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EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR A CLASS OF ASYMPTOTICALLY PERIODIC SCHRÖDINGER-POISSON SYSTEMS

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Abstract. In this article, by using variational method, we study the existence of a positive ground state solution for the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u+V(x)u+K(x)\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\ -\Delta\phi=K(x)u^2, \quad x\in\mathbb{R}^3, \end{gathered}$$

where $V(x), K(x)$ and $f(x,u)$ are asymptotically periodic functions in x at infinity.

1. Introduction and statement of results

For past decades, much attention has been paid to the nonlinear Schrödinger-Poisson system

$$\begin{gathered} i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi+U(x)\Psi+\phi(x)\Psi-|\Psi|^{q-1}\Psi, \quad x\in\mathbb{R}^3,\ t\in\mathbb{R}\\ -\Delta\phi=|\Psi|^2, \quad x\in\mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $\hbar$ is the Planck constant. Equation (1.1) derived from quantum mechanics. For this equation, the existence of stationary wave solutions is often sought, that is, the following form of solution

$$\Psi(x,t)=e^{it}u(x), x\in\mathbb{R}^3, \quad t\in\mathbb{R}.$$

Therefore, the existence of the standing wave solution of the equation (1.1) is equivalent to finding the solution of the following system ($m=\frac{1}{2}$, $\hbar=1$, $V(x)=U(x)+1$)

$$\begin{gathered} -\Delta u+V(x)u+\phi u=|u|^{q-1}u, \quad x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, \quad x\in\mathbb{R}^3. \end{gathered} \tag{1.2}$$

As far as we know, the first result on Schrödinger-Poisson system was obtained in [6]. Thereafter, using the variational method, there is a series of work to discuss the existence, non existence, radially symmetric solutions, non-radially symmetric

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solutions and ground state to Schrödinger-Poisson system (1.2) [1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 32, 33, 37, 40, 41, 44, 45, 47, 48, 49].

To the best of our knowledge, Azzollini and Pomponio [5] firstly obtained the ground state solution to the Schrödinger-Poisson system (1.2). The conclusion they got was that if V is a positive constant and $2 < q < 5$, or V is non-constant, possibly unbounded below and $3 < q < 5$, system (1.2) has a ground state solution.

Alves, Souto and Soares [1] studied Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f(u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where V is bounded locally Hölder continuous and satisfies:

(1) $V(x) \geq \alpha > 0$, $x \in \mathbb{R}^3$;
(2) $\lim_{|x|\to\infty} |V(x) - V_0(x)| = 0$, where V_0 satisfy $V_0(x) = V_0(x+y)$ for all $x \in \mathbb{R}^3$ and all $y \in \mathbb{Z}^3$;
(3) $V(x) \leq V_0(x)$ for all $x \in \mathbb{R}^3$, and there exists an open set $\Omega \subset \mathbb{R}^3$ with $m(\Omega) > 0$ such that $V(x) < V_0(x)$ for all $x \in \Omega$.

Alves et al. studied the ground state solutions to system (1.3) in case the asymptotically periodic condition under conditions (1)–(3).

In case $p \in (3, 5)$, Cerami and Vaira [9] studied the existence of positive solutions for the following non-autonomous Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + K(x)\phi(x)u = a(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.4}$$

where a, K are nonnegative functions such that $\lim_{|x|\to\infty} a(x) = a_\infty > 0$, and $\lim_{|x|\to\infty} K(x) = 0$.

Zhang, Xu and Zhang [48] considered existence of positive ground state solution for the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = f(x, u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.5}$$

In their paper, V and K satisfy:

- $V, K \in L^\infty(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} V > 0$, $\inf_{\mathbb{R}^3} K > 0$, and $V - V_p, K - K_p \in \mathcal{F}$, where V_p and K_p satisfy $V_p(x+z) = V_p(x)$, $K_p(x+z) = K_p(x)$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^3$, here $\mathcal{F} = \{g \in L^\infty(\mathbb{R}^3) : \forall \varepsilon > 0\}$, the set $\{x \in \mathbb{R}^3 : |g(x)| \geq \varepsilon\}$ has finite Lebesgue measure$\}$.

