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APPROXIMATE SOLUTIONS TO BBM EQUATIONS WITH BILINEAR CONTROL IN A SLOWLY VARYING MEDIUM

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Abstract. This article concerns an approximate solution for the Benjamin-Bona-Mahony (BBM) equation with a bilinear control in slowly varying medium. By a sharp estimation of the error term, a suitable approximate solution for this equation is established.

1. Introduction

In this article, we consider the following BBM equation with internal bilinear control in slowly varying medium

$$(1-\lambda\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + m_\varepsilon u^2) = f(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R} \tag{1.1}$$

where $u = u(t,x)$ is a real-valued function, $\lambda \in (0,1)$ is a constant, and the interior control f is given by the bilinear control (or feedback law) $f(t,x) = n(t,x)u(t,x)$ with $n(t,\cdot) \in C^3(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Concerning slowly varying medium $m_\varepsilon = m(\varepsilon x)$, we always assume that there exist positive constants k and γ such that

$$\begin{gathered} 1 < m(s) < 2, \quad m'(s) > 0, \quad \forall s \in \mathbb{R} \\ 0 < m(s) - 1 < ke^{\gamma s}, \quad \forall s \le 0 \\ 0 < 2 - m(s) < ke^{-\gamma s}, \quad \forall s > 0. \end{gathered} \tag{1.2}$$

Clearly, it is inferred from (1.2) that $\lim_{s\to-\infty} m(s) = 1$ and $\lim_{s\to+\infty} m(s) = 2$.

Let us briefly review some results concerning the related control problems and stability of solitons for the BBM, KdV and gKdV equations. The BBM equation model [3] was proposed by Benjamin, Bona and Mahony. Ko and Kuehl [10] considered the approximate solution of the soliton for the variable coefficient KdV equation under given initial conditions, and obtained the unavoidable loss of the solitary wave energy in the propagation process. Bona, Pritchard and Scott [4] introduced solitary wave interaction in the dispersion medium numerically. Albert [1, 2] investigated the global existence, the long time behavior and the decay of solutions about the gBBM equation. Weinstein [27] studied the existence and dynamics stability of solitary wave. Weinstein [17] also obtained asymptotic stability

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of regularized long wave equations. Russell and Zhang [23, 24] studied controllability and stabilizability of the third-order linear dispersion equation and KdV equation, and showed smoothing and decay properties of the KdV equation on a periodic domain [25]. The asymptotic stability of the soliton solution of the BBM equation in H^1 space is studied in [18]. Dejak and Jonsson [5] considered the long-time dynamics of the variable coefficient modified KdV solitary waves. In the same year, Dejak and Sigal [6] studied the KdV solitary waves over a variable bottom similarly. Martel [11] researched asymptotic N-solitons-like solutions of the subcritical and critical gKdV equations. Martel and Merle has been making a great contribution for the BBM, KdV and gKdV equations. They proved that the soliton solution near Q_c of gKdV equation with a general nonlinearity is asymptotically stable in H^1 space [12]. They also studied the inelastic interaction of nearly equal solitons for the BBM and gKdV equations in [13, 14]. They estimated the error term between the approximate solution and the exact solution. They also gave descriptions of the inelastic collision of two solitary waves for the BBM and quartic gKdV equations [15, 16], and estimated some non-zero residual items accurately. *Muñoz* explored soliton dynamics and the existence and global properties under slowly varying medium for the gKdV equation, and proved that there is no pure soliton solution [19]. On this basis, he introduced inelastic character of solitons of slowly varying gKdV equations in [20], and dealt with approximate controllability of the gKdV solitons with bilinear control in [21, 22]. Holmer introduced dynamics of the KdV solitons in the presence of a slowly varying potential, and obtained an explicit description of the trajectory of the soliton parameters of scale and position on the dynamically relevant time scale, together with an estimate on the error [8].

Unlike the previous study, in this paper, we focus on a bilinear control problem for a given BBM soliton in slowly varying medium. The main difficulties to our problem are the suitable construction and the decomposition of an approximate solution. An essential step of the proof is the error term between the approximate solution and exact solution can be controlled under $O(\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|})$ during an interval of time $[0,T]$. Finally, we introduce a cut-off function $\zeta\in C^\infty(\mathbb{R})$ and redefine the new approximate solution $\widetilde{K}=\zeta_\varepsilon(y)K(t,x)$ to solve the problem of $K\notin L^2(\mathbb{R})$.

