diffusion coefficient $a(x)$ and the damping coefficient $B(x)$ are degenerate on the boundary, we explore not only the existence of weak solution, but also the uniqueness of weak solutions without any boundary value condition.

1. Introduction

Consider the evolution p-Laplacian equation with the nonlinear gradient term

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) - B(x)|\nabla u|^q, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

with the initial-boundary value conditions:

$$u(x,t) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a C^2 smooth boundary, $p > 1$, $q < p$, $a(x)$ and $B(x) \in C^1(\overline{\Omega})$ satisfy

$$a(x)\big|_{x\in\partial\Omega} = 0, \quad a(x)\big|_{x\in\Omega} > 0, \quad ba(x) \geq B(x) \geq 0. \tag{1.4}$$

Here and in what follows, b is a positive constant.

Equation (1.1) arises in several scientific fields such as mechanics, physics and biology [7, 14]. If $a(x) \geq c > 0$, and there exists a point $x_0 \in \Omega$ such that $B(x_0) > 0$, then in general the uniqueness of the solution is not true [2, 3, 5, 6, 13, 16, 17]. In [1], the equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)u^\gamma, \quad (x,t) \in Q_T, \tag{1.5}$$

with $0 < \gamma < 1$, was studied. It shows that the uniqueness of the solution of equation (1.5) is not true, provided that $q(x) \geq 0$ and there exists a point $x_0 \in \Omega$

2010 *Mathematics Subject Classification.* 35L65, 35K85, 35R35.
Key words and phrases. Evolution p-Laplacian equation; weak solution; uniqueness; boundary value condition.

Submitted April 9, 2017. Published December 31, 2017.

such that $q(x_0) > 0$. Recently, Zhan [15] considered the equation

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2}\nabla u) + f(u,x,t), \quad (x,t) \in Q_T, \tag{1.6}$$

and proved that the weak solution of equation (1.6) with the initial value (1.2) has the stability

$$\int_\Omega |u(x,t) - v(x,t)|dx \le c\int_\Omega |u_0(x) - v_0(x)|dx, \tag{1.7}$$

where $\rho(x) = dist(x, \partial\Omega)$ and $f(u,\cdot,\cdot)$ is a Lipschitz function. The inequality (1.7) also indicates that the solution of (1.6) with the initial condition (1.2) is unique. However, if $f(u,\cdot,\cdot)$ is not a Lipschitz function, for an example,

$$f(u,x,t) = q(x)u^\gamma,$$

as given in (1.5), the problem whether the solution u of (1.6) has the stability (1.7) or not, remains to be an open problem.

By the above short reviews, when the diffusion coefficient $a(x)$ is degenerate on the boundary, the uniqueness of the solution for the initial-boundary value problem (1.1)-(1.3) has been an interesting topic. In this study, we assume that the damping coefficient $B(x)$ is also degenerate on the boundary, and establish the uniqueness of weak solution. This result is different from those presented in the literature [2, 3, 5, 6, 13, 16, 17].

To introduce the weak solution of equation (1.1), we set

$$V = L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{and} \quad V^* = L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Definition 1.1. A function $u(x,t) \in L^\infty(Q_T)$, satisfying $a(x)|\nabla u|^p \in L^1(Q_T)$, is said to be a weak solution of equation (1.1) with the initial condition (1.2), provided that $u_t \in V^* + L^{p/q}(Q_T)$ and

$$\int_0^T \langle u_t, \phi\rangle dt + \int_0^T\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\phi dxdt = \int_0^T\int_\Omega B|\nabla u|^q\phi dxdt, \tag{1.8}$$

holds for all $\phi(x,t) \in V \cap L^{\frac{p}{p-q}}(Q_T)$. The initial value condition is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega u(x,t)\varphi(x)dx = \int_\Omega u_0(x)\varphi(x)dx, \tag{1.9}$$

for any $\varphi(x) \in C_0^\infty(\Omega)$.

Definition 1.2. The function $u(x,t)$ is said to be the weak solution of the initial-boundary value problem (1.1)-(1.3), if $u(x,t)$ satisfies Definition 1.1, and the boundary value condition (1.3) is satisfied in the sense of trace.

Now, we state our main results on the existence and uniqueness.

Theorem 1.3. *If $p > 1$, $0 < q < p$, $a(x)$ and $B(x)$ satisfy* (1.4), *and*

$$u_0 \in L^\infty(\Omega), \quad a(x)|\nabla u_0|^p \in L^1(\Omega), \tag{1.10}$$

then there exists a weak solution of equation (1.1) *with the initial condition* (1.2).

Theorem 1.4. *If $p > 1$, $0 < q < p$, $a(x)$ and $B(x)$ satisfy* (1.4), *and*

$$\int_\Omega a(x)^{-\frac{1}{p-1}}(x)dx < \infty, \tag{1.11}$$

then the initial-boundary value problem (1.1)-(1.3) *has a solution in the sense of Definition 1.2*

Theorem 1.5. *Let $p > 1$ and $0 < q < p$. Suppose that u and v be two solutions of* (1.1) *with the initial value $u_0(x) = v_0(x)$, and with the same homogeneous boundary value condition* (1.3). *If conditions* (1.4) *and* (1.11) *are true, then $u = v$.*

In general, if condition (1.11) is not true, i.e.,

$$\int_\Omega a(x)^{-\frac{1}{p-1}} dx = \infty,$$

then weak solutions of equation (1.1) may lack the regularity to have a trace on the boundary. Accordingly, we can not impose the usual boundary value condition (1.3). However, because of condition (1.4), we are able to prove the uniqueness of the weak solution of equation (1.1) without any boundary value condition. In other words, the degeneracy of $a(x)$ and $B(x)$ on the boundary may take place of the boundary value condition (1.3). This is the key feature of this paper.