On the other hand, when $K = 0$ the Schrödinger-Poisson system (1.5) becomes the standard Schrödinger equation (replace $\mathbb{R}^3$ with $\mathbb{R}^N$)

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.6}$$

The Schrödinger equation (1.6) has been widely investigated by many authors, see [2, 7, 13, 14, 18, 23, 29, 24, 34, 35, 36, 42, 43, 46] and reference their. Especially, in [23, 29, 34, 42, 43], they studied the nontrivial solution and ground state solution for problem (1.6) in which V or f satisfy the asymptotically periodic condition. In the other context about asymptotically periodic condition, we refer the reader to [22, 25, 26, 39] and reference their.

Motivated by above results, in this paper we study positive ground state solutions to system (1.5) under reformative condition about asymptotically periodic case of V, K and f at infinity.

To state our main results, we assume that:

(A1) $V,\ V_p \in L^\infty(\mathbb{R}^3)$, $0 \le V(x) \le V_p(x)$ and $V(x) - V_p(x) \in A_0$, where $A_0 := \{k(x) :\ \text{for any } \varepsilon > 0,\ m\{x \in B_1(y) : |k(x)| \ge \varepsilon\} \to 0 \text{ as } |y| \to \infty\}$ and V_p satisfies $V_0 := \inf_{x\in\mathbb{R}^3} V_p > 0$ and $V_p(x+z) = V_p(x)$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^3$. $K,\ K_p \in L^\infty(\mathbb{R}^3)$, $0 < K(x) \le K_p(x)$, $K(x) - K_p(x) \in A_0$ and K_p satisfies $K_0 := \inf_{x\in\mathbb{R}^3} K_p > 0$ and $K_p(x+z) = K_p(x)$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^3$;

and $f \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R})$ satisfies

(A2) $\lim_{s\to 0^+} \frac{f(x,s)}{s} = 0$ uniformly for $x \in \mathbb{R}^3$,

(A3) $\lim_{s\to +\infty} \frac{f(x,s)}{s^5} = 0$ uniformly for $x \in \mathbb{R}^3$,

(A4) $\frac{f(x,s)}{s^3}$ is nondecreasing on $(0, +\infty)$,

(A5) there exists $f_p \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R})$ such that
  (i) $f(x,s) \ge f_p(x,s)$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}^+$ and $f(x,s) - f_p(x,s) \in A$, where $A := \{h(x,s) : \text{for any } \varepsilon > 0,\ m\{x \in B_1(y) : |h(x,s)| \ge \varepsilon\} \to 0 \text{as } |y| \to \infty \text{ uniformly for } |s| \text{bounded}\}$,
  (ii) $f_p(x+z,s) = f_p(x,s)$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}^+$ and $z \in \mathbb{Z}^3$,
  (iii) $\frac{f_p(x,s)}{s^3}$ is nondecreasing on $(0, +\infty)$,
  (iv) $\lim_{s\to+\infty} \frac{F_p(x,s)}{s^4} = +\infty$ uniformly for $x \in \mathbb{R}^3$, where $F_p(x,s) = \int_0^s f_p(x,t)dt$.

Remark 1.1. (i) Functional sets A_0 in (A1) and A in (A5) were introduced by [29] in which Liu, Liao and Tang studied positive ground state solution to Schrödinger equation (1.6).

(ii) Since $\mathcal{F} \subset A_0$, our assumptions on V and K are weaker than in [48]. Furthermore, in our paper $V(x) \ge 0$ but in [48] they assumed $V(x) > 0$.

(iii) In [48], to obtain the positive ground state to system (1.5), they firstly consider the periodic system

$$\begin{gathered} -\Delta u + V_p(x)u + K_p(x)\phi u = f_p(x,u) \quad x \in \mathbb{R}^3, \\ -\Delta\phi = K_p(x)u^2 \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.7}$$

Then a solution of system (1.5) was obtained by applying inequality between the energy of periodic system (1.7) and that of system (1.5). In this paper, we do not using methods that of [48] and we proof the Theorem 1.2 directly.

Since we are looking for a positive solution, we may assume that $f(x,s) = f_p(x,s) = 0$ for all $(x,s) \in (\mathbb{R}^3 \times \mathbb{R}^-)$. The next theorems are the main results of the present paper.

