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TRIGONOMETRIC POLYNOMIAL SOLUTIONS OF EQUIVARIANT TRIGONOMETRIC POLYNOMIAL ABEL DIFFERENTIAL EQUATIONS

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Abstract. Let $A(\theta)$ non-constant and $B_j(\theta)$ for $j = 0, 1, 2, 3$ be real trigonometric polynomials of degree at most $\eta \geq 1$ in the variable x. Then the real equivariant trigonometric polynomial Abel differential equations $A(\theta)y' = B_1(\theta)y+B_3(\theta)y^3$ with $B_3(\theta) \neq 0$, and the real polynomial equivariant trigonometric polynomial Abel differential equations of second kind $A(\theta)yy' = B_0(\theta)+B_2(\theta)y^2$ with $B_2(\theta) \neq 0$ have at most 7 real trigonometric polynomial solutions. Moreover there are real trigonometric polynomial equations of these type having these maximum number of trigonometric polynomial solutions.

1. Introduction and statement of the main results

Abel differential equations of first kind

$$a(x)\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3 \tag{1.1}$$

with $b_3(x) \neq 0$ appear in many textbooks of ordinary differential equations as one of first non-trivial examples of nonlinear differential equations, see for instance [11]. Here the dot denotes the derivative with respect to the independent variable x. If $b_3(x) = b_0(x) = 0$ or $b_2(x) = b_0(x) = 0$ the Abel differential equation reduces to a Bernoulli differential equation, while if $b_3(x) = 0$ the Abel differential equation reduces to a Riccati differential equation.

The Abel differential equations (1.2) have been studied intensively, either calculating their solutions (see for instance [8, 12, 15, 16]), or classifying their centers (see [2, 3, 4]), and recently in [7, 9, 10, 13] the authors studied the polynomial solutions of the differential equations $y' = \sum_{i=0}^{n} a_i(x)y^i$, or similar.

The analysis of particular solutions (as polynomial or rational solutions) of the differential equations is important for understanding the set of solutions of a differential equation. In 1936 Rainville [17] characterized the Riccati differential equations $\dot{y} = b_0(x) + b_1(x)y + y^2$, with $b_0(x)$ and $b_1(x)$ polynomials in the variable x, having polynomial solutions.

In 1954, Campbell and Golomb [5] provided an algorithm for determining the polynomial solutions of the Riccati differential equation $a(x)y' = b_0(x) + b_1(x)y +$

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$b_2(x)y^2$, where a, b_0, b_1, b_2 are polynomials in the variable x. In 2006, Behloul and Cheng [1] gave a different algorithm for finding the rational solutions of the differential equations $a(x)y' = \sum_{i=0}^{n} b_i(x)y^i$, where a, b_i are polynomials in the variable x.

The case in which the Abel differential equations (1.2) where $a(x) \in \mathbb{R}[x] \setminus \{0\}$, $b_i(x) \in \mathbb{R}[x]$, $i = 0, 1, 2, 3$ with $b_3(x) \neq 0$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{F}[x] := \mathbb{R}[x]$ is the ring of polynomials in the variable x with coefficients in $\mathbb{F}$, being either $a(x)$ constant or not and with the equivariant symmetry (see below) were studied in [14].

Here we go a step beyond and we consider the Abel differential equations (1.1) for real trigonometric polynomials, that is

$$A(\theta)Y' = B_0(\theta) + B_1(\theta)Y + B_2(\theta)Y^2 + B_3(\theta)Y^3, \tag{1.2}$$

where the prime denotes derivative with respect to θ and where $A(\theta) \in \mathbb{R}_t(\theta) \setminus \{0\}$, $B_i(\theta) \in \mathbb{R}_t(\theta)$, $i = 0, 1, 2, 3$, $B_3(\theta) \neq 0$, being $\mathbb{R}_t[\theta] := \mathbb{R}[\cos\theta, \sin\theta]$ the ring of trigonometric polynomials in the variables $\cos\theta, \sin\theta$ with coefficients in $\mathbb{R}$. We also assume that $A(\theta)$ is not constant. The case $A(\theta)$ constant has been studied in [10]. We also have $\eta := \max\{\alpha, \beta_0, \beta_1, \beta_2, \beta_3\}$, where α is the degree of $A(\theta)$, β_i is the degree of $B_i(\theta)$ for $i = 0, 1, 2, 3$. We say that the Abel trigonometric polynomial differential equation (1.2) has degree η.