Theorem 1.6. *Let $p > 1$ and $0 < q < p$. Suppose that u and v be two solutions of* (1.1) *with the initial value $u_0(x) = v_0(x)$. If condition* (1.4) *is true, and for small $\lambda > 0$ there holds*

$$\frac{1}{\lambda}\Big(\int_{\Omega_\lambda} a(x)|\nabla a(x)|^p dx\Big)^{1/p} \le c, \tag{1.12}$$

where $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$, then $u = v$, i.e., the solution of the initial value problem (1.1)-(1.2) *is unique.*

Theorem 1.6 tells us that for the uniqueness of the solution of equation (1.1) with the initial value $u_0(x) = v_0(x)$, the condition $\int_\Omega a(x)^{-\frac{1}{p-1}} dx < \infty$ may not be necessary.

The paper is organized as follows. In Section 2, we prove the existence of the solution to equation (1.1) with the initial condition (1.2). In Section 3, we present the proof of Theorem 1.5. Section 4 is dedicated to the proof of Theorem 1.6 and the uniqueness of the solution without any boundary value condition.

2. Proofs of main results

Lemma 2.1 ([8]). *Let $\theta(s) = se^{\eta s^2}$, $s \in \mathbb{R}$, where $\eta \ge \frac{b^2}{4a^2}$ is fixed, and let $\Theta(s) = \int_0^s \theta(\tau)d\tau$. Then $\theta(0) = 0$ and*

$$\Theta(s) \ge 0, \quad a\theta'(s) - b|\theta(s)| \geqslant \frac{a}{2}, \quad \forall s \in R, \tag{2.1}$$

where b is the constant as in (1.4), *and a is a constant to be determined.*

Lemma 2.2 ([8]). *Assume that $\pi : R \to R$ is a piecewise C^1 with $\pi(0) = 0$ and $\pi' = 0$ outside a compact set. Let $\Pi(s) = \int_0^s \pi(\sigma)d\sigma$. If $u \in V$ with $u_t \in V^* + L^1(Q_T)$, then*

$$\begin{aligned}\int_0^T \langle u_t, \pi(u)\rangle dt :=& \langle u_t, \pi(u)\rangle_{V^*+L^1(Q_T), V\cap L^\infty(Q_T)}\\ =& \int_\Omega \Pi(u(T))dx - \int_\Omega \Pi(u(0))dx.\end{aligned} \tag{2.2}$$

Proof of Theorem 1.3. Consider the approximation equation

$$\frac{\partial u_n}{\partial t} - \operatorname{div}\big[\big(a(x) + \frac{1}{n}\big)|\nabla u_n|^{p-2}\nabla u_n\big] = B(x)\min\{|\nabla u_n|^q, n\}, \quad (x,t) \in Q_T, \tag{2.3}$$

with the initial-boundary value conditions (1.2)-(1.3). The existence of the weak solution $u_n \in L^\infty$ follows from the standard methods (for instance, the pseudo-monotonicity operator theory [9, 10, 11], or the difference and variation methods [12]). By the maximal theory, we have the uniform bound:

$$\|u_n(x,t)\|_{L^\infty(Q_T)} \leqslant \|u_0\|_{L^\infty(\Omega)}. \tag{2.4}$$

Our goal is to show that a subsequence of the approximate solution sequence $\{u_n\}$ converges to a measurable function u, which coincides with the weak solution of the problem (1.1)-(1.2).

Step 1: Weak convergence We choose $\theta(u_n)$ as a test function in (2.3), then

$$\begin{aligned} &\int_0^T \langle \frac{\partial u_n}{\partial t}, \theta(u_n)\rangle dt + \iint_{Q_T} \Big(a(x)+\frac{1}{n}\Big)|\nabla u_n|^p \theta'(u_n)\,dx\,dt \\ &= \iint_{Q_T} B\min\{|\nabla u_n|^q, n\}\theta(u_n)\,dx\,dt. \end{aligned} \tag{2.5}$$

From Lemma 2.2, we have

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, \theta(u_n)\rangle dt = \int_\Omega [\Theta(u_n(T)) - \Theta(u_0)]dx.$$

By Young's inequality, (2.5) becomes

$$\begin{aligned} &\int_\Omega \Theta(u_n(T))dx + \iint_{Q_T} \Big(a(x)+\frac{1}{n}\Big)\nabla u_n|^p \theta'(u_n)\,dx\,dt \\ &\leqslant \int_\Omega \Theta(u_0)dx + \iint_{Q_T} B|\nabla u_n|^q|\theta(u_n)|\,dx\,dt \\ &\leqslant \int_\Omega \Theta(u_0)dx + \iint_{Q_T} \big(\frac{q}{p}B|\nabla u_n|^p + \frac{p-q}{p}|\theta(u_n)|\big)\,dx\,dt \\ &\leqslant \int_\Omega \Theta(u_0)dx + \iint_{Q_T} \big(B|\nabla u_n|^p + \frac{p-q}{p}|\theta(u_n)|\big)\,dx\,dt. \end{aligned}$$

We rewrite the above inequality as

$$\begin{aligned} &\int_\Omega \Theta(u_n(T))dx + \iint_{Q_T} \Big[\theta'(u_n) - B\Big(a(x)+\frac{1}{n}\Big)^{-1}|\theta(u_n)|\Big] \\ &\times \Big(a(x)+\frac{1}{n}\Big)|\nabla u_n|^p\,dx\,dt \\ &\leqslant \int_\Omega \Theta(u_0)dx + c. \end{aligned} \tag{2.6}$$