Theorem 1.2. *Suppose that* (A1)–(A5) *are satisfied. Then system* (1.5) *has a positive ground state solution.*

Theorem 1.3. *Suppose that* $V(x) \equiv V_p(x)$, $K(x) \equiv K_p(x)$ *satisfy* (A1), *and* $f(x,s) \equiv f_p(x,s)$ *satisfies* (A2)–(A5). *Then system* (1.5) *has a positive ground state solution.*

2. Variational framework and preliminary results

The letter C and C_i will be repeatedly used to denote various positive constants whose exact values are irrelevant. $B_R(z)$ denotes the open ball centered at z with radius R. We denote the standard norm of L^p by $|u|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{1/p}$ and $|u|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |u|$.

The Sobolev space $H^1(\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_H^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx.$$

The space $D^{1,2}(\mathbb{R}^3)$ is endowed with the standard norm

$$\|u\|_{D^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Let $E := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} V(x)u^2 dx < \infty\}$ be the Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx.$$

Lemma 2.1. [29] *Suppose* (A1) *holds. Then there exists two positive constants* C_1 *and* C_2 *such that* $C_1\|u\|_H^2 \le \|u\| \le C_2\|u\|_H^2$ *for all* $u \in E$. *Moreover,* $E \hookrightarrow L^p(\mathbb{R}^3)$ *for any* $p \in [2, 6]$ *is continuous.*

System (1.5) can be transformed into a Schrödinger equation with a nonlocal term. In fact, for all $u \in E$ (then $u \in H^1(\mathbb{R}^3)$), considering the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v dx.$$

According to the Hölder inequality and lemma (2.1), one has that

$$|L_u(v)| \le |K|_\infty |u|_{12/5}^2 |v|_6 \le C\|u\|^2 \|v\|_{D^{1,2}}. \tag{2.1}$$

So, by the Lax-Milgram theorem exists an unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\, dx = (\phi_u, v)_{D^{1,2}} = L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v\, dx,$$

for any $v \in D^{1,2}(\mathbb{R}^3)$ and $\|\phi_u\|_{D^{1,2}} \le C\|u\|^2$. Namely, ϕ_u is the unique solution of

$$-\Delta\phi = K(x)u^2,\ x \in \mathbb{R}^3.$$

Moreover, ϕ_u can be expressed as

$$\phi_u = C\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy.$$

Substituting ϕ_u into the system (1.5), we obtain

$$-\Delta u + V(x)u + K(x)\phi_u u = f(x, u), \quad x \in \mathbb{R}^3. \tag{2.2}$$

By (2.1), we get

$$\Big|\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx\Big| \le C\|u\|^4. \tag{2.3}$$

So the energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ corresponding to (2.2) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx,$$

where $F(x, s) = \int_0^s f(x, t)dt$.

Moreover, under our condition, I belongs to C^1, so the Fréchet derivative of I is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)dx + \int_{\mathbb{R}^3} K(x)\phi_u uv dx - \int_{\mathbb{R}^3} f(x,u)v dx$$

and $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of system (1.5) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I and $\phi = \phi_u$.

For all $u \in E$, let $\tilde{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ is unique solution of the following equation

$$-\Delta\phi = K_p(x)u^2, \ x \in \mathbb{R}^3.$$

Moreover, $\widetilde{\phi}_u$ can be expressed as

$$\widetilde{\phi}_u = C\int_{\mathbb{R}^3} \frac{K_p(y)u^2(y)}{|x-y|}dy.$$

Let

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_p(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_u u^2 dx - \int_{\mathbb{R}^3} F_p(x,u)dx,$$

where $F_p(x,s) = \int_0^s f_p(x,t)dt$. Then I_p is the energy functional corresponding to the equation

$$-\Delta u + V_p(x)u + K_p(x)\widetilde{\phi}_u u = f_p(x,u), \quad x \in \mathbb{R}^3. \tag{2.4}$$

It is easy to see that $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of periodic system (1.7) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I_p and $\phi = \widetilde{\phi}_u$.

Lemma 2.2. *Suppose* (A1) *holds. Then*

$$\int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_{u(\cdot+z)}u^2(\cdot+z)dx = \int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_u u^2 dx, \quad \forall z \in \mathbb{Z}^3, u \in E.$$

Lemma 2.3. *Suppose that* (A2), (A4), (A5) *hold. Then*

(i) $\frac{1}{4}f(x,s)s \geq F(x,s) \geq 0$ *for all* $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$,
(ii) $\frac{1}{4}f_p(x,s)s \geq F_p(x,s) \geq 0$ *for all* $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.