Equation (1.2) is *reversible* with respect to the change of variables $(\theta, Y) \to (\theta, -Y)$ if the equation

$$-A(\theta)Y' = -(B_0(\theta) - B_1(\theta)Y + B_2(\theta)Y^2 - B_3(\theta)Y^3)$$

coincides with equation (1.2). In particular this implies $B_1(\theta) = B_3(\theta) = 0$, and since $B_3(\theta) = 0$ we do not consider these reversible differential equations.

The Abel differential equation (1.2) is *equivariant* with respect to the change of variables $(\theta, Y) \to (\theta, -Y)$ if the following equation

$$-A(\theta)Y' = B_0(\theta) - B_1(\theta)Y + B_2(\theta)Y^2 - B_3(\theta)Y^3$$

coincides with equation (1.2). This implies $B_0(\theta) = B_2(\theta) = 0$. In this paper first we focus our study in these kind of *equivariant trigonometric polynomial Abel equations*, i.e. in the equations

$$A(\theta)Y' = B_1(\theta)Y + B_3(\theta)Y^3. \tag{1.3}$$

Theorem 1.1. *Real equivariant trigonometric polynomial Abel differential equations with $B_3(\theta) \neq 0$ and $A(\theta)$ non-constant, have at most 7 trigonometric polynomial solutions. Moreover there are equations of this type having these maximum number of trigonometric polynomial solutions.*

The proof of Theorem 1.1 is given in section 3.

Our second objective in this paper is on the Abel trigonometric polynomial differential equations of second kind, i.e. on the equations of the form

$$A(\theta)YY' = B_0(\theta) + B_1(\theta)Y + B_2(\theta)Y^2, \tag{1.4}$$

where again the prime denotes derivative in the variable θ, $A(\theta), B_i(\theta) \in \mathbb{R}_t(\theta)$ for $i = 0, 1, 2$, with $A(\theta)$ and $B_2(\theta)$ non-zero. We also consider the ones that are equivariant with respect to the change $(\theta, Y) \to (\theta, -Y)$. Then we have that $B_1(\theta) = 0$ and so equation (1.4) becomes

$$A(\theta)YY' = B_0(\theta) + B_2(\theta)Y^2. \tag{1.5}$$

We also assume that $B(\theta) \neq 0$ (otherwise would be linear) and $A(\theta)$ is not constant, because the case $A(\theta)$ constant has been studied in [7]. We say that system (1.5) is an *equivariant trigonometric polynomial Abel differential equation of second kind*. The study of the number of polynomial solutions of equivariant Abel polynomial differential equations of the second kind

$$a(x)y\dot{y} = b_0(x) + b_2(x)y^2$$

where the dot means derivative with respect to x, $a(x), b_0(x), b_2(x) \in \mathbb{R}[x]$ with $a(x)$ non constant and $b_0(x)b_2(x) \neq 0$ was done in [14].

Theorem 1.2. *Real equivariant trigonometric polynomial Abel differential equations of second kind with $B_2(\theta) \neq 0$ and $A(\theta)$ non-constant, have at most 7 trigonometric polynomial solutions. Moreover there are equations of this type having these maximum number of trigonometric polynomial solutions.*

The proof of Theorem 1.2 is given in section 4.