Let $a=1$ in Lemma 2.1. Then

$$\theta'(u_n) - B\Big(a(x)+\frac{1}{n}\Big)^{-1}|\theta(u_n)| \geqslant \theta'(u_n) - b|\theta(u_n)| \geqslant \frac{1}{2},$$

so we deduce that

$$\frac{1}{2}\iint_{Q_T} \Big(a(x)+\frac{1}{n}\Big)|\nabla u_n|^p\,dx\,dt \leqslant \int_\Omega \Theta(u_0)dx. \tag{2.7}$$

By (1.4) and (2.7), we have $|\nabla u_n| \in L^p_{\rm loc}(Q_T)$. By the Hölder inequality and $ba(x) \geq B(x)$, we have

$$B(x)|\nabla u_n|^q \in L^1_{\rm loc}(Q_T). \tag{2.8}$$

By (2.4), (2.7) and (2.8), there exists a function u and an n-dimensional vector function $\vec{\zeta} = (\zeta_1, \cdots, \zeta_n)$ satisfying

$$u \in L^\infty(Q_T), \quad |\vec{\zeta}| \in L^{\frac{p}{p-1}}(Q_T),$$

and

$$\begin{gathered} u_n \rightharpoonup *u, \quad \text{weakly star in } L^\infty(Q_T), \\ \Big(a(x) + \frac{1}{n}\Big)|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \vec{\zeta} \quad \text{in } L^{\frac{p}{p-1}}(Q_T), \\ B(x)|\nabla u_n|^q \rightharpoonup \nu, \quad \text{in } L^{p/q}(Q_T). \end{gathered}$$

Step 2: Strong convergence Clearly, by (2.7) and (2.8), $\frac{\partial u_n}{\partial t}$ is bounded in the space

$$L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{p/q}(Q_T).$$

For a fixed s such that $s > \frac{N}{2} + 1$, the following holds:

(1) When $s > \frac{N}{2}$, we have $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$.
(2) When $s - 1 > \frac{N}{2}$, we have $H_0^s(\Omega) \hookrightarrow W^{1,p}(\Omega)$, consequently, $W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega)$.

As a result, we have

$$\|\frac{\partial u_n}{\partial t}\|_{L^1(0,T;H^{-s}(\Omega))} \leqslant c,$$

where c is independent of n. For any given $\varphi \in C_0^1(\Omega)$, we have

$$\|\frac{\partial(\varphi u_n)}{\partial t}\|_{L^1(0,T;H^{-s}(\Omega))} \leqslant C \tag{2.9}$$

for $\varphi u_n \in W_0^{1,p}(\Omega)$ and

$$\begin{aligned} \int_\Omega |\nabla(\varphi u_n)|^p dx &\le c\Big[\int_\Omega |\nabla\varphi|^p|u_n|^p dx + \int_\Omega |\varphi|^p|\nabla u_n|^p dx\Big] \\ &\le c + c\int_\Omega \frac{|\varphi|^p}{a(x)} a(x)|\nabla u_n|^p dx \\ &\le c + c_1\int_\Omega a(x)|\nabla u_n|^p dx, \end{aligned} \tag{2.10}$$

where $c_1 = \max_{x\in\overline{\Omega_\varphi}} |\varphi|^p/a(x) > 0$ is a constant independent of n, and $\overline{\Omega_\varphi}$ is the support set of φ. Notice that $W_0^{1,p}(\Omega) \hookrightarrow^{compact} L^p(\Omega) \hookrightarrow H^{-s}(\Omega)$. It follows Simon's compactness theorem [8] that $\varphi u_n \to \varphi u$, strongly in $L^p(0,T;L^p(\Omega))$.

Step 3: Almost everywhere convergence In step 2, by the arbitrariness of φ, we can let $\{u_n\}$ be a subsequence of $\{u_\varepsilon\}$ such that $u_n \to u$ a.e. in Q_T. According to Egoroff's theorem, for the fixed $\delta > 0$, there is a closed set $E_\delta \subset Q_T$ such that

(1) The measure $\mu(Q_T - E_\delta) \le \delta$;
(2) $u_n \rightrightarrows u$ uniformly on E_δ. It follows that $|u_n - u_m| < k$, for the fixed $k > 0$ and sufficiently large m and n.

Suppose that ζ $(0 \le \zeta \le 1)$ is a cut-off function satisfying $\zeta \in C_0^\infty(Q_T)$, and $\zeta = 1$ on E_δ. Let $T_k(s)$ be the usual truncation function defined as

$$T_k(s) = \begin{cases} s, & |s| < k, \\ k, & s \ge k, \\ -k, & s \leqslant -k. \end{cases}$$

For $n \neq m$, we have

$$\begin{aligned} \frac{\partial(u_n - u_m)}{\partial t} = \operatorname{div} \Big[& \Big(a(x) + \frac{1}{n}\Big)|\nabla u_n|^{p-2}\nabla u_n \\ & - \Big(a(x) + \frac{1}{m}\Big)|\nabla u_m|^{p-2}\nabla u_m\Big] - b(x)(|\nabla u_n|^q - |\nabla u_m|^q). \end{aligned} \tag{2.11}$$