The rest of this paper is organized as follows. In Section 2, we introduce the soliton solution of the BBM equation and the associated control system. In Section 3, we construct an approximate solution and prove that the error term can be raised to $O(\varepsilon^2)$ by analyzing the first order term. In Section 4, we introduce a cut-off function $\zeta\in C^\infty(\mathbb{R})$ in order to resolve the case of $K\notin L^2(\mathbb{R})$.

2. Preliminaries

We consider the equation

$$(1-\lambda\partial_x^2)u_t+(u_{xx}-u+m_\varepsilon u^2)_x=nu, \tag{2.1}$$

which has solitary wave solutions:

$$u(t,x)=Q_c(x-ct), \tag{2.2}$$

and

$$Q_c(x)=(1+c)Q\Big(\sqrt{\frac{1+c}{1+\lambda c}}x\Big), \tag{2.3}$$

where the parameter $c>0$ describes the wave speed of the soliton.

Differentiating (2.3) with respect to x gives

$$\big(Q_c(x)\big)'_x = \frac{(1+c)^{3/2}}{(1+\lambda c)^{1/2}} Q'(\xi), \quad \xi = \sqrt{\frac{1+c}{1+\lambda c}}x. \tag{2.4}$$

Differentiating (2.3) with respect to c leads to

$$\wedge\, Q_c(x) = \big(Q_c(x)\big)'_c = \frac{1}{1+c}\Big(Q_c(x) + \frac{1-\lambda}{2(1+\lambda c)} x\big(Q_c(x)\big)'_x\Big). \tag{2.5}$$

By (2.3), we have

$$Q(x) = \frac{1}{1+c} Q_c\Big(\sqrt{\frac{1+\lambda c}{1+c}}x\Big), \tag{2.6}$$

with

$$Q(x) = \frac{3}{2}\cosh^{-2}(\frac{x}{2}) \quad \text{which satisfies } Q'' + Q^2 = Q, \tag{2.7}$$

and

$$(1+\lambda c)Q_c'' + Q_c^2 = (1+c)Q_c. \tag{2.8}$$

We now introduce the control $n(t,x) = -\varepsilon n_0'(x) Q_{c(t)}(x - \rho(t))$. For any $\varepsilon > 0$ small enough, we define

$$\alpha := -\frac{4}{3}\frac{\int_R Q^3}{\int_R Q^2} > 0, \tag{2.9}$$

$$n_\infty := -\frac{1}{\alpha}\log c. \tag{2.10}$$

We choose a smooth function n_0 satisfying

$$\begin{gathered} n_0 \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ |n_0(x)| \le k e^{\gamma_0 x}, \quad \text{for } x \le -1, \\ |n_\infty - n_0(x)| \le k e^{-\gamma_0 x}, \quad \text{for } x \ge 1, \\ |n_0^{(p)}(x)| \le k e^{-\gamma_0 |x|}, \quad x \in \mathbb{R},\ p = 1,2,3, \\ {n'}_0(x) > 0 \quad \text{if } n_\infty > 0; \quad {n'}_0(x) < 0 \quad \text{if } n_\infty < 0. \end{gathered} \tag{2.11}$$

for a fixed positive constant γ_0. Note that with this choice, it holds $\|n_0\|_\infty = |n_\infty|$.