2. Preliminary results

As we will see the proof of Theorems 1.1 and 1.2 are based on divisibility arguments in the ring of polynomials. In the ring of trigonometric polynomials we do not have a *Unique Factorization Domain*. This can be seen for instance using the identity $\cos^2\theta + \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$. So, $\cos\theta$ divides the right hand expression but it does not divide the left hand expression. This difficulty can be overcome by using the isomorphism $\Phi\colon R_t(\theta) \to R(x)$ given by

$$(\cos\theta, \sin\theta) \mapsto \Big(\frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2}\Big)$$

between the fields $\mathbb{R}_t(\theta) = \mathbb{R}(\cos\theta, \sin\theta)$ and $R(x)$ being $\mathbb{R}(x)$ the ring of rational functions. In fact we have the following well-known result.

Lemma 2.1. *Let $P(\theta) \in \mathbb{R}_t[\theta]$ with $\deg(P) = \eta$. Then*

$$\Phi(P(\theta)) = \frac{p(x)}{(1+x^2)^\eta},$$

where $\gcd(p(x), 1+x^2) = 1$ and $\deg(p(x)) \leq 2\eta$. Conversely, any rational function $g(x)/(1+x^2)^\eta$ with $g(x)$ an arbitrary polynomial of degree at most 2η can be written as a trigonometric polynomial through the inverse change Φ^{-1}.

Another result that we will us is the following theorem proved in [6].

Theorem 2.2. *Let $p, q \in \mathbb{R}[x]$ be polynomials satisfying $\gcd(p,q) = 1$ and*

$$p^2 + q^2 = r^2, \quad p^2 + \alpha^2 q^2 = s^2 \tag{2.1}$$

where $r, s \in \mathbb{R}[x]$ and $\alpha \in \mathbb{R}$. Then either $\alpha = 0$ or $\alpha^2 = 1$.

Now we write how equation (1.3) can be written in terms of $a(x), b_1(x), b_3(x) \in \mathbb{R}[x]$.

Lemma 2.3. *If $Y(\theta)$ is a nonconstant real trigonometric polynomial solution of* (1.3)*, set*

$$Y(\theta) = \frac{y(x)}{(1+x^2)^{\eta_0}}, \quad A(\theta) = \frac{a(x)}{(1+x^2)^{\eta_1}}, \tag{2.2}$$

$$B_1(\theta) = \frac{b_1(x)}{(1+x^2)^{\eta_2}}, \quad B_3(\theta) = \frac{b_3(x)}{(1+x^2)^{\eta_3}}$$

with $\deg(y) \le 2\eta_0$, $\deg(a) \le 2\eta_1$, $\deg(b_1) \le 2\eta_2$, $\deg(b_3) \le 2\eta_3$ *and* $\gcd(y, 1+x^2) = \gcd(a, 1+x^2) = \gcd(b_1, 1+x^2) = \gcd(b_3, 1+x^2) = 1$. *Then equation* (1.3) *becomes*

$$\begin{aligned} &\frac{a(x)}{2(1+x^2)^{\eta_1}}(\dot{y}(x)(1+x^2) - 2\eta_0 x y(x)) \\ &= \frac{b_1(x)}{(1+x^2)^{\eta_2}} y(x) + \frac{b_3(x)}{(1+x^2)^{\eta_3+2\eta_0}} y(x)^3, \end{aligned} \tag{2.3}$$

where the dot denotes derivative with respect to x.

Proof. From the diffeomorphism Φ we have that

$$x' = \frac{dx}{d\theta} = \frac{1+x^2}{2}$$

and so

$$Y(\theta)' = \frac{\dot{y}(x)(1+x^2) - 2\eta_0 x y(x)}{2(1+x^2)^{\eta_0}},$$

where the dot and the prime denote the derivative with respect to x and θ, respectively. So, equation (1.3) becomes (2.3). $\square$

In the same manner as in the proof of Lemma 2.3 we can see how equation (1.5) can be written in terms of $a(x), b_0(x), b_2(x) \in \mathbb{R}[x]$.