By choosing $\zeta a T_k(u_n - u_m)$ as a test function and using

$$\iint_{Q_T} a(x)|\nabla u_n|^p \,dx\,dt \le c, \quad \iint_{Q_T} a(x)|\nabla u_m|^p \,dx\,dt \le c,$$

by the Hölder inequality it is not difficult to deduce that

$$\begin{aligned} &\iint_{Q_T} [a(x) + \frac{1}{n}](|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \\ &\times (\nabla u_n - \nabla u_m)\zeta a T_k'(u_n - u_m)\,dx\,dt \\ &\le k\int_\Omega \int_0^T a|\zeta_t||u_n - u_m|\,dt\,dx \\ &\quad + k\iint_{Q_T} a(x)|(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_n|^{p-2}\nabla u_n)||\nabla(a\zeta)|\,dx\,dt \\ &\quad + |\frac{1}{n} - \frac{1}{m}| \iint_{Q_T} a|\nabla u_m|^{p-1}[(|\nabla u_n| + |\nabla u_m|)\zeta T_k'(u_n - u_m) \\ &\quad + |\nabla(\zeta a)|T_k(u_n - u_m)]\,dx\,dt \\ &\quad + k\iint_{Q_T} a(x)B(x)||\nabla u_n|^q - |\nabla u_m|^q|\zeta\,dx\,dt \\ &\le kc(\delta) + c|\frac{1}{n} - \frac{1}{m}|. \end{aligned} \tag{2.12}$$

In view of $T_k' \ge 0$, $T_k'(s) = 1$ on $|s| < k$ and the fact that u_n converges uniformly on E_δ, we have

$$\begin{aligned} &\iint_{E_\delta} a^2(x)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\big)(\nabla u_n - \nabla u_m)\,dx\,dt \\ &= \iint_{E_\delta} a^2(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \\ &\quad \times (\nabla u_n - \nabla u_m))T_k'(u_n - u_m)\,dx\,dt \\ &\le \iint_{Q_T} [a(x) + \frac{1}{n}](|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \\ &\quad \times (\nabla u_n - \nabla u_m)\zeta a T_k'(u_n - u_m)\,dx\,dt. \end{aligned} \tag{2.13}$$

By the arbitrariness of k, from (2.12)-(2.13), we get

$$\begin{aligned}&\limsup_{n,m\to+\infty}\iint_{E_\delta} a^2(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m)\\ &\times(\nabla u_n-\nabla u_m)\,dx\,dt=0.\end{aligned} \tag{2.14}$$

Using this equality and following [8, 4], we have

$$\iint_{E_\delta} a^2(x)|\nabla u_n-\nabla u_m|^p\,dx\,dt\to 0. \tag{2.15}$$

For any $\varphi\in C_0^\infty(\Omega)$ with $0\le\varphi\le 1$ such that

$$\varphi\big|_{\Omega_{2\lambda}}=1,\quad \varphi\big|_{\Omega\setminus\Omega_\lambda}=0,$$

since $a^2(x)\ge c(\lambda)>0$ on Ω_λ, it follows from (2.15) that

$$\iint_{E_\delta}|\nabla(\varphi u_n)-\nabla(\varphi u_m)|^p\,dx\,dt\to 0.$$

Thus, $\{\nabla\varphi u\}$ is a Cauchy sequence in $(L^p(E_\delta))^N$. We may assume that $\nabla\varphi u_n\to\alpha$, strongly in $(L^p(E_\delta))^N$. Since $\varphi u_n\to\varphi u$ strongly in $L^s(\Omega)$ with $s>1$, it is easy to see that $\varphi u_n\to\varphi u$ strongly in $L^p(E_\delta)$. From the above analysis, we see that $\alpha=\varphi u$. By the arbitrariness of λ, $\nabla u_n\to\nabla u$ a.e. in E_δ, and by the arbitrariness of δ, $\nabla u_n\to\nabla u$ a.e. in Q_T.

Step 4: Convergence Let θ be the function defined in Lemma 2.1. It follows that $\theta(u_n-u_m)\in L^\infty(Q_T)\cap V$ since $u_n,u_m\in L^\infty(Q_T)\cap V$. Thus, for any $0\le\varphi(x)\in C_0^1(\Omega)$, we can take $\varphi\theta(u_n-u_m)$ as a test function in (2.11). Then

$$\begin{aligned}&\int_0^T\langle\frac{\partial(u_n-u_m)}{\partial t},\varphi\theta(u_n-u_m)\rangle\\ &+\iint_{Q_T}\Big[\Big(a(x)+\frac1n\Big)|\nabla u_n|^{p-2}\nabla u_n-\Big(a(x)+\frac1m\Big)|\nabla u_m|^{p-2}\nabla u_m\Big]\\ &\times(\nabla u_n-\nabla u_m)\theta'(u_n-u_m)\varphi\,dx\,dt\\ &+\iint_{Q_T}\Big[\Big(a(x)+\frac1n\Big)|\nabla u_n|^{p-2}\nabla u_n-\Big(a(x)+\frac1m\Big)|\nabla u_m|^{p-2}\nabla u_m\Big]\\ &\times(\nabla u_n-\nabla u_m)\nabla\varphi\theta(u_n-u_m)\,dx\,dt\\ &=\iint_{Q_T}B(\min\{|\nabla u_n|^q,n\}-\min\{|\nabla u_m|^q,m\})\theta(u_n-u_m)\varphi dx.\end{aligned} \tag{2.16}$$

Using (2.2) to estimate the first term on the left-hand side of (2.16) yields

$$\int_0^T\langle\frac{\partial(u_n-u_m)}{\partial t},\varphi\theta(u_n-u_m)\rangle dt=\int_\Omega\varphi\Theta(u_n-u_m)(T)dx\geqslant 0. \tag{2.17}$$

