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# LARGE TIME BEHAVIOUR FOR NON-SIMPLE THERMOELASTICITY WITH SECOND SOUND

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Abstract. We prove that the non-simple thermoelastic model, with Cattaneo's or Gurtin-Pipkin's law, is indifferent to the presence of the inertial term. That is, considering or not the irrotational term, there is a lack of exponential stability. Additionally, we show that the semigroup is polynomially stable and that the rate of decay of the solution (both optimal) are the same with or without the rotational term.

## 1. Introduction

The Euler Bernoulli thermoelastic model is

$$\rho u_{tt} - \gamma u_{xxtt} + \alpha u_{xxxx} - \beta\theta_{xx} = 0, \quad \text{in } ]0,\ell[\times\mathbb{R}_+\,, \tag{1.1}$$

$$c\theta_t + q_x + \beta u_{xxt} = 0, \quad \text{in } ]0,\ell[\times\mathbb{R}_+\,. \tag{1.2}$$

In Graselli's article [7] is proved that the thermoelastic plate ($\gamma = 0$) with heat flux given by the theory of Gurtin and Pipkin, $q = -\int_0^\infty g(s)\theta_x(t-s)ds$, is not exponential stable, but when the irrotational term ($\gamma > 0$) is inserted, the model becomes exponentially stable. Another case of the same phenomenon occurs when the flux is defined by Cattaneo's law: $\tau q_t + q + K\theta_x = 0$. System (1.1)–(1.2) with $\gamma = 0$, does not have exponential stability, but when $\gamma$ is positive, the resulting model is exponentially stable, see [8].

Here we consider the same problem to non-simple thermoelastic model, which mathematically is analogous to model (1.1)–(1.2). The difference is due to the coupling. Whereas in model (1.1)–(1.2) the coupling terms are of second order, in non-simple thermoelastic model, they are of first order. The non-simple thermoelasticity with second sound is

$$\rho u_{tt} = T_x - S_{xx}, \quad T = \mu u_x + \beta\theta, \quad S = \alpha u_{xx}. \tag{1.3}$$

The balance of the energy is give by

$$\rho T_0\Theta_t = q_x, \quad \rho\Theta = -\beta u_x + c\theta, \tag{1.4}$$

where $q$ is the heat flux. Therefore the system of field equations are

$$\rho u_{tt} - \gamma u_{xxtt} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = 0, \quad \text{in } ]0,\ell[\times\mathbb{R}_+\,, \tag{1.5}$$

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$$c\theta_t + q_x - \beta u_{xt} = 0, \quad \text{in } ]0,\ell[\times\mathbb{R}_+ . \tag{1.6}$$

Here we consider both, the second sound constitutive equation

$$\tau q_t + q + \kappa\theta_x = 0, \quad \text{in } ]0,\ell[\times\mathbb{R}_+ \tag{1.7}$$

and the Gurtin-Pipkin's law [10]

$$q = \int_0^\infty g(s)\theta_x(t-s)\,ds. \tag{1.8}$$

The memory kernel $g : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be positive, such that $|g(s)| \le Ce^{-\gamma s}$. For both models we consider the following boundary and initial conditions

$$u(0,t) = u_{xx}(0,t) = u(\ell,t) = u_{xx}(\ell,t) = 0, \quad \theta_x(0) = \theta_x(\ell) = 0, \tag{1.9}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x) . \tag{1.10}$$

When $\tau = 0$ in (1.7) (Fourier law) it was proved in [9], that the system is exponentially stable. The main result of this paper is that the non-simple thermoelastic model (1.5)–(1.6) with Cattaneo's law (1.7) or Gurtin and Pipkin's law (1.8) are not exponentially stable for $\gamma \ge 0$. Still, we prove that the inertial term does not improve uniform stability at all. That is, the decay rate is equal to $t^{-1/2}$ for $\gamma \ge 0$.