Lemma 2.4. *If* $Y(\theta)$ *is a nonconstant real trigonometric polynomial solution of* (1.5)*, set* $Y(\theta), A(\theta)$ *as in* (2.2) *and*

$$B_0(\theta) = \frac{b_0(x)}{(1+x^2)^{\eta_2}}, \quad B_2(\theta) = \frac{b_2(x)}{(1+x^2)^{\eta_3}}$$

with $\deg(y) \le 2\eta_0$, $\deg(a) \le 2\eta_1$, $\deg(b_0) \le 2\eta_2$, $\deg(b_2) \le 2\eta_3$, *and* $\gcd(y, 1+x^2) = \gcd(a, 1+x^2) = \gcd(b_0, 1+x^2) = \gcd(b_2, 1+x^2) = 1$. *Then equation* (1.5) *becomes*

$$\begin{aligned} &\frac{a(x)}{2(1+x^2)^{\eta_1}}(y(x)\dot{y}(x)(1+x^2) - 2\eta_0 x y(x)^2) \\ &= \frac{b_0(x)}{(1+x^2)^{\eta_2-2\eta_0}} y(x) + \frac{b_2(x)}{(1+x^2)^{\eta_3}} y(x)^2, \end{aligned} \tag{2.4}$$

where the dot denotes derivative with respect to x.

3. Proof of Theorem 1.1

First we recall that if $Y(\theta) \neq 0$ is a solution of (1.3), then $-Y(\theta)$ is also a solution of equation (1.3) which is different from $Y(\theta)$.

Lemma 3.1. *Let* $Y_0(\theta) \neq 0$, $Y_1(\theta), Y_2(\theta)$ *be polynomial solutions of equation* (1.3) *such that* $Y_1(\theta) \not\equiv 0$, $Y_2(\theta) \not\equiv 0$ *and* $Y_2(\theta) \neq -Y_1(\theta)$. *Set*

$$Y_i(\theta) = \frac{y_i(x)}{(1+x^2)^{\eta_i}}, \quad i = 0, 1, 2$$

where $\eta_i = \deg(Y_i)$ *and* $\deg(y_i) \le 2\eta_i$, $\eta_1 \le \eta_2$ *and* $\gcd(y_i, 1+x^2) = 1$ *for* $i = 0, 1, 2$. *We write* $y_1(x) = g(x)\tilde{y}_1(x)$ *and* $y_2(x) = g(x)\tilde{y}_2(x)$ *where* $g = \gcd(y_1, y_2)$. *Then,* *except the solution $Y=0$, all the other polynomial solutions of equation* (1.3) *can be expressed as*

$$y_0(\theta;c)=\pm\frac{\tilde y_1(x)\tilde y_2(x)g(x)}{\big(c\tilde y_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)}+(1-c)\tilde y_2^2(x)\big)^{1/2}}, \tag{3.1}$$

where c is a constant and $\big(c\tilde y_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)}+(1-c)\tilde y_2^2(x)\big)^{1/2}$ is a polynomial.

Proof. Let Y be a nonzero trigonometric polynomial solution of (1.3). The functions $Z_0=1/Y_0^2$, $Z_1=1/Y_1^2$ and $Z_2=1/Y_2^2$ are solutions of a linear differential equation and satisfy

$$-A(\theta)Z_i(\theta)'=2B_1(\theta)Z_i+2B_3(\theta),\quad i=0,1,2.$$

Therefore we have

$$\frac{Z_0(\theta)'-Z_1(\theta)'}{Z_0(\theta)-Z_1(\theta)}=\frac{Z_2(\theta)'-Z_1(\theta)'}{Z_2(\theta)-Z_1(\theta)}.$$