Since $u_n\to u$, $u_m\to u$ a.e. in Q_T, the right-hand side of (2.17) can be estimated as follows:

$$\begin{aligned}&\iint_{Q_T}B(\min\{|\nabla u_n|^q,n\}-\min\{|\nabla u_m|^q,m\})\theta(u_n-u_m)\varphi\,dx\,dt\\ &\leqslant b\iint_{Q_T}a(x)(|\nabla u_n|^p+|\nabla u_m|^p)|\theta(u_n-u_m)|\varphi\,dx\,dt\\ &\le b\iint_{Q_T}a(x)(|\nabla u_n|^{p-2}\nabla u_n\nabla u_m+|\nabla u_m|^{p-2}\nabla u_m\nabla u_n)|\theta(u_n-u_m)|\varphi\,dx\,dt\end{aligned}$$

$$
\begin{aligned}
&+ b \iint_{Q_T} \big[\frac{1}{n}|\nabla u_n|^{p-2}\nabla u_n - \frac{1}{m}|\nabla u_m|^{p-2}\nabla u_m\big](\nabla u_n - \nabla u_m)|\theta(u_n-u_m)|\varphi\,dx\,dt \\
&+ b \iint_{Q_T} \big[\big(a(x)+\frac{1}{n}\big)|\nabla u_n|^{p-2}\nabla u_n - \big(a(x)+\frac{1}{m}\big)|\nabla u_m|^{q-2}\nabla u_m\big] \\
&\times (\nabla u_n - \nabla u_m)|\theta(u_n-u_m)|\varphi\,dx\,dt.
\end{aligned}
$$

Hence, (2.16) can be rewritten as:

$$
\begin{aligned}
&\iint_{Q_T} [(a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - (a(x)+\frac{1}{m})|\nabla u_m|^{p-2}\nabla u_m](\nabla u_n - \nabla u_m) \\
&\times [\theta'(u_n-u_m) - b|\theta(u_n-u_m)|]\varphi\,dx\,dt \\
&+ \iint_{Q_T} \Big((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - (a(x)+\frac{1}{m})|\nabla u_m|^{p-2}\nabla u_m\Big) \\
&\quad \times \nabla\varphi\theta((u_n-u_m))\,dx\,dt \\
&\leqslant b \iint_{Q_T} a(x)||\nabla u_n|^{p-2}\nabla u_n\nabla u_m + |\nabla u_m|^{p-2}\nabla u_m\nabla u_n||\theta(u_n-u_m)|\,dx\,dt \\
&+ b \iint_{Q_T} \big[\frac{1}{n}|\nabla u_n|^{p-2}\nabla u_n - \frac{1}{m}|\nabla u_m|^{p-2}\nabla u_m\big](\nabla u_n - \nabla u_m) \\
&\quad \times |\theta(u_n-u_m)|\varphi\,dx\,dt.
\end{aligned} \tag{2.18}
$$

Clearly, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty} \iint_{Q_T} [(a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - (a(x)+\frac{1}{m})|\nabla u_m|^{p-2}\nabla u_m] \\
&\times (\nabla u_n - \nabla u_m)\nabla\varphi\theta(u_n-u_m)\,dx\,dt = 0,
\end{aligned} \tag{2.19}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} b \iint_{Q_T} &\big[\frac{1}{n}|\nabla u_n|^{p-2}\nabla u_n - \frac{1}{m}|\nabla u_m|^{p-2}\nabla u_m\big] \\
&\times (\nabla u_n - \nabla u_m)|\theta(u_n-u_m)|\varphi\,dx\,dt = 0.
\end{aligned} \tag{2.20}
$$

With the help of (2.1) in Lemma 2.1 (with $a = 1$), since $\nabla u_n \to \nabla u$ a.e. in Q_T, and $\varphi(x) \in C_0^1(\Omega)$, we may utilize Fatou's Lemma in (2.18) as $m \to +\infty$ to obtain that

$$
\begin{aligned}
&\iint_{Q_T} ((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)\varphi\,dx\,dt \\
&\leqslant cb \iint_{Q_T} a(x)[(|\nabla u_n|^{p-2}\nabla u_n\nabla u + |\nabla u|^{p-2}\nabla u\nabla u_n)]|\theta(u_n-u)|\,dx\,dt + o\big(\frac{1}{n}\big) \\
&\leqslant cb|a^{\frac{q-1}{p}}|\nabla u_n|^{p-2}\nabla u_n|_{L^{p'}(x)}(Q_T)|a^{1/p}\theta(u_n-u)\nabla u|_{L^p(Q_T)} \\
&\quad + cb||a^{\frac{p-1}{p}}\nabla u|^{p-2}\nabla u\theta(u_n-u)|_{L^{p'(x)}}(Q_T)|a^{1/p}\nabla u_n|_{L^q(Q_T)} + o(\frac{1}{n}) \\
&\leqslant C\Big(\iint_{Q_T} a|\nabla u_n|^p\,dx\,dt\Big)^{1/p'}(\iint_{Q_T} a|\theta(u_n-u)|^p|\nabla u|^q\,dx\,dt)^{1/p} \\
&\quad + C\Big(\iint_{Q_T} a(x)|\theta(u_n-u)|^{p'}|\nabla u|^p\,dx\,dt\Big)^{1/p'}\Big(\iint_{Q_T} a(x)|\nabla u_n|^p\,dx\,dt\Big)^{1/p} \\
&\quad + o(\frac{1}{n})
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\Big(\iint_{Q_T} a|\theta(u_n-u)|^p|\nabla u|^q\,dx\,dt\Big)^{1/p}\\
&\quad + C\Big(\iint_{Q_T} a|\theta(u_n-u)|^{p'}|\nabla u|^q\,dx\,dt\Big)^{1/p'} + o(\frac{1}{n}).
\end{aligned}
$$