The proof of the above lemma is similar to that in [31], so we omitted here.

Lemma 2.4. *Operator I' is weakly sequentially continuous. Namely if $u_n \rightharpoonup u$ in E, $I'(u_n) \rightharpoonup I'(u)$ in E^{-1}.*

The proof of the above lemma is similar to that of in [48], so we omitted here.

Lemma 2.5 ([29])**.** *Suppose that* (A2), (A3), (A5)(i) *hold. Assume that $\{u_n\}$ is bounded in E and $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^3)$, for any $s \in [2,6)$. Then up to a subsequence, one has*

$$\int_{\mathbb{R}^3} (F(x,u_n) - F_p(x,u_n))dx = o_n(1).$$

Lemma 2.6 ([29])**.** *Suppose that* (A1), (A2), (A3) (A5)(i) *hold. Assume that $\{u_n\}$ is bounded in E and $|z_n| \to \infty$. Then any $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has*

$$\int_{\mathbb{R}^3} (V_p(x) - V(x))u_n\varphi(\cdot - z_n)dx = o_n(1),$$

$$\int_{\mathbb{R}^3} (f(x,u_n) - f_p(x,u_n))\varphi(\cdot - z_n)dx = o_n(1).$$

Lemma 2.7. *Suppose that* (A1), (A2), (A3), (A5)(i) *hold. Assume that* $u_n \rightharpoonup 0$ *in* E. *Then up to a subsequence, one has*

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi(\cdot - z_n) - K_p(x)\widetilde{\phi}_{u_n} u_n \varphi(\cdot - z_n))dx = o_n(1),$$

where $|z_n| \to \infty$ *and* $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. Set $h(x) := K(x) - K_p(x)$. By (A1), we have $h(x) \in A_0$. Then for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$m\{x \in B_1(y) : |h(x)| \geq \varepsilon\} < \varepsilon, \quad \text{for any } |y| \geq R_\varepsilon.$$

We cover $\mathbb{R}^3$ by balls $B_1(y_i)$, $i \in \mathbb{N}$. In such a way that each point of $\mathbb{R}^3$ is contained in at most $N+1$ balls. Without any loss of generality, we suppose that $|y_i| < R_\varepsilon,\ i = 1, 2, \dots, n_\varepsilon$ and $|y_i| \geq R_\varepsilon, i = n_\varepsilon + 1, n_\varepsilon + 2, n_\varepsilon + 3, \dots, +\infty$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi(\cdot - z_n) - K_p(x)\widetilde{\phi}_{u_n} u_n \varphi(\cdot - z_n))dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_p(y)u_n(y)\varphi(y - z_n)}{|x-y|} dy h(x) u_n^2(x) dx \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_p(y)u_n^2(y)}{|x-y|} dy h(x) u_n(x)\varphi(x - z_n) dx \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{h(y)u_n^2(y)}{|x-y|} dy h(x) u_n(x)\varphi(x - z_n) dx \\
&:= E_1 + E_2 + E_3
\end{aligned}$$

As in [48], we define

$$\begin{aligned}
H(x) &:= \int_{\mathbb{R}^3} \frac{K_p(y)u_n(y)\varphi(y - z_n)}{|x-y|} dy \\
&= \int_{\{y:|x-y|\leq 1\}} \frac{K_p(y)u_n(y)\varphi(y - z_n)}{|x-y|} dy \\
&\quad + \int_{\{y:|x-y|>1\}} \frac{K_p(y)u_n(y)\varphi(y - z_n)}{|x-y|} dy.
\end{aligned}$$

By the Hölder inequality and the Sobolev embedding, we have

$$\begin{aligned}
|H(x)| &\leq |K_p|_\infty |u_n|_3 |\varphi|_6 \Big(\int_{\{y:|x-y|\leq 1\}} \frac{1}{|x-y|^2} dy\Big)^{1/2} \\
&\quad + |K_p|_\infty |u_n|_2 |\varphi|_4 \Big(\int_{\{y:|x-y|>1\}} \frac{1}{|x-y|^4} dy\Big)^{1/4} \\
&\leq C\Big(\int_{\{z:|z|\leq 1\}} \frac{1}{|z|^2} dz\Big)^{1/2} + C\Big(\int_{\{z:|z|>1\}} \frac{1}{|z|^4} dz\Big)^{1/4}.
\end{aligned}$$