## 2. Semigroup approach

The semigroup approach to Gurtin and Pipkin's law follows the same ideas as in [10]. We introduce the summed past history of $\theta$ (cf. [2]), defined as

$$\eta(s,t) = \int_0^s \theta(t-\sigma)\,d\sigma, \quad (t,s) \in [0,\infty[\times\mathbb{R}_+$$

Therefore integrating by parts, $q$ can be rewritten as

$$q = \int_0^\infty \kappa(s)\eta_x(t-s)\,ds, \quad \kappa(s) = -g'(s) \tag{2.1}$$

with $\eta$ satisfying the following conditions

$$\eta_t + \eta_s = \theta, \quad \text{in } ]0,\ell[, \quad (t,s) \in [0,\infty[\times\mathbb{R}_+ ,$$
$$\eta(0) = 0, \quad \eta(s,0) = \eta_0(s).$$

Therefore the corresponding resolvent model for $\gamma \ge 0$ is

$$i\lambda u - v = f_1 , \tag{2.2}$$

$$i\lambda\rho v - i\lambda\gamma v_{xx} - \mu u_{xx} + \alpha u_{xxxx} - \beta\theta_x = \rho f_2 + \gamma f_{2,xx} , \tag{2.3}$$

$$i\lambda c\theta + q_x - \beta v_x = f_3 . \tag{2.4}$$

In the case of Cattaneo's law, we additionally have

$$i\lambda\tau q + q + \kappa\theta_x = f_4 . \tag{2.5}$$

For Gurtin and Pipkin law (2.1) we have

$$i\lambda\eta + \eta_s - \theta = f_4 . \tag{2.6}$$

For $\gamma > 0$, $f_2 \in H_0^1$, where $H_0^1 = H_0^1(0,\ell)$, $L^2 = L^2(0,\ell)$ and so on. The space for $\eta$ is $\mathcal{M}_1 = L^\infty_\kappa(\mathbb{R}_+; H^1_*)$, where

$$H^1_* = H^1 \cap L^2_*, \quad L^2_* = \big\{f \in L^2(0,\ell) : \int_0^\ell f(s)\,ds = 0\big\}.$$

The main tool to show the asymptotic properties is the next theorem.

**Theorem 2.1.** *Let $e^{At}$ be contraction semigroup. Then the exponential* [5] *and polynomial characterization* [1] *are*

$$\|e^{At}\| \le Ce^{-\gamma t} \Leftrightarrow i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda I - \mathcal{A})^{-1}\| \le C,\ \forall \lambda \in \mathbb{R}\,, \tag{2.7}$$

$$\|e^{At}\mathcal{A}^{-1}\| \le \frac{C}{t^{1/\alpha}} \Leftrightarrow i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda I - \mathcal{A})^{-1}\| \le C|\lambda|^{\alpha},\ \forall \lambda \in \mathbb{R}\,. \tag{2.8}$$

Let $\mathcal{H}_0^1$, $\mathcal{H}_0^2$ be the phase space to Cattaneo and Gurtin-Pipkin law respectively for $\gamma = 0$, where

$$\mathcal{H}_0^i = H^2 \cap H_0^1 \times L^2 \times L_*^2 \times \mathcal{V}_i, \quad i = 1, 2, \quad \mathcal{V}_1 = L^2(0,\ell), \quad \mathcal{V}_2 = \mathcal{M}_1.$$

The corresponding domain of the infinitesimal generator $\mathcal{A}$ for $\gamma = 0$ is

$$D(\mathcal{A}_{0,i}) = H^4 \cap H_0^1 \times H^2 \cap H_0^1 \times H_*^1 \times \mathcal{W}_i\,,$$

where

$$\mathcal{W}_1 = H_0^1, \quad \mathcal{W}_2 = \big\{\eta \in \mathcal{M}_1 : \eta_s \in \mathcal{M}_1,\ \ \eta(0) = 0,\ \int_0^\infty \kappa(s)\eta_x\,ds \in H_0^1\big\}$$

Instead when $\gamma > 0$, the phase space is of the form

$$\mathcal{H}_\gamma^i = H^2 \cap H_0^1(0,\ell) \times H_0^1(0,\ell) \times L_*^2(0,\ell) \times \mathcal{V}_i$$