Since $\theta(u_n - u)$ is uniformly bounded, by the Lebesgue dominated convergence theorem we have

$$
\iint_{Q_T} ((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n-\nabla u)\varphi\,dx\,dt \to 0,
$$

which implies

$$
\iint_{Q_T} (a(x)+\frac{1}{n})(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n-\nabla u)\varphi\,dx\,dt \to 0, \tag{2.21}
$$

because

$$
\lim_{n\to\infty}\frac{1}{n}\iint_{Q_T} |\nabla u|^{p-2}\nabla u\cdot(\nabla u_n-\nabla u)\varphi\,dx\,dt = 0.
$$

Following [8], by (2.21), we arrive at

$$
\iint_{Q_T} \Big(a(x)+\frac{1}{n}\Big)|\nabla u_n-\nabla u|^p\varphi\,dx\,dt \to 0, \tag{2.22}
$$

which implies

$$
|\nabla u_n-\nabla u|_{L^p(\Omega_1\times[0,T))} \to 0,
$$

where Ω is any compact subset including in Ω. That is, $u_n \to u$ strongly in $L^p(0,T;W^{1,p}_{\rm loc}(\Omega))$.

Step 5: Passing to the limit. By (2.22) and the property of Nemytskii operator ([10, 11]), the generalized Lebesgue dominated convergence theorem yields

$$
\begin{gathered}
|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u, \quad \text{strongly in } L^{p'}_{\rm loc}(Q_T),\\
\min\{|\nabla u_n|^p, n\} \to |\nabla u|^p, \quad \text{strongly in } L^1_{\rm loc}(Q_T).
\end{gathered}
$$

For each $\varphi\in C_0^\infty(Q_T)$, we get

$$
\begin{aligned}
&\big|\big\langle -\operatorname{div}((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u|^{p-2}\nabla u),\varphi\big\rangle\big|\\
&= \big|\iint_{Q_T}(a(x)+\frac{1}{n})(|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u|^{p-2}\nabla u)\cdot\nabla\varphi\,dx\,dt\big|\\
&\leqslant \big||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|_{L^{p'}(\Omega_\varphi\times(0,T))}|a(x)\nabla\varphi\big|_{L^p(Q_T)}\\
&\quad + \frac{1}{n}\int_0^T\int_{\Omega_\varphi}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,dx\,dt.
\end{aligned}
$$

It follows that

$$
\|-\operatorname{div}((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\|_{\mathcal{D}'} \to 0.
$$

Thus, for the principal term in the approximate equation (2.3), we have

$$
-\operatorname{div}((a(x)+\frac{1}{n})|\nabla u_n|^{p-2}\nabla u_n) \to -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u), \quad \text{strongly in } \mathcal{D}'.
$$

Meanwhile,

$$
\lim_{n\to\infty}\langle B(x)|\nabla u_n|^p - B(x)|\nabla u|^p,\varphi\rangle = 0.
$$

As a consequence, one has $u_{nt} \to u_t$, strongly in $\mathcal{D}'$, it follows that $u_n(x,0) \to u_0(x)$ in the sense of measure. This proves that $u \in L^\infty(Q_T)$ is a weak solution to equation (1.1) with the initial value condition (1.2). □

Lemma 2.3. *If $\int_\Omega a(x)^{-\frac{1}{p-1}} dx < \infty$, u is a weak solution of equation* (1.1) *with the initial condition* (1.2). *Then the trace of u on the boundary $\partial\Omega$ can be defined in the traditional way.*

Proof. Clearly, we have

$$\begin{aligned}
&\iint_{Q_T} |\nabla u|\, dx\, dt \\
&= \iint_{\{(x,t)\in Q_T; a^{\frac{1}{p-1}}|\nabla u|\leqslant 1\}} |\nabla u|\, dx\, dt + \iint_{\{(x,t)\in Q_T; a^{\frac{1}{p-1}}|\nabla u|> 1\}} |\nabla u|\, dx\, dt \\
&\leqslant \iint_{Q_T} a^{-\frac{1}{p-1}}\, dx\, dt + \iint_{Q_T} (a^{\frac{1}{p-1}}|\nabla u|)^{p-1}|\nabla u|\, dx\, dt \\
&= \iint_{Q_T} a^{-\frac{1}{p-1}}\, dx\, dt + \iint_{Q_T} a|\nabla u|^p\, dx\, dt \leqslant c.
\end{aligned}$$

The last inequality is because of the assumption that $\int_\Omega a(x)^{-\frac{1}{p-1}} dx \le c$. So u has the trace on the boundary $\partial\Omega$. □

By Lemma 2.3 and Theorem 1.3, we arrive at Theorem 1.4 immediately, so here we omit its proof.