So, $\sup_{x\in\mathbb{R}^3} |H(x)| < \infty$. Then, we obtain

$$\begin{aligned}
E_1 &= \int_{\mathbb{R}^3} H(x)h(x)u_n^2(x)dx \\
&\leq \int_{\{x:|h(x)|\geq\varepsilon\}} |H(x)h(x)u_n^2(x)|dx + \int_{\{x:|h(x)|<\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&:= Q_1 + Q_2
\end{aligned}$$

$$\begin{aligned}
Q_1 &= \int_{\{x:|h(x)|\geq\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&= \int_{\{x:|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\quad + \int_{\{x:|h(x)|\geq\varepsilon,|x|\leq R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\quad + 2\sup_{x\in\mathbb{R}^3}|H(x)||K_p|_\infty \int_{B_{R_\varepsilon+1}} |u_n(x)|^2dx \\
&:= Q_{11}+Q_{12}
\end{aligned}$$

$$\begin{aligned}
Q_{11} &= \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq 2\sup_{x\in\mathbb{R}^3}|H(x)||K_p|_\infty \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |u_n^2(x)|dx \\
&\leq C\sum_{n_\varepsilon+1}^{\infty} \Big(m\{x\in B_1(y):|h(x)|\geq\varepsilon\}\Big)^{2/3} \\
&\quad \times \Big(\int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |u_n^6(x)|dx\Big)^{1/3} \\
&\leq C_1\varepsilon^{2/3}\sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} (|\nabla u_n|^2+u_n^2)dx \\
&\leq C_1(N+1)\varepsilon^{2/3}\int_{\mathbb{R}^3}(|\nabla u_n|^2+u_n^2)dx \leq C_2\varepsilon^{2/3}.
\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain $Q_{11}\to 0$.

Since $u_n \rightharpoonup 0$, one has that $Q_{12}\to 0$. So, $Q_1=Q_{11}+Q_{12}\to 0$.

$$\begin{aligned}
Q_2 &= \int_{\{x:|h(x)|<\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq \varepsilon\sup_{x\in\mathbb{R}^3}|H(x)|\int_{\mathbb{R}^3}|u_n^2(x)|dx \leq C\varepsilon.
\end{aligned}$$

Let $\varepsilon\to 0$, we have $Q_2\to 0$. Therefore, from the above fact we get that $E_1\to 0$. In the same way, we can prove $E_2\to 0$ and $E_3\to 0$. □

We define $\mathcal{N}:=\{u\in E\backslash\{0\}:(I'(u),u)=0\}$. Then $\mathcal{N}$ is a Nehari type associate to I, and set $c:=inf_{u\in\mathcal{N}}I$. Let $F:=\{u\in E:u^+\neq 0\}$, where $u^\pm=\max\{\pm u,0\}$. In fact

$$\mathcal{N}=\{u\in F:(I'(u),u)=0\}.$$

Lemma 2.8. *Suppose that* (A1)–(A5) *hold. For any $u\in F$, there is a unique $t_u>0$ such that $t_u u\in\mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t\geq 0$ is achieved.*

Proof. Define $g(t) := I(tu)$, $t \geq 0$. Using (A2), (A3) and (A5), we can prove that $g(0) = 0$, $g(t) > 0$ for t small and $g(t) < 0$ for t large. In fact, by (A2) and (A3), for all $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$|f(x,s)| \leq \varepsilon|s| + C_\varepsilon|s|^5, \quad |F(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{6}|s|^6, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} g(t) &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,tu)dx \\ &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,tu)dx \\ &\geq \frac{t^2}{2}\|u\|^2 - \varepsilon t^2\int_{\mathbb{R}^3} |u|^2 dx - C_\varepsilon t^6 \int_{\mathbb{R}^3} |u|^6 dx \\ &\geq \frac{t^2}{2}\|u\|^2 - C\varepsilon t^2\|u\|^2 - C_\varepsilon t^6\|u\|^6. \end{aligned}$$

Hence, $g(0) = 0$, $g(t) > 0$ for t small.