3. Approximate solution

In the aforementioned context, we consider the function $V_{c(t)}(\varepsilon t, x) \in L^\infty(\mathbb{R})$ satisfying the following hypothesis :

(H1) $V'_{c(t)}(\varepsilon t, x) \in L^2(\mathbb{R})$, $\partial_c V_{c(t)}(\varepsilon t, x) \in L^\infty(\mathbb{R})$

(H2) For all $t \in \mathbb{R}$, there exist positive constants k and γ satisfying

$$\|V_{c(t)}(\varepsilon t, x)\|_{L^\infty(\mathbb{R})} \le k e^{-\gamma\varepsilon|\rho(t)|}.$$

Equation (1.1) remains invariant under space and time translations at the H^1-level, usually called mass conservation and energy conservation:

$$M(u(t)) = \int_R (\frac{1}{2}u^2 - \lambda u_x^2)(t,x)dx = M(u(0)), \tag{3.1}$$

and

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\int_R (1-\lambda\partial_x^2)\big((u_x)^2+u^2\big)(t,x)dx - \frac{1}{3}\int_R (1-\lambda\partial_x^2)u^3(t,x)dx \\ &= E(u(0)). \end{aligned} \tag{3.2}$$

We introduce the time of interaction for any given $\varepsilon > 0$ and $\delta_0 > 0$ small enough:

$$T := \min\{T_0, \varepsilon^{-1-\delta_0}\}, \tag{3.3}$$

where $T_0 > 0$ is the maximal time of existence for the solution $u(t)$.

Let

$$y := x - \rho(t), \quad R(t,x) := \frac{Q_{c(t)}(y)}{m(\varepsilon\rho(t))}, \tag{3.4}$$

with

$$\rho(t) = \int_0^t c(\varepsilon s)ds, \quad \partial_t\rho(t) = c(\varepsilon t).$$

Define $K(t,x)$ as the approximate solution of equation (2.1):

$$K(t,x) := R(t,x) + W(t,x), \quad W(t,x) := \varepsilon n_0'(\varepsilon\rho(t))V_{c(t)}(\varepsilon t, y), \tag{3.5}$$

where $V_{c(t)}$ satisfies the hypothesis (H1)-(H2).

Then we can reduce the error by introducing $K(t,x)$ defined in (3.5). Set

$$S[K](t,x) := (1-\lambda\partial_x^2)K_t + (K_{xx} - K + m_\varepsilon K^2)_x + \varepsilon n_0'(\varepsilon x)Q_{c(t)}(y)K, \tag{3.6}$$

with $m_\varepsilon = m(\varepsilon x)$.

Theorem 3.1. *There exists a function $V_{c(t)} \in L^\infty(\mathbb{R})$ such that $K(t,x)$ defined by (3.5) satisfies*

$$S[K](t,x) = (1-\lambda\partial_x^2)\big(c'\partial_c K - c\partial_y K\big) + S_0[K](t,x) \tag{3.7}$$

for some $n \in \mathbb{R}$, and

$$\|S_0[K](t,x)\|_{H^1(y>-\frac{2}{\varepsilon})} \le \varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|} + \varepsilon^3 \tag{3.8}$$

and

$$\begin{aligned} &\Big|\int_{\mathbb{R}} (1-\lambda\partial_x^2)Q_{c(t)}(y)S_0[K](t,x)dx\Big| + \Big|\int_{\mathbb{R}} (1-\lambda\partial_x^2)yQ_{c(t)}(y)S_0[K](t,x)dx\Big| \\ &\le \varepsilon^2 e^{-\varepsilon\gamma|\rho(t)|} + \varepsilon^3. \end{aligned} \tag{3.9}$$

Following the strategy in [21], *from* (3.6) *we obtain*

$$S[K] = I + II + III, \tag{3.10}$$

where $I = S[R](t,x)$,

$$II = (1-\lambda\partial_x^2)W_t + (W_{xx} - W + 2m_\varepsilon RW)_x + \varepsilon n_0'(\varepsilon x)Q_{c(t)}(y)W,$$

and $III = (m_\varepsilon W^2)_x$, where

$$R(t,x) = \frac{Q_{c(t)}(y)}{m(\varepsilon\rho(t))}, \quad y = x - \rho(t).$$

For proving Theorem 3.1, we discuss I, II and III, separately.