3. Proof of Theorem 1.5

Let u and v be two solutions of equation (1.1) with the initial values $u_0(x) = v_0(x)$ and with the same homogeneous boundary value condition (1.3). We will prove the uniqueness of the solutions by the way of contradiction. Suppose that

$$\operatorname{ess\,sup}_{x\in\Omega} |u - v| > 0. \tag{3.1}$$

For the function θ defined in Lemma 2.1, it follows that $\theta(u-v) \in L^\infty(Q_T) \cap V$, since $u, v \in L^\infty(Q_T) \cap V$. Thus, $\theta(u-v)$ can be taken as a test function in (1.8) such that

$$\begin{aligned}
&\Big\langle \frac{\partial(u-v)}{\partial t}, \theta(u-v)\Big\rangle + \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\theta'(u-v)dx \\
&= \int_\Omega B(x)(|\nabla u|^q - |\nabla v|^q)\theta(u-v)dx.
\end{aligned} \tag{3.2}$$

Using (2.2) to estimate the first term on the left-hand side of (3.2) yields

$$\Big\langle \frac{\partial(u-v)}{\partial t}, \varphi\theta(u-v)\Big\rangle = \frac{\partial}{\partial t}\int_\Omega \Theta(u-v)dx. \tag{3.3}$$

By Young's inequality, we have

$$\begin{aligned} &|\int_\Omega B(x)(|\nabla u|^q-|\nabla v|^q)\theta(u-v)dx|\\ &\leq \int_\Omega B(x)\big[\frac{q}{p}(|\nabla u|^p+|\nabla v|^p)+\frac{p-q}{p}\big]|\theta(u-v)|dx\\ &\leq \frac{q}{p}\int_\Omega B(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|dx+\frac{p-q}{p}\int_\Omega B(x)|\theta(u-v)|dx, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\frac{q}{p}\int_\Omega B(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|dx\\ &\leqslant b\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|dx\\ &\leq b\iint_\Omega a(x)(|\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u)|\theta(u-v)|dx\\ &\quad +b\iint_\Omega a(x)[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v](\nabla u-\nabla v)|\theta(u-v)|dx. \end{aligned} \tag{3.5}$$

By (3.3)-(3.5), we have

$$\begin{aligned} &\frac{\partial}{\partial t}\int_\Omega \Theta(u-v)dx\\ &+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)(\nabla u-\nabla v)[\theta'(u-v)-b|\theta(u-v)|]dx\\ &\leq \frac{p-q}{p}\int_\Omega B(x)|\theta(u-v)|dx\\ &+b\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u)|\theta(u-v)|dx. \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} &\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u)|\theta(u-v)|dx\\ &\leq c\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|dx. \end{aligned}$$

So, by (3.6), we have

$$\begin{aligned} &\int_\Omega \Theta(u-v)dx-\int_\Omega \Theta(u_0-v_0)dx\\ &\leq \frac{p-q}{p}\int_0^t\int_\Omega B(x)|\theta(u-v)|\,dx\,dt\\ &\quad +c\int_0^t\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|\,dx\,dt. \end{aligned} \tag{3.7}$$

Since $u_0=v_0$, by (3.7), we find that

$$\begin{aligned} \int_\Omega \Theta(u-v)dx &\leq \frac{p-q}{p}\int_0^t\int_\Omega B(x)|\theta(u-v)|\,dx\,dt\\ &\quad +c\int_0^t\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|\,dx\,dt. \end{aligned} \tag{3.8}$$

Notice that $\Theta(s) = \frac{1}{2\eta}e^{\eta s^2}$ is an even function. Without loss of generality, we may assume that $\theta(u-v) \geq 0$. If

$$\frac{p-q}{p}\int_\Omega B(x)|\theta(u-v)|dx \leq c\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)|\theta(u-v)|dx,$$

then, by (3.8), we have

$$\int_\Omega \Theta(u-v)dx \leq c\int_0^t\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)|\theta(u-v)|\,dx\,dt.$$

That is,

$$\begin{aligned}\frac{1}{2\eta}\int_\Omega e^{\eta(u-v)^2}dx &\leq c\int_0^t\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)|u-v|e^{\eta(u-v)^2}\,dx\,dt\\ &\leq c\int_0^t \operatorname{ess\,sup}_{x\in\Omega}|u-v|\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)e^{\eta(u-v)^2}\,dx\,dt.\end{aligned}$$

This is impossible when $\eta \to 0$.

If

$$\frac{p-q}{p}\int_\Omega B(x)|\theta(u-v)|dx \geq c\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p)|u-v|e^{\eta(u-v)^2}dx,$$

then, (3.8) yields

$$\frac{1}{2\eta}\int_\Omega e^{\eta(u-v)^2}dx \leq c\int_\Omega B(x)|u-v|e^{\eta(u-v)^2}dx.$$

This is impossible when $\eta \to 0$. Consequently, we have

$$\operatorname{ess\,sup}_{x\in\Omega}|u-v| = 0,$$

which implies that the solution is unique.

4. Uniqueness without any boundary value condition

Proof of Theorem 1.6. For any $0 \leq \varphi(x) \in C_0^1(\Omega)$, we take $\varphi\theta(u-v)$ as a test function in (1.8) as

$$\begin{aligned}&\Big\langle\frac{\partial(u-v)}{\partial t}, \varphi\theta(u-v)\Big\rangle\\ &+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\theta'(u-v)\varphi dx\\ &+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\nabla\varphi\theta(u-v)dx\\ &=\int_\Omega B(x)(|\nabla u|^q - |\nabla v|^q)\theta(u-v)\varphi dx.\end{aligned} \tag{4.1}$$

Using (2.2) to estimate the first term on the left-hand side of (4.1) leads to

$$\Big\langle\frac{\partial(u-v)}{\partial t}, \varphi\theta(u-v)\Big\rangle = \frac{\partial}{\partial t}\int_\Omega \varphi\Theta(u-v)dx. \tag{4.2}$$