Set $\Omega := \{x \in \mathbb{R}^3 : u(x) > 0\}$, by using Fatou lemma and (A5), we have

$$\liminf_{t\to+\infty} \int_\Omega \frac{F(x,tu)}{(tu)^4} u^4 dx \geq \liminf_{t\to+\infty} \int_\Omega \frac{F_p(x,tu)}{(tu)^4} u^4 dx = +\infty.$$

Hence

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{g(t)}{t^4} \\ &= \limsup_{t\to+\infty} \frac{1}{2t^2}\|u\|^4 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \liminf_{t\to+\infty}\int_{\mathbb{R}^3} \frac{F(x,tu)}{t^4} dx \\ &= \limsup_{t\to+\infty} \frac{1}{2t^2}\|u\|^4 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \liminf_{t\to+\infty}\int_\Omega \frac{F(x,tu)}{(tu)^4} u^4 dx = -\infty, \end{aligned}$$

which deduces $g(t) \to -\infty$ as $t \to +\infty$. Therefore, there exists a t_u such that $I(t_u u) = \max_{t>0} I(tu)$ and $t_u u \in \mathcal{N}$. Suppose that there exist $t'_u > t_u > 0$ such that $t'_u u$, $t_u u \in \mathcal{N}$. Then, We have

$$\frac{1}{(t'_u)^2}\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = \int_{\mathbb{R}^3} \frac{f(x,t'_u u)u^4}{(t'_u u)^3} dx$$

and this identity is also true if t'_u is replaced by t_u. Therefore,

$$\Big(\frac{1}{(t'_u)^2} - \frac{1}{(t_u)^2}\Big)\|u\|^2 = \int_{\mathbb{R}^3} (\frac{f(x,t'_u u)}{(t'_u u)^3} - \frac{f(x,t_u u)}{(t_u u)^3})u^4 dx,$$

which is absurd in view of (A4) and $t'_u > t_u > 0$. □

Remark 2.9. As in [36, 46], we have

$$c = \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in F}\max_{t>0} I(tu) = \inf_{\gamma(t)\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0$$

where

$$\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Lemma 2.10. *Suppose that* (A1), (A2)–(A5) *hold. Then there exists a nonnegative and bounded sequence* $\{u_n\} \in E$ *such that*

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{E^{-1}} \to 0.$$

Proof. From the proof of Lemma 2.8, it is easy to see that I satisfies the mountain pass geometry. By [38], there exists an $\{u_n\}$ such that $I(u_n) \to c$ and $(1+\|u_n\|)\|I'(u_n)\|_{E^{-1}} \to 0$. By Lemma (2.3), we have

$$\begin{aligned} c &= I(u) - \frac{1}{4}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f(x,u_n)u_n - F(x,u_n))dx + o_n(1) \\ &\geq \frac{1}{4}\|u_n\|^2 + o_n(1). \end{aligned}$$

Therefore, $\{u_n\}$ is bounded. Moreover, we have

$$\langle I'(u_n), u_n^-\rangle = \|u_n^-\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^-)^2 dx = o_n(1).$$

Then $\|u_n^-\|^2 = o(1)$ and $\int_{\mathbb{R}^3} K(x)\phi_{u_n^-}(u_n^-)^2 = o(1)$. Therefore, we can infer that $I(u_n^+) \to c$ and $\|I'(u_n^+)\|_{E^{-1}} \to 0$. Hence, we may always assume that $\{u_n\}$ is nonnegative and the prove is fished. □

Lemma 2.11. *Suppose that* (A1)–(A5) *hold. If $u \in \mathcal{N}$ and $I(u) = c$, then u is a solution of system* (1.5).