Lemma 3.2. *we have*
$$I = \varepsilon A_1(t,y) + \varepsilon^2 A_2(t,y),$$
where
$$\begin{aligned} A_1(t,y) &= \frac{c'}{m(\varepsilon\rho)}(1-\lambda\partial_x^2)\wedge Q_{c(t)}(y) - \frac{m'(\varepsilon\rho)c}{m^2(\varepsilon\rho)}(1-\lambda\partial_x^2)Q_{c(t)}(y) \\ &\quad + \frac{m'(\varepsilon\rho)}{m^2(\varepsilon\rho)}\big(yQ_{c(t)}^2(y)\big)_x + \frac{n_0'(\varepsilon\rho)}{m(\varepsilon\rho)}Q_{c(t)}^2(y) \end{aligned} \tag{3.11}$$
and
$$A_2(t,y) = \frac{m''(\varepsilon\rho)}{2m^2(\varepsilon\rho)}\big(y^2Q_{c(t)}^2(y)\big)_x + \frac{n_0''(\varepsilon\rho)}{m(\varepsilon\rho)}yQ_{c(t)}^2(y) \tag{3.12}$$
for all $t\in[0,T]$*, and* $A_2(t)$ *satisfies* $\|A_2(t)\|_{H^1(\mathbb{R})} \le e^{-\varepsilon\gamma|\rho(t)|} + \varepsilon$.

Proof. We have
$$\begin{aligned} I &= (1-\lambda\partial_x^2)R_t + (R_{xx} - R + m_\varepsilon R^2)_x + \varepsilon n_0'(\varepsilon x)Q_{c(t)}R \\ &= (1-\lambda\partial_x^2)\frac{(\wedge Q_c c'\varepsilon - Q_c'c)m(\varepsilon\rho) - Q_c m'(\varepsilon\rho)\varepsilon c}{m^2(\varepsilon\rho)} + \frac{1}{m(\varepsilon\rho)}Q_c''' \\ &\quad - \frac{1}{m(\varepsilon\rho)}Q_c' + \frac{1}{m^2(\varepsilon\rho)}(m(\varepsilon x)Q_c^2)_x + \frac{1}{m(\varepsilon\rho)}\varepsilon n_0'(\varepsilon x)Q_c^2. \end{aligned}$$
By the Taylor expansion, we obtain
$$\begin{aligned} (m(\varepsilon x)Q_c^2)_x &= (m(\varepsilon\rho(t))Q_c^2)_x + \varepsilon m'(\varepsilon\rho(t))(yQ_c^2)_x \\ &\quad + \frac{1}{2}\varepsilon^2 m''(\varepsilon\rho(t))(y^2Q_c^2)_x + O_{H^1(\mathbb{R})}(\varepsilon^3) \end{aligned}$$
and
$$n_0'(\varepsilon x)Q_c^2 = n_0'(\varepsilon\rho(t))Q_c^2 + \varepsilon n_0''(\varepsilon\rho(t))yQ_c^2 + O_{H^1(\mathbb{R})}(\varepsilon^2),$$
which implies
$$\begin{aligned} I &= (1-\lambda\partial_x^2)\frac{(\wedge Q_c c'\varepsilon - Q_c'c)m(\varepsilon\rho) - Q_c m'(\varepsilon\rho)\varepsilon c}{m^2(\varepsilon\rho)} \\ &\quad + \frac{1}{m^2(\varepsilon\rho)}\big[(m(\varepsilon\rho)Q_c^2)_x + \varepsilon m'(\varepsilon\rho)(yQ_c^2)_x + \frac{1}{2}\varepsilon^2 m''(\varepsilon\rho)(y^2Q_c^2)_x\big] \\ &\quad + \frac{Q_c'''}{m(\varepsilon\rho)} - \frac{Q_c'}{m(\varepsilon\rho)} + \frac{1}{m(\varepsilon\rho)}\varepsilon\big(n_0'(\varepsilon\rho)Q_c^2 + \varepsilon n_0''(\varepsilon\rho)yQ_c^2\big) + O_{H^1(\mathbb{R})}(\varepsilon^3). \end{aligned}$$
Then, using $(1+\lambda c)Q_c'' + Q_c^2 = (1+c)Q_c$, we obtain
$$\begin{aligned} I &= \frac{1}{m(\varepsilon\rho)}[Q_c^2 - (1+c)Q_c + (1+\lambda c)Q_c'']_x \\ &\quad + \varepsilon\Big[\frac{c'}{m(\varepsilon\rho)}(1-\lambda\partial_x^2)\wedge Q_c - \frac{m'(\varepsilon\rho)c}{m^2(\varepsilon\rho)}(1-\lambda\partial_x^2)Q_c + \frac{m'(\varepsilon\rho)}{m^2(\varepsilon\rho)}(yQ_c^2)_x \\ &\quad + \frac{n_0'(\varepsilon\rho)}{m(\varepsilon\rho)}Q_c^2\Big] + \varepsilon^2\Big[\frac{m''(\varepsilon\rho)}{2m^2(\varepsilon\rho)}(y^2Q_c^2)_x + \frac{n_0''(\varepsilon\rho)}{m(\varepsilon\rho)}yQ_c^2\Big] + O_{H^1(\mathbb{R})}(\varepsilon^3) \\ &= \varepsilon\Big[\frac{c'}{m(\varepsilon\rho)}(1-\lambda\partial_x^2)\wedge Q_c - \frac{m'(\varepsilon\rho)c}{m^2(\varepsilon\rho)}(1-\lambda\partial_x^2)Q_c + \frac{m'(\varepsilon\rho)}{m^2(\varepsilon\rho)}(yQ_c^2)_x \\ &\quad + \frac{n_0'(\varepsilon\rho)}{m(\varepsilon\rho)}Q_c^2\Big] + \varepsilon^2\Big[\frac{m''(\varepsilon\rho)}{2m^2(\varepsilon\rho)}(y^2Q_c^2)_x + \frac{n_0''(\varepsilon\rho)}{m(\varepsilon\rho)}yQ_c^2\Big] + O_{H^1(\mathbb{R})}(\varepsilon^3). \end{aligned}$$