Integrating this equality we obtain

$$Z_0(\theta)=Z_1(\theta)+c(Z_2(\theta)-Z_1(\theta)),$$

with c an arbitrary constant. So the general solution of equation (1.3) is

$$\begin{aligned}Y_0^2(\theta)&=\frac{1}{Z_0(\theta)}=\frac{1}{Z_1(\theta)+c(Z_2(\theta)-Z_1(\theta))}\\&=\frac{Y_1^2(\theta)Y_2^2(\theta)}{cY_1^2(\theta)+(1-c)Y_2^2(\theta)}\end{aligned}$$

In other words,

$$\frac{y_0(x)^2}{(1+x^2)^{2(\eta_0-\eta_1)}}=\frac{g(x)^2\tilde y_1^2(x)\tilde y_2^2(x)}{c(1+x^2)^{2(\eta_2-\eta_1)}\tilde y_1^2(x)+(1-c)\tilde y_2^2(x)}. \tag{3.2}$$

Since the right-hand side of equation (3.2) is not divisible by $1+x^2$ we must have that $\eta_0\ge\eta_1$. However, if $\eta_0>\eta_1$ since neither $\tilde y_2$ nor y_0 (and when $\eta_1=\eta_2$ then also $\tilde y_1$) do not divide $1+x^2$, we get a contradiction with (3.2). In short we must have $\eta_0=\eta_1$. Then equation (3.2) becomes

$$y_0(x)^2=\frac{g(x)^2\tilde y_1^2(x)\tilde y_2^2(x)}{c(1+x^2)^{2(\eta_2-\eta_1)}\tilde y_1^2(x)+(1-c)\tilde y_2^2(x)}$$

with c an arbitrary constant. □

In view of Lemma 2.3, if $Y_1(\theta),Y_2(\theta)$ are trigonometric polynomial solutions of equation (1.3) such that $Y_1(\theta)\not\equiv 0$, $Y_2(\theta)\not\equiv 0$, $Y_2(\theta)\ne Y_1(\theta)$, then any other trigonometric polynomial solution different from them is of the form given in (3.1) for some appropriate constant c such that $c\notin\{0,1\}$. In particular, $c\tilde y_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)}+(1-c)\tilde y_2^2(x)$, or $\tilde y_3^2(x)+(1-c)\tilde y_2^2(x)/c$, where $\tilde y_3(x)=(1+x^2)^{\eta_2-\eta_1}\tilde y_1(x)$ is a square of a polynomial p and p divides g. In view of Theorem 2.2 there is at most one constant $c\notin\{0,1\}$ such that $c\tilde y_1^2(1+x^2)^{2(\eta_2-\eta_1)}+(1-c)\tilde y_2^2$ is a square of a polynomial meaning that equation (1.3) has at most seven different trigonometric polynomial solutions 0, $\pm Y_1$, $\pm Y_2$ and Y_0.

Example 3.2. Note that in view of Lemma 2.3, the polynomial solutions y_1,y_2 and y_0 can always be taken of the form

$$y_1=\pm\frac{rs(r^2+s^2)}{2\sqrt c},\quad y_2=\pm\frac{rs(r^2-s^2)}{2\sqrt{c-1}},\quad y_0=\pm\frac{(r^2+s^2)(r^2-s^2)}{4\sqrt c\sqrt{c-1}}$$

with r^2-s^2 and r^2+s^2 coprime. For instance one can take $r=1$, $s=2x$, $c=2$, and then

$$y_1=\pm\Big(2\sqrt{2}x^3+\frac{x}{\sqrt{2}}\Big),\quad y_2=\pm(x-4x^3),\quad y_0=\pm\Big(\frac{1}{4\sqrt{2}}-2\sqrt{2}x^4\Big).$$

We recall that these are polynomial solutions of the equation

$$a(x)\dot{y}=b_1(x)y+b_3(x)y^3,$$

with

$$\begin{gathered}a(x)=-2x+48x^3-768x^7+512x^9,\\ b_1(x)=2(-1+96x^4-1536x^6+768x^8),\\ b_3(x)=64.\end{gathered}$$