Proof. The proof is similar to that of [29, 30]. Suppose by contradiction, that u is not a solution of system (1.5). Hence, there exists $\varphi \in E$ such that

$$\langle I'(u), \varphi\rangle < -1.$$

Choose $\varepsilon \in (0,1)$ small enough such that for all $|t-1| \leq \varepsilon$ and $|\sigma| \leq \varepsilon$,

$$\langle I'(tu + \sigma\varphi), \varphi\rangle \leq -\frac{1}{2}.$$

Let $\zeta(t) \in [0,1]$ satisfies $\zeta(t) = 1$ for $|t-1| \leq \frac{\varepsilon}{2}$ and $\zeta(t) = 0$ for $|t-1| \geq \varepsilon$. for all $t > 0$, let $\gamma(t)$ be a curve such that $\gamma(t) = tu$ for $|t-1| \geq \varepsilon$ and $\gamma(t) = tu + \varepsilon\zeta(t)\varphi$ for $|t-1| < \varepsilon$. Obviously, $\gamma(t)$ is a continuous curve, furthermore, $\|\gamma(t)\| > 0$ for $|t-1| < \varepsilon$ in which ε small enough. Next we will prove $I(\gamma(t)) < c$, for all $t > 0$. In fact, if $|t-1| \geq \varepsilon$, $I(\gamma(t)) = I(tu) < I(u) = c$. If $|t-1| < \varepsilon$, for all $\sigma \in [0,\varepsilon]$, we define $A : \sigma \mapsto I(tu + \sigma\zeta(t)\varphi)$. Obviously, $A \in C^1$. By the mean value therm, there exists $\overline{\sigma} \in (0,\varepsilon)$ such that

$$I(tu + \varepsilon\zeta(t)\varphi) = I(tu) + \langle I'(tu + \overline{\sigma}\zeta(t)\varphi), \varepsilon\zeta(t)\varphi\rangle \leq I(tu) - \frac{\varepsilon}{2}\zeta(t) < c.$$

Set $\nu(u) := \langle I'(u), u\rangle$, then $\nu(\gamma(1-\varepsilon)) = \nu((1-\varepsilon)u) > 0$ and $\nu(\gamma(1+\varepsilon)) = \nu((1+\varepsilon)u) < 0$. According to the continuity of $t \to \nu(\gamma(t))$, there exists $t' \in (1-\varepsilon, 1+\varepsilon)$ such that $\nu(\gamma(t')) = 0$. Thus $\gamma(t') \in \mathcal{N}$ and $I(\gamma(t')) < c$, which is a contradiction. □

Define

$$\mathcal{N}_p = \{u \in F \setminus \{0\} : \langle I_p'(u), u\rangle = 0\} \text{ and } c_p = \inf_{u \in \mathcal{N}_p}.$$

In fact, $c_p = \inf_{u\in F} \max_{t>0} I_p(tu)$.

Remark 2.12. For any $u \in F$, by Lemma 2.8, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$ and then $I(t_u u) \geq c$. Using $V(x) \leq V_p(x)$ and $F(x,s) \geq F_p(x,s)$, we have $c \leq I(t_u u) \leq I_p(t_u u) \leq \max_{t>0} I_p(tu)$. Then we obtain $c \leq c_p$.

3. Proof of main results

Proof. According to Lemma 2.10, there exist a nonnegative and bounded sequence $\{u_n\} \in E$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{E^{-1}} \to 0$. Then there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. By lemma 2.4, we have that

$$0 = \langle I'(u_n), v\rangle + o_n(1) = \langle I'(u), v\rangle, \ \forall v \in E,$$

that is u is a solution of system (1.5). We next distinguish the following two case to prove system (1.5) have a nonnegative ground state solution.

Case 1: $u \neq 0$. Then $I(u) \geq c$. By Lemma 2.3 and the Fatou lemma, we obtain

$$\begin{aligned} c &= \liminf_{n\to\infty}(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle) \\ &= \liminf_{n\to\infty} \Big(\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4}f(x,u_n)u_n - F(x,u_n))dx\Big) \\ &\geq \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4}f(x,u)u - F(x,u))dx \\ &= I(u) - \frac{1}{4}\langle I'(u), u\rangle = I(u). \end{aligned}$$

Therefore, $I(u) = c$ and $I'(u) = 0$.