From equation (3.12), we have

$$A_2(t,y) \in S(\mathbb{R}), \quad \|A_2(t,y)\|_{H^1(\mathbb{R})} \le e^{-\varepsilon\gamma|\rho(t)|} + \varepsilon.$$

This completes the proof. □

We now consider a linear elliptic operator L, for any fixed $c > 0$. Let

$$LB = -(1+\lambda c)B'' + (1+c)B - 2Q_c B, \tag{3.13}$$

where

$$Q_c(x) = (1+c)Q\Big(\sqrt{\frac{1+c}{1+\lambda c}}x\Big). \tag{3.14}$$

Lemma 3.3. *Assume that V_c satisfies the hypothesis* (H1)-(H2). *Then*

$$\begin{aligned} II &= (1-\lambda\partial_x^2)(c'\partial_c W - c\partial_y W) - (LW)_y \\ &\quad + \varepsilon^2\big[(1-\lambda\partial_x^2)\big(n_0''(\varepsilon\rho)cV_c + n_0'(\varepsilon\rho)c'\partial_c V_c\big)\big] + \varepsilon^2 A_3(t,y), \end{aligned}$$

where

$$\begin{aligned} A_3(t,y) &= (1-\lambda\partial_x^2)n_0'(\varepsilon\rho)\partial_t V_c + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}(yQ_c n_0'(\varepsilon\rho)V_c)_x \\ &\quad + O_{H^1(\mathbb{R})}(e^{-\varepsilon\gamma|\rho(t)|}). \end{aligned}$$

Proof. Let $B_c(t,y) = W$ be a smooth function with $y = x - \rho(t)$, then by using

$$II(B) = (1-\lambda\partial_x^2)B_t + (B_{xx} - B + 2m_\varepsilon RB)_x + \varepsilon n_0'(\varepsilon x)Q_{c(t)}B$$

we obtain

$$\begin{aligned} II(B) &= (1-\lambda\partial_x^2)[c'\partial_c B + B_t - (\rho' - c)B_y] \\ &\quad + \Big[B_{yy} - (1-\lambda\partial_x^2)cB - B + 2Q_{c(t)}B + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}\varepsilon y Q_{c(t)}B]_x \\ &\quad + O(\varepsilon n_0'(\varepsilon x)Q_{c(t)}B) \\ &= (1-\lambda\partial_x^2)(c'\partial_c B + B_t) + [(1+\lambda c)B_{xx} - (1+c)B + 2Q_{c(t)}B]_x \\ &\quad + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}\varepsilon(yQ_{c(t)}B)_x + O_{H^1(\mathbb{R})}\big(\varepsilon^2 e^{-\varepsilon\gamma|\rho(t)|}\big). \end{aligned}$$