The domain of the infinitesimal generator $\mathcal{A}$ for $\gamma > 0$ is given by

$$D(\mathcal{A}_{\gamma,i}) = H^3 \cap H_0^1 \times H^2 \cap H_0^1 \times H_*^1 \times \mathcal{W}_i$$

and the corresponding norm we use to get a contraction semigroup is

$$\|\Phi\|_{\mathcal{H}_\gamma^1}^2 = \int_0^\ell \rho|v|^2 + \gamma|v_{xx}|^2 + \mu|u_x|^2 + \alpha|u_{xx}|^2 + c|\theta|^2 + \frac{\tau}{\kappa}|q|^2\,dx\,,$$

$$\|\Phi\|_{\mathcal{H}_\gamma^2}^2 = \int_0^\ell \rho|v|^2 + \gamma|v_{xx}|^2 + \mu|u_x|^2 + \alpha|u_{xx}|^2 + c|\theta|^2 + \int_0^\infty \kappa|\eta_x|^2\,ds\,dx$$

for any $\Phi^t = (u, v, \theta, q) \in \mathcal{H}_\gamma^1$ and $\Phi^t = (u, v, \theta, \eta) \in \mathcal{H}_\gamma^2$. It is not difficult to see that

$$\mathrm{Re}(\mathcal{A}_{\gamma,1}\Phi, \Phi)_{\mathcal{H}} = -\frac{1}{\kappa}\int_0^\ell |q|^2\,dx, \quad \mathrm{Re}(\mathcal{A}_{\gamma,2}\Phi, \Phi)_{\mathcal{H}} = -\int_0^\ell \int_0^\infty \kappa'(s)|\eta_x|^2\,ds\,dx.$$

Therefore the above inequalities imply

$$\int_0^\ell |q|^2\,dx = \kappa(\Phi, F)_{\mathcal{H}_\gamma^1}, \quad \int_0^\ell \int_0^\infty \kappa'(s)|\eta_x|^2\,ds\,dx = \kappa(\Phi, F)_{\mathcal{H}_\gamma^2}\,. \tag{2.9}$$

## 3. Asymptotic behaviour

In this section we prove the lack of exponential stability and the polynomial decay to zero.

**Theorem 3.1.** *The semigroups $S_1 = e^{\mathcal{A}_\gamma^c t}$ and $S_2 = e^{\mathcal{A}_\gamma^p t}$ are not exponentially stable for $\gamma \ge 0$. That is, for $\gamma \ge 0$ there exists sequences $\lambda_\nu \in \mathbb{R}$ such that*

$$\|(Ii\lambda_\nu - \mathcal{A}_{\gamma,1})^{-1}\| \ge C|\lambda_\nu|^2, \quad \|(Ii\lambda_\nu - \mathcal{A}_{\gamma,2})^{-1}\| \ge C|\lambda_\nu|^2$$

*Proof.* Let us take $\ell = \pi$, $f_1 = f_3 = f_4 = 0$ and $f_2 = \sin(\nu x)$ when $\gamma = 0$ and $f_2 = 1/\nu \sin(\nu x)$ to $\gamma > 0$. Because of the boundary conditions, we can assume that the solution is

$$u = A\sin(\nu x), \quad v = i\lambda A\sin(\nu x), \quad \theta = B\cos(\nu x), \quad q = C\sin(\nu x)$$

Note that $A = A_\nu$ to simplify, we omit this dependence. To find the solution we solve system (2.2)–(2.5) for $F = (f_1, \dots, f_4)$:

$$\begin{gathered} p(\lambda)A + \beta\nu B = m\,, \\ -i\lambda\beta\nu A + ic\lambda B + \nu C = 0\,, \\ -\kappa\nu B + (i\lambda\tau + 1)C = 0\,, \end{gathered}$$

where $p(\lambda) = -\lambda^2\rho + \gamma\nu^2\lambda^2 + \mu\nu^2 + \alpha\nu^4$ and $m = \rho$ or $m = \rho/\nu + \gamma\nu$ if $\gamma = 0$ or $\gamma > 0$ respectively. Solving for $A$ we obtain

$$A = \frac{[-\lambda^2 c\tau + i\lambda c + \kappa\nu^2]m}{\underbrace{p(\lambda)\nu^2 - (\tau\lambda^2 - i\lambda)(cp(\lambda) - \beta^2\nu^2)}_{:=\Delta}} \tag{3.1}$$

Now, for $\gamma = 0$ we take $\lambda$ such that $cp(\lambda) - \beta^2\nu^2 = \frac{\rho\beta^2}{c\alpha\tau}$, therefore we have

$$-c\rho\lambda^2 + c\mu\nu^2 + c\alpha\nu^4 - \beta^2\nu^2 = \frac{\rho\beta^2}{c\alpha\tau} \;\Rightarrow\; \lambda^2 = \frac{c\mu - \beta^2}{c\rho}\nu^2 + \frac{\alpha}{\rho}\nu^4 - \frac{\beta^2}{c^2\alpha\tau}.$$