Therefore, in view of Lemmas 2.3 and 2.1 if we set

$$\begin{gathered}A(\theta)=\frac{2a(x)}{(1+x^2)^5}=\frac{-2x+48x^3-768x^7+512x^9}{(1+x^2)^5},\\ B_0(\theta)=\frac{b_0(x)(1+x^2)-4xa(x)}{(1+x^2)^5}=\frac{-2+6x^2-2880x^6+1536x^8-540x^{10}}{(1+x^2)^5},\\ B_2(\theta)=\frac{b_2(x)}{(1+x^2)^2}=\frac{64}{(1+x^2)^2},\end{gathered}$$

then, the seven solutions

$$Y_1(\theta)=\pm\frac{y_1(x)}{(1+x^2)^2},\quad Y_2(\theta)=\pm\frac{y_2(x)}{(1+x^2)^2},\quad Y_0(\theta)=\pm\frac{y_0(x)}{(1+x^2)^2},\quad Y_3(\theta)=0$$

are trigonometric polynomial solutions of equation (1.5).

4. Proof of Theorem 1.2

First we recall that if $Y(\theta)\neq 0$ is a solution of (1.5), then $-Y(\theta)$ is also a solution of (1.5) which is different from $Y(\theta)$.

Lemma 4.1. *Let $Y_0(\theta)\neq 0$, $Y_1(\theta),Y_2(\theta)$ be polynomial solutions of equation* (1.5) *such that $Y_1(\theta)\not\equiv 0$, $Y_2(\theta)\not\equiv 0$ and $Y_2(\theta)\neq -Y_1(\theta)$. Set*

$$Y_i(\theta)=\frac{Y_i(x)}{(1+x^2)^{\eta_i}},\quad i=1,2$$

where $\eta_i=\deg(Y_i)$ and $\deg(y_i)\le 2\eta_i$, $\eta_1\le\eta_2$ and $\gcd(y_i,1+x^2)=1$ for $i=1,2$. We write $y_1(x)=g(x)\tilde{y}_1(x)$ and $y_2(x)=g(x)\tilde{y}_2(x)$ where $g=\gcd(y_1,y_2)$. Then, except the solution $Y=0$, all the other trigonometric polynomial solutions of equation (1.5) *can be expressed as*

$$y_0(\theta;c)=\pm g(x)\big(c\tilde{y}_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)}+(1-c)\tilde{y}_2^2(x)\big)^{1/2},\tag{4.1}$$

where c is a constant.

Proof. Let Y be a nonzero trigonometric polynomial solution of equation (1.3). The functions $Z_0=Y_0^2$, $Z_1=Y_1^2$ and $Z_2=Y_2^2$ are solutions of a linear differential equation and satisfy

$$A(\theta)Z_i'=2B_0(\theta)+2B_2(\theta)Z_i,\quad i=0,1,2.$$

Therefore

$$\frac{Z_0(\theta)' - Z_1(\theta)'}{Z_0(\theta) - Z_1(\theta)} = \frac{Z_2(\theta)' - Z_1(\theta)'}{Z_2(\theta) - Z_1(\theta)}.$$

Integrating this equality we obtain

$$Z_0(\theta) = Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta)),$$

with c an arbitrary constant. So the general solution of equation (1.3) is

$$Y_0^2(\theta) = Z_0(\theta) = Z_1(\theta) + c(Z_2(\theta) - Z_1(\theta)) = (1-c)Y_1^2(\theta) + cY_2^2(\theta),$$

where c is an arbitrary constant. Hence, we have

$$\begin{aligned}\frac{y_0^2}{(1+x^2)^{2\eta_0}} &= \frac{(1-c)y_1^2(x)}{(1+x^2)^{2\eta_1}} + \frac{cy_2^2(x)}{(1+x^2)^{2\eta_2}}\\ &= \frac{(1-c)y_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)} + cy_2(x)^2}{(1+x^2)^{2\eta_2}}.\end{aligned}$$