Applying the identity $W(t,x) = \varepsilon n_0'(\varepsilon\rho(t))V_{c(t)}(\varepsilon t, y)$ defined in (3.5), we have

$$\begin{gathered} \partial_c W = \varepsilon n_0'(\varepsilon\rho(t))\partial_c V_c, \\ W_t = \varepsilon\big[\varepsilon n_0''(\varepsilon\rho(t))\rho'(t)V_c + n_0'(\varepsilon\rho(t))\big(c'\varepsilon\partial_c V_c + \varepsilon\partial_t V_c - c\partial_y V_c\big)\big], \end{gathered}$$

and

$$\begin{aligned} II(W) &= (1-\lambda\partial_x^2)(c'\partial_c W + W_t) - (LW)' + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}\varepsilon(yQ_{c(t)}W)_x \\ &\quad + O_{H^1(\mathbb{R})}(\varepsilon^2 e^{-\varepsilon\gamma|\rho(t)|}) \\ &= \varepsilon^2\Big[(1-\lambda\partial_x^2)(n_0''(\varepsilon\rho)\rho' V_c + n_0'(\varepsilon\rho)c'\partial_c V_c + n_0'(\varepsilon\rho)\partial_t V_c) \\ &\quad + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}(yQ_c n_0'(\varepsilon\rho)V_c)_x\Big] + \varepsilon n_0'(\varepsilon\rho)(1-\lambda\partial_x^2)(c'\partial_c V_c - c\partial_y V_c) \\ &\quad - \varepsilon n_0'(\varepsilon\rho)(LV_c)_y + O_{H^1(\mathbb{R})}(\varepsilon^2 e^{-\varepsilon\gamma|\rho(t)|}) \\ &= (1-\lambda\partial_x^2)(c'\partial_c W - c\partial_y W) - (LW)_y + \varepsilon^2\big[(1-\lambda\partial_x^2)\big(n_0''(\varepsilon\rho)cV_c \end{aligned}$$

$$+ n_0'(\varepsilon\rho)c'\partial_c V_c\big)\big] + \varepsilon^2\Big[(1-\lambda\partial_x^2)n_0'(\varepsilon\rho)\partial_t V_c + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}(yQ_c n_0'(\varepsilon\rho)V_c)_x + O_{H^1(\mathbb{R})}(e^{-\varepsilon\gamma|\rho(t)|})\Big].$$

This completes the proof. □

Lemma 3.4. *For any $t \in [0,T]$,*

$$III = O_{H^1(\mathbb{R})}(\varepsilon^2 e^{-\varepsilon\gamma|\rho(t)|}).$$

Proof. Recall that $III := (m_\varepsilon W^2)_x$. Then we obtain

$$III = \varepsilon^2\big(n_0'(\varepsilon\rho)\big)^2(m(\varepsilon x)V_c^2)_x = \varepsilon^2(n_0'(\varepsilon\rho))^2\big[\varepsilon m'(\varepsilon x)V_c^2 + m(\varepsilon x)(V_c^2)'\big].$$

Since V_c satisfies the hypothesis (H1)-(H2), $(V_c^2)' \in S(\mathbb{R})$ holds. By taking the space derivative, we can see the desired result. □

Proof of Theorem 3.1. According to the estimates from Lemmas 3.2, 3.3 and 3.4, we obtain

$$S[K] = (1-\lambda\partial_x^2)(c'\partial_c K - c\partial_y K) + S_0[K], \tag{3.15}$$

where

$$\begin{aligned} S_0[K] &= \varepsilon[A_1(t,y) - n_0'(\varepsilon\rho)(LV_c)_y] \\ &\quad + \varepsilon^2\big[(1-\lambda\partial_x^2)(n_0''(\varepsilon\rho)cV_c + n_0'(\varepsilon\rho)c'\partial_c V_c + n_0'(\varepsilon\rho)\partial_t V_c)\big] \\ &\quad + \varepsilon^2\big[\frac{m''(\varepsilon\rho)}{2m^2(\varepsilon\rho)}(y^2Q_c^2)_x + \frac{n_0''(\varepsilon\rho)}{m(\varepsilon\rho)}yQ_c^2 + \frac{2m'(\varepsilon\rho)}{m(\varepsilon\rho)}(yQ_c n_0'(\varepsilon\rho)V_c)_x\big] \\ &\quad + \varepsilon^2 O_{H^1(\mathbb{R})}(e^{-\varepsilon\gamma|\rho(t)|} + \varepsilon). \end{aligned} \tag{3.16}$$