Note that $\lambda \approx \sqrt{\frac{\alpha}{\rho}}\nu^2$ for large values of $\nu$. Substitution of $\lambda$ into the definition of $\Delta$ yields

$$\Delta = \frac{\beta^2}{c}\nu^4 + \frac{\rho\beta}{c^2\alpha\tau}\nu^2 - (\tau\lambda^2 + i\lambda)\frac{\rho\beta^2}{c\alpha\tau} \approx c_0\nu^2 + \frac{\sigma\beta^2}{c\alpha}\,.$$

Therefore we have that $A \approx \frac{\alpha c\tau\nu^4}{c_0\nu^2} = c_1\nu^2$, where $c_1$ does not depend on $\nu$. Therefore

$$\|\Phi\|^2_{\mathcal{H}^1_0} \geq \int_0^\pi \alpha|u_{xx}|^2\,dx = \alpha A^2\nu^4\int_0^\pi |\sin(\nu x)|^2\,dx = \frac{1}{2}\alpha c_1^2\nu^8 \approx \alpha_0|\lambda|^4 \to \infty.$$

For $\gamma > 0$ we choose $p(\lambda) = \xi\nu^2$ hence $\lambda$ is given by

$$(\rho + \gamma\nu^2)\lambda^2 = \mu\nu^2 + \alpha\nu^4 - \xi\nu^2 \;\Leftrightarrow\; \lambda^2 = \lambda_\nu^2 \approx \frac{\alpha}{\gamma}\nu^2$$

Taking $\xi$ such that $(\gamma - \alpha c\tau)\xi = -\tau\alpha\beta^2$ we have

$$\begin{aligned} \Delta &= p(\lambda)(\nu^2 - c\tau\lambda^2) + \tau\lambda^2\beta^2\nu^2 + i\lambda(cp(\lambda) - \beta^2\nu^2) \\ &\approx \xi(1 - \frac{\alpha c\tau}{\gamma})\nu^4 + \frac{\tau\alpha\beta^2}{\gamma}\nu^4 + i\lambda(c\xi - \beta^2)\nu^2 \approx c_2\nu^3. \end{aligned}$$

Substitution on (3.1) we obtain $A \approx \xi_0$, that is $A$ is asymptotically equals to a constant for $\nu$ large. Recalling the definition of $\lambda$ we obtain

$$\|\Phi\|^2_{\mathcal{H}^1_\gamma} \geq \int_0^\pi \alpha|u_{xx}|^2\,dx = \alpha A^2\nu^4\int_0^\pi |\sin(\nu x)|^2\,dx = \frac{1}{2}\alpha\xi_0^2\nu^4 \approx \alpha_0|\lambda|^4 \to \infty.$$

So the result follow in case of Cattaneo law. Let us consider the Gurtin-Pipkin law. We take $f_i$, $i = 1, \dots, 4$ as above and $\kappa(t) = Ke^{-\sigma t}$. Therefore, the solution is of the form

$$u = A\sin(\nu x), \quad v = i\lambda A\sin(\nu x), \quad \theta = B\cos(\nu x), \quad \eta = \varphi\cos(\nu x)$$

Solving (2.6) we obtain

$$\varphi = \frac{B}{i\lambda}(1-e^{-i\lambda s}) \;\Rightarrow\; \int_0^\infty \kappa(s)\eta_{xx}(t-s)\,ds = \frac{KB\nu^2}{\sigma(i\lambda+\sigma)}\cos(\nu x)$$

To find the exact solutions we solve the system

$$\begin{gathered} p(\lambda)A + \beta\nu B = m \\ -\beta\nu A + (c + \frac{K\nu^2}{\sigma(-\lambda^2+\sigma i\lambda)})B = 0 \end{gathered}$$

so we have

$$A = \frac{[c\sigma(-\lambda^2+\sigma i\lambda) + K\nu^2]m}{p(\lambda)K\nu^2 + (cp+\beta^2\nu^2)\sigma(-\lambda^2+\sigma i\lambda)}$$