In other words,

$$\frac{y_0(x)^2}{(1+x^2)^{2(\eta_0-\eta_2)}} = g(x)^2((1-c)\tilde{y}_1(x)^2(1+x^2)^{(2(\eta_2-\eta_1)} + c\tilde{y}_2(x)^2). \tag{4.2}$$

Since the right-hand side of equation (4.2) is a polynomial and $y_0(x)$ does not divide $1+x^2$, we must have that $\eta_0 \le \eta_2$. However, if $\eta_0 < \eta_2$ since $\tilde{y}_2$ does not divide $1+x^2$ we get a contradiction with (4.2). In short we must have $\eta_0 = \eta_2$. Then equation (4.2) becomes

$$y_0(x)^2 = g(x)^2((1-c)\tilde{y}_1(x)^2(1+x^2)^{2(\eta_2-\eta_1)} + c\tilde{y}_2(x)^2)$$

with c an arbitrary constant. $\square$

In view of Lemma 2.3, if $Y_1(\theta), Y_2(\theta)$ are trigonometric polynomial solutions of equation (1.5) such that $Y_1(\theta) \not\equiv 0$, $Y_2(\theta) \not\equiv 0$ and $Y_2(\theta) \neq -Y_1(\theta)$ then any other trigonometric polynomial solution is of the form as in (4.1) for some appropriate constant c. In particular, $c\tilde{y}_1^2(x)(1+x^2)^{2(\eta_2-\eta_1)} + (1-c)\tilde{y}_2^2(x)$ is a square of a polynomial pP. In view of Theorem 2.2 this c is unique and we conclude that (1.5) has at most seven different trigonometric polynomial solutions 0, $\pm Y_1$, $\pm Y_2$ and Y_0.

Example 4.2. Note that in view of Lemma 4.1, the polynomial solutions y_1, y_2 and y_0 can always be taken of the form

$$y_1 = \pm\frac{r^2+s^2}{\sqrt{c}}, \quad y_2 = \pm\frac{r^2-s^2}{\sqrt{c-1}}, \quad y_0 = \pm 2rs$$

with r^2-s^2 and r^2+s^2 coprime. For instance one can take

$$r = \sqrt{2}x, \quad s = \frac{1}{\sqrt{2}}, \quad c = 2,$$

and then

$$y_1 = \pm\Big(\sqrt{2}x^2 + \frac{1}{2\sqrt{2}}\Big), \quad y_2 = \pm\Big(2x^2 - \frac{1}{2}\Big), \quad y_0 = \pm 2x.$$

We recall that these are polynomial solutions of the equation

$$a(x)y\dot{y} = b_0(x)y + b_2(x)y^2,$$

with

$$a(x) = 2x^4 - 3x^2 + \frac{1}{8},$$

$$b_0(x) = \frac{x}{2} - 8x^5,$$
$$b_2(x) = 4x^3 - 3x.$$

Therefore, in view of Lemmas 2.4 and 2.1, if we set

$$A(\theta) = \frac{2a(x)}{(1+x^2)^5} = \frac{4x^4 - 6x^2 + 1/4}{(1+x^2)^5},$$
$$B_0(\theta) = \frac{b_0(x)(1+x^2) - 2xa(x)}{(1+x^2)^5} = \frac{x(1+26x^2 - 48x^4 - 32x^6)}{4(1+x^2)^5},$$
$$B_2(\theta) = \frac{b_2(x)}{(1+x^2)^2} = \frac{4x^3 - 3x}{(1+x^2)^2},$$

then, the solutions

$$Y_1(\theta) = \frac{y_1(x)}{1+x^2}, \quad Y_2(\theta) = \frac{y_2(x)}{1+x^2}, \quad Y_0(\theta) = \frac{2x}{1+x^2}, \quad Y_3(\theta) = 0$$

are trigonometric polynomial solutions of equation (1.5).

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