The next step is the resolution of the linear differential equation about the first order term in ε. From (3.16), we want to solve

$$n_0'(\varepsilon\rho(t))(LV_c)_y = A_1(t,y). \tag{3.17}$$

Obviously, accuracy of the error term can be raised to $O(\varepsilon^2)$ now. It is not difficult to check that, for all $y \in \mathbb{R}$ and t fixed, A_1 satisfies the regularity conditions:

$$\int_{\mathbb{R}} A_1(t,y)Q_c(y)dy = 0. \tag{3.18}$$

Lemma 3.5. *The operator L defined on $L^2(\mathbb{R})$ by (3.13) satisfies*

1. *The kernel of L is Q_c', that is $LQ_c' = 0$.*
2. *For any $f = f(y) \in L^2(\mathbb{R})$ which satisfies $\int_{\mathbb{R}} fQ_c'(y)dy = 0$, and there exists a unique function f_0 such that $\int_{\mathbb{R}} f_0 Q_c'(y)dy = 0$ and $Lf_0 = f$. Furthermore, if f is odd, then f_0 is odd.*

Set $c > 0$ and define

$$\psi(x) := -\frac{Q'(x)}{Q(x)}, \quad \psi_c(x) := -\frac{Q_c'(x)}{Q_c(x)} = \sqrt{\frac{1+c}{1+\lambda c}}\psi(\sqrt{\frac{1+c}{1+\lambda c}}x). \tag{3.19}$$

A direct computation yields

$$\lim_{x\to-\infty}\psi(x) = -1, \quad \lim_{x\to+\infty}\psi(x) = 1. \tag{3.20}$$

Lemma 3.6. *There exists a unique solution $V_c = V_{c(t)}(t,y)$ satisfying*

$$n_0'(\varepsilon\rho)(LV_c)_y = A_1(t,y)$$

such that, for every t,

$$V_c(t,y) := \alpha_c(t)\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) + \beta_c(t)Q_c'(y) + V_1(t,y) + \sigma_c(t), \tag{3.21}$$

and

$$\lim_{y\to-\infty} V_c(t,y) = -2\sqrt{\frac{1+c}{1+\lambda c}}\alpha_c(t); \quad |V_c(y)| \le ke^{-\gamma y}, \text{ as } y \to +\infty, \tag{3.22}$$

with $V_1(y) \in \mathrm{S}(\mathrm{R})$ for all $t, \alpha_c(t), \beta_c(t)$ and $\sigma_c(t) \in \mathbb{R}$. Moreover, we have

$$\alpha_c(t) := \frac{1}{2n_0'(\varepsilon\rho(t))}\sqrt{\frac{1+\lambda c}{1+c}}\int_{\mathbb{R}} A_1(t,y)dy \neq 0. \tag{3.23}$$

Proof. The proof of this lemma is divided into three steps.

Step 1. The first step is to prove the existence of $n_0'(\varepsilon\rho(t))(LV_c)_y = A_1(t,y)$, where V_c was established in [19]. For equation (3.21), we have

$$LV_c(y) = \alpha_c(t)L\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) + \beta_c(t)LQ_c'(y) + LV_1(y) + L\sigma_c(t).$$

So

$$LV_1(y) = H(y) - \alpha_c(t)L\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) - \theta_c(t),$$

with

$$H(y) = \frac{1}{n_0'(\varepsilon\rho)}\int_{\mathbb{R}} A_1(t,y)dy, \quad \theta_c(t) = L\sigma_c(t).$$