Note that $A$ have the same estructure of (3.1), therefore using the same arguments we obtain that

$$\|\Phi\|^2_{\mathcal{H}^2_\gamma} \geq \alpha_0|\lambda_\nu|^4 \to \infty.$$

Since $\Phi = (I\lambda_\nu - \mathcal{A}_{\gamma,i})^{-1}F_\nu$, them item (2.8) of Theorem 2.1 implies the result. $\square$

Now we are able to show the polynomial rate of decay

**Theorem 3.2.** *The optimal rate of decay of the semigroup $S_i(t) = e^{\mathcal{A}_{\gamma,i}t}$ is given by*

$$\|e^{\mathcal{A}_{\gamma,i}t}\Phi_0\| \leq \frac{C}{\sqrt{t}}\|\mathcal{A}_{\gamma,i}\Phi_0\|_{\mathcal{H}^i_\gamma}, \quad \gamma \geq 0, \quad i = 1,2.$$

*Proof.* Here we use relation (2.8) of Theorem 2.1. Since $D(\mathcal{A}_{\gamma,i})$ has compact embedding over the phase space $\mathcal{H}^i_\gamma$, then the corresponding resolvent operators are compact. It is not difficult to see that $0 \in \rho(\mathcal{A}_{\gamma,i})$. Therefore to show that $i\mathbb{R} \subset \rho(\mathcal{A}_{\gamma,i})$ it is enough to prove that there is no imaginary eigenvalues. Suppose that there exists $W \neq 0$ such that $i\lambda W - \mathcal{A}_{\gamma,i}W = 0$. Using (2.9) we conclude that flux $q = 0$, from equations (2.5) or (2.6) we conclude that $\theta = 0$. Using that $q = 0$ and $\theta = 0$ in (2.4) we conclude that $v = 0$, therefore $W = 0$. This is the contradiction that implies that $i\mathbb{R} \subset \rho(\mathcal{A}_{\gamma,i})$. Next we prove that the resolvent operator is bounded. Multiplying (2.4) by $\int_0^x \overline{\theta}\,ds$ we obtain

$$\begin{aligned} &\kappa\int_0^\ell |\theta|^2\,dx \\ &= -\frac{\tau}{c}\int_0^\ell q\int_0^x c\overline{i\lambda\theta}\,ds\,dx + \int_0^\ell q\int_0^x \overline{\theta}\,ds\,dx - \int_0^\ell f_4\int_0^x \overline{\theta}\,ds\,dx \\ &= -\frac{\tau}{c}\int_0^\ell q\overline{(q-\beta v)}\,dx + \int_0^\ell q\int_0^x \overline{\theta}\,ds\,dx + R \\ &\leq C\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|v\|\|q\|, \end{aligned} \tag{3.2}$$

where $R$ is such that $|R| \leq C\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$. Using (2.6) we conclude that

$$\int_0^\ell |\theta_x|^2\,dx \leq c(1+|\lambda|^2)\int_0^\ell |q|^2\,dx + c\int_0^\ell |f_4|^2\,dx.$$

Therefore

$$\int_0^\ell |\theta_x|^2\,dx \leq c|\lambda|^2\|\Phi\|\|F\|| + \|F\|^2. \tag{3.3}$$

Multiplying (1.7) by $\int_0^x \overline{v}\,dx$ we have

$$\begin{aligned}\beta\int_0^\ell |v|^2\,dx &= c\int_0^\ell c\omega_{xx}\int_0^x c\overline{i\lambda v}\,ds\,dx + \int_0^\ell q\overline{v}\,dx - \int_0^\ell f_3\int_0^x \overline{v}\,ds\,dx\\ &= c\mu\int_0^\ell \theta\overline{u_x}\,dx - c\alpha\int_0^\ell \theta_x\overline{u_{xx}}\,dx + c\beta\int_0^\ell |\theta|^2\,dx + R\,,\end{aligned}$$

where $\omega$ is the solution of $\omega_{xx} = \theta$, $\omega_x(0) = \omega_x(\ell) = 0$. Using (3.3) we obtain

$$\beta\int_0^\ell |v|^2\,dx \le C_\epsilon|\lambda|^2\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \epsilon\int_0^\ell |u_{xx}|^2\,dx \tag{3.4}$$

for $\lambda$ large. Multiplying (1.4) by $\overline{u}$ we obtain

$$\int_0^\ell \alpha|u_{xx}|^2\,dx + \int_0^\ell \mu|u_x|^2\,dx = \int_0^\ell \rho|v|^2\,dx + \gamma\int_0^\ell \rho|v_x|^2\,dx - \int_0^\ell \beta\theta\overline{u_x}\,dx + R\,.$$