By Young's inequality, we have

$$\begin{aligned}
&\big|\int_\Omega B(x)(|\nabla u|^q-|\nabla v|^q)\theta(u-v)\varphi dx\big| \\
&\leq \int_\Omega B(x)[\frac{q}{p}(|\nabla u|^p+|\nabla v|^p)+\frac{p-q}{p}]|\theta(u-v)\varphi|dx \\
&\leq \frac{q}{p}\int_\Omega B(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)\varphi|dx+\frac{p-q}{p}\int_\Omega B(x)|\theta(u-v)\varphi|dx,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\frac{q}{p}\int_\Omega B(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)\varphi|dx \\
&\leqslant b\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|\varphi\,dx\,dt \\
&\leq b\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u)|\theta(u-v)|\varphi\,dx\,dt \\
&\quad +b\int_\Omega a(x)[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v](\nabla u-\nabla v)|\theta(u-v)|\varphi\,dx\,dt.
\end{aligned} \tag{4.4}$$

Then, by (4.2)–(4.4), we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega \varphi\Theta(u-v)dx \\
&+\int_\Omega a(x)[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v](\nabla u-\nabla v)[\theta'(u-v)-b|\theta(u-v)|]\varphi dx \\
&+\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi\theta(u-v)dx \\
&\leqslant b\iint_\Omega a(x)\varphi||\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u||\theta(u-v)|dx.
\end{aligned} \tag{4.5}$$

By Young's inequality, we have

$$\begin{aligned}
&\int_\Omega a(x)\varphi[|\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u||\theta(u-v)\,dx\,dt \\
&\leqslant c\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)|\theta(u-v)|\varphi\,dx\,dt.
\end{aligned} \tag{4.6}$$

In view of $\varphi\leq c$, we have

$$\int_\Omega a(x)\varphi||\nabla u|^{p-2}\nabla u\nabla v||\theta(u-v)|\varphi dx\leq c(\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p dx, \tag{4.7}$$

$$|\int_\Omega a(x)||\nabla u|^p|\nabla v|^{p-2}\nabla v\nabla u||\theta(u-v)|\varphi dx|\leq c(\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p dx. \tag{4.8}$$

For a small positive constant $\lambda>0$, let $\Omega_\lambda=\{x\in\Omega: a(x)>\lambda\}$. By a process of limit, we set

$$\varphi=\phi_\lambda(x)=\begin{cases}1, & \text{if} x\in\Omega_\lambda,\\ \frac{1}{\lambda}a(x), & x\in\Omega\setminus\Omega_\lambda.\end{cases} \tag{4.9}$$

It follows from (1.12) that

$$
\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi\theta(u-v)dx\Big|\\
&=\Big|\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi\theta(u-v)dx\Big|\\
&\le c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p}+|\nabla v|^{p})dx\Big)^{\frac{p-1}{p}}\frac{1}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^{p}dx\Big)^{1/p}\\
&\le c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p}+|\nabla v|^{p})dx\Big)^{\frac{p-1}{p}},
\end{aligned}
\tag{4.10}
$$

which approaches zero as $\lambda\to 0$.

By integrating (4.5) from 0 to t, we have

$$
\begin{aligned}
&\int_\Omega \varphi\Theta(u-v)dx-\int_\Omega\varphi\Theta(u_0-v_0)dx\\
&+\int_0^t\int_\Omega a(x)[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v](\nabla u-\nabla v)\\
&\times[\theta'(u-v)-b|\theta(u-v)|]\varphi\,dx\,dt\\
&+\int_0^t\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi\theta(u-v)\,dx\,dt\\
&\leqslant b\int_0^t\int_\Omega a(x)\varphi\big||\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u\big||\theta(u-v)|\,dx\,dt.
\end{aligned}
\tag{4.11}
$$

Let $\lambda\to 0$ in (4.11). Then

$$
\begin{aligned}
&\int_\Omega \Theta(u-v)dx-\int_\Omega\Theta(u_0-v_0)dx\\
&+\lim_{\lambda\to0}\int_0^t\int_\Omega a(x)[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v](\nabla u-\nabla v)\\
&\times[\theta'(u-v)-b|\theta(u-v)|]\,dx\,dt\\
&+\lim_{\lambda\to0}\int_0^t\int_\Omega a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\varphi\theta(u-v)\,dx\,dt\\
&\leqslant b\int_0^t\int_\Omega a(x)\big||\nabla u|^{p-2}\nabla u\nabla v+|\nabla v|^{p-2}\nabla v\nabla u\big||\theta(u-v)|\,dx\,dt\\
&\leqslant c\int_0^t\int_\Omega a(x)(|\nabla u|^{p}+|\nabla v|^{p}|\theta(u-v)|\,dx\,dt.
\end{aligned}
\tag{4.12}
$$

By (4.5)-(4.12), noticing that $\Theta(s)=\frac{1}{2\eta}e^{\eta s^2}$, we have

$$
\begin{aligned}
&\int_\Omega \Theta(u-v)dx-\int_\Omega\Theta(u_0-v_0)dx\\
&\le c\int_\Omega B(x)|\theta(u-v)|dx+c\int_\Omega a(x)(|\nabla u|^{p}+|\nabla v|^{p}||u-v|e^{\eta(u-v)^2}dx.
\end{aligned}
\tag{4.13}
$$

Since $u_0 = v_0$, by (4.13), we have

$$\begin{aligned}\int_\Omega \Theta(u-v)dx \le c\int_\Omega B(x)|\theta(u-v)|dx \\ + c\int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p||u-v|e^{\eta(u-v)^2}dx.\end{aligned} \tag{4.14}$$

From this inequality, similar to the proof of Theorem 1.5, one can obtain $u = v$. Consequently, the proof is complete. □

Acknowledgments. This work was supported by the NSF of Fujian Province 2015J01592.

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