Case 2: $u = 0$. Let

$$\beta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_1(z)} u_n^2 dx.$$

If $\beta = 0$, by using the Lions lemma [27, 28], we have $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$. From the conditions of (A2) and (A3), for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $\frac{1}{2}f(x,u)u - F(x,u) \leq \varepsilon(|u|^2 + |u|^6) + C_\varepsilon|u|^\alpha$ for any $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$ and $\alpha \in (2,6)$. Let ε small enough, we have that

$$\begin{aligned} c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= -\frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx + \int_{\mathbb{R}^3} (\frac{1}{2}f(x,u_n)u_n - F(x,u_n))dx + o_n(1) \\ &\leq -\frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx + \int_{\mathbb{R}^3} (\varepsilon(|u_n|^2 + |u_n|^6) + C_\varepsilon|u_n|^\alpha)dx + o_n(1) \leq 0, \end{aligned}$$

which is a contradiction with $c > 0$. So $\beta > 0$. Up to a subsequence, there exists $R > 0$ and $\{z_n\} \subset \mathbb{Z}^3$ such that

$$\int_{B_R} u_n(x+z_n)^2 dx = \int_{B_R(z_n)} u_n^2 dx > \frac{\beta}{2}.$$

Set $w_n := u_n(x+z_n)$. Hence, there exists a nonnegative function $w \in E$ such that, up to a subsequence, $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\rm loc}(\mathbb{R}^3)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^3$. Obviously, $w \neq 0$. If $\{z_n\}$ is bounded, $\exists\ R'$ such that

$$\int_{B_{R'}(0)} u_n^2 dx \geq \int_{B_R(z_n)} u_n^2 dx \geq \frac{\beta}{2},$$

which contradicts with the fact $u_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3)$. Hence $\{z_n\}$ is unbounded. Up to a subsequence, we have $z_n \to \infty$. For all $\varphi \in C_0^\infty(\mathbb{R}^3)$, by Lemmas 2.6 and 2.7, we have

$$0 = \langle I'(u_n, \varphi(\cdot - z_n))\rangle + o_n(1)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla \varphi(\cdot - z_n) dx + V(x) u_n \varphi(\cdot - z_n)) dx + \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n \varphi(\cdot - z_n) dx \\
&\quad - \int_{\mathbb{R}^3} f(x, u_n) \varphi(\cdot - z_n) dx + o_n(1) \\
&= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla \varphi(\cdot - z_n) + V_p(x) u_n \varphi(\cdot - z_n)) dx + \int_{\mathbb{R}^3} K_p(x) \widetilde{\phi}_{u_n} u_n \varphi(\cdot - z_n) dx \\
&\quad - \int_{\mathbb{R}^3} f_p(x, u_n) \varphi(\cdot - z_n) dx + o_n(1) \\
&= \int_{\mathbb{R}^3} (\nabla w_n \cdot \nabla \varphi + V_p(x) w_n \varphi) dx + \int_{\mathbb{R}^3} K_p(x) \widetilde{\phi}_{w_n} w_n \varphi dx \\
&\quad - \int_{\mathbb{R}^3} f_p(x, w_n) \varphi dx + o_n(1) \\
&= \langle I_p'(w), \varphi \rangle,
\end{aligned}
$$

that is, w is a solution of periodic system (1.7). By Lemma (2.3), Lemma 2.5, (A5) and Fatou lemma, we have

$$
\begin{aligned}
c &= I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle + o_n(1) \\
&= \frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f(x, u_n) u_n - F(x, u_n)) dx + o_n(1) \\
&\geq \frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f_p(x, u_n) u_n - F_p(x, u_n)) dx + o_n(1) \\
&= \frac{1}{4} \|w_n\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f_p(x, w_n) w_n - F_p(x, w_n)) dx + o_n(1) \\
&\geq \frac{1}{4} \|w\|^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f_p(x, w) w - F_p(x, w)) dx + o_n(1) \\
&= I_p(w) - \frac{1}{4} \langle I_p'(w), w \rangle \\
&= I_p(w) \geq c_p.
\end{aligned}
$$

Using Remark (2.12), $I_p(w) = c_p = c$. By the properties of c and $\mathcal{N}$, there exits $t_w > 0$ such that $t_w w \in \mathcal{N}$. Thus, we obtain $c \leq I(t_w w) \leq I_p(t_w w) \leq I_p(w) = c$. So c is achieved by $t_w w$. By Lemma 2.11, we have $I'(t_w w) = 0$. Therefore, $u = t_w w$ is a nonnegative ground state solution for system (1.5). Similar to that of discussed in [48], by the maximum principle discussed , $u > 0$. □

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