Therefore, using (3.4) we obtain (with $\gamma = 0$)

$$\int_0^\ell \alpha|u_{xx}|^2\,dx + \int_0^\ell \mu|u_x|^2\,dx \le C\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{3.5}$$

Finally, summing inequalities (3.2), (3.4), (3.5) we obtain

$$\|\Phi\|_{\mathcal{H}} \le C|\lambda|^2\|F\|_{\mathcal{H}}.$$

So our conclusion follows for Cattaneo's law with $\gamma = 0$. Now let us consider $\gamma > 0$. Multiplying (2.4) by $\overline{q_x}$ we have

$$\int_0^\ell |q_x|^2\,dx = i\lambda c\int_0^\ell \theta\overline{q_x}\,dx - \beta\int_0^\ell v_{xx}\overline{q}\,dx + \int_0^\ell f_3\overline{q_x}\,dx\,.$$

Using (1.4) we obtain

$$\int_0^\ell |q_x|^2\,dx \le C|\lambda|^2\int_0^\ell |\theta|^2\,dx + C_\epsilon|\lambda|^2\|U\|\|F\| + \epsilon\int_0^\ell |u_{xx}|^2\,dx + C\|F\|^2.$$

On the other hand, multiplying (1.7) by $\overline{v_x}$ we have

$$\beta\int_0^\ell |v_x|^2\,dx = i\lambda c\int_0^\ell \theta\overline{v_x}\,dx + \int_0^\ell q_x\overline{v_x}\,dx + R\,.$$

Therefore

$$\beta\int_0^\ell |v_x|^2\,dx \le C|\lambda|^2\int_0^\ell |\theta|^2\,dx + C\int_0^\ell |q_x|^2\,dx + R\,.$$

Using (2.3) with $\gamma > 0$ we obtain

$$|\lambda|\|v\|_{L^2} \le C\|u_{xx}\| + C\|\theta\|_{-1} + C\|F\| \;\Rightarrow\; |\lambda|\|v\|_{L^2} \le C\|U\| + C\|F\|\,.$$

The above inequality and (3.2) imply

$$C|\lambda|^2\int_0^\ell |\theta|^2\,dx \le C_\epsilon|\lambda|^2\|U\|\|F\| + C\|F\|^2 + \epsilon\|U\|^2\,.$$

So we have

$$\|U\|^2 \le C_\epsilon|\lambda|^2\|U\|\|F\| + C\|F\|^2 + \epsilon\|U\|^2\,.$$

Therefore our conclusion follows. Finally, for Gurtin-Pipkin's model, inequality (2.9) implies

$$\int_0^\ell |q|^2\,dx \le \int_0^\infty \kappa\,ds\int_0^\ell \kappa|\eta_x|^2\,dx \le C\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\,. \tag{3.6}$$

Multiplying (2.4) by $\int_0^\infty \kappa\overline{\eta}\, ds$ and using (2.5), we obtain

$$\int_0^\infty \kappa\, ds \int_0^\ell |\theta|^2\, dx \le C\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|\eta\|_{\mathcal{M}_1}\|v\|\,.$$

Differentiating (2.6) with respect to $x$ and multiplying by $\kappa\theta_x$ and using (2.9) we obtain

$$\int_0^\infty \kappa\, ds \int_0^\ell |\theta_x|^2\, dx \le C|\lambda|^2\|\Phi\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}\,.$$

Therefore, to estimate $u$ and $v$ we follows same above reasoning, so our conclusion follows. Finally, the optimality follows from Theorem 2.1 and Theorem 3.1. In fact, let us suppose that the rate of decay can be improved, for example as $t^{-1/(2-\epsilon)}$. Then relation (2.8) of Theorem 2.1 implies that

$$\|(i\lambda I - \mathcal{A}_{\gamma,i})^{-1}\| \le C|\lambda|^{2-\epsilon}, \quad \forall \lambda \in \mathbb{R}.$$

This is a contradiction to Theorem 3.1. Hence the rate can not be improved. The proof is complete. □

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