Without lose of generality, we assume

$$\theta_c(t) = 2\alpha_c(t)\sqrt{\frac{1+c}{1+\lambda c}}.$$

The solvability of equation (3.17) is equivalent to

$$\begin{aligned}
&\int_{\mathbb{R}} LV_1(y)Q_c'(y)dy \\
&= \int_{\mathbb{R}} \Big[H(y) - \alpha_c L\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) - 2\alpha_c\sqrt{\frac{1+c}{1+\lambda c}}\Big]Q_c'(y)dy \\
&= \int_{\mathbb{R}} \Big[H(y) - \alpha_c\Big(L\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) + 2\sqrt{\frac{1+c}{1+\lambda c}}\Big)\Big]Q_c'(y)dy \\
&= \int_{\mathbb{R}} H(y)Q_c'(y)dy = -\int_{\mathbb{R}} Q_c(y)dH \\
&= -\frac{1}{n_0'(\varepsilon\rho)}\int_{\mathbb{R}} Q_c(y)A_1(y)dy = 0.
\end{aligned}$$

Since $LQ_c'(y) = 0$ and $\int_{\mathbb{R}} LV_1(y)Q_c'(y)dy = 0$, according to Lemma 3.5, there exists a function $V_1(y)$ satisfying $\int_{\mathbb{R}} V_1(y)Q_c'(y)dy = 0$. Note that

$$\lim_{y\to-\infty}\Big[H(y) - \alpha_c\Big(L\Big(\psi_c(y) - \sqrt{\frac{1+c}{1+\lambda c}}\Big) + 2\sqrt{\frac{1+c}{1+\lambda c}}\Big)\Big] = 0$$

and

$$\begin{aligned}&\lim_{y\to+\infty}\Big[H(y)-\alpha_c\Big(L\Big(\psi_c(y)-\sqrt{\frac{1+c}{1+\lambda c}}\Big)+2\sqrt{\frac{1+c}{1+\lambda c}}\Big)\Big]\\&=\frac{1}{n_0'(\varepsilon\rho)}\int_{\mathbb{R}}A_1(y)dy-2\alpha_c\sqrt{\frac{1+c}{1+\lambda c}}=0,\end{aligned}$$

So we have $V_1(y)\in S(R)$ and

$$\alpha_c=\frac{1}{2n_0'(\varepsilon\rho)}\sqrt{\frac{1+\lambda c}{1+c}}\int_{\mathbb{R}}A_1(t,y)dy.$$

Step 2. In this step we show that

$$\begin{aligned}\alpha_c=\frac{\int_{\mathbb{R}}Q_c(y)dy}{2n_0'(\varepsilon\rho)}\sqrt{\frac{1+\lambda c}{1+c}}\Big[&\frac{1+2\lambda c+\lambda}{2(1+\lambda c)(1+c)}\frac{c'}{m(\varepsilon\rho)}-\frac{m'(\varepsilon\rho)c}{m(\varepsilon\rho)}\\&+\frac{n_0'(\varepsilon\rho)(1+c)}{m(\varepsilon\rho)}\Big].\end{aligned}$$

From equation (3.11), one has

$$\begin{aligned}A_1(t,y)=&\frac{c'}{m(\varepsilon\rho)}(1-\lambda\partial_x^2)\wedge Q_c(y)-\frac{m'(\varepsilon\rho)c}{m^2(\varepsilon\rho)}(1-\lambda\partial_x^2)Q_c(y)\\&+\frac{m'(\varepsilon\rho)}{m^2(\varepsilon\rho)}(yQ_c^2(y))_x+\frac{n_0'(\varepsilon\rho)}{m(\varepsilon\rho)}Q_c^2(y).\end{aligned}$$

Next we consider the following four integrals:

$$\int_{\mathbb{R}}(1-\lambda\partial_x^2)\wedge Q_c(y)dy,\quad\int_{\mathbb{R}}(1-\lambda\partial_x^2)Q_c(y)dy,\quad\int_{\mathbb{R}}(yQ_c^2(y))_x dy,\quad\int_{\mathbb{R}}Q_c^2(y)dy.$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}}(1-\lambda\partial_x^2)\wedge Q_c(y)dy\\&=\frac{1}{1+c}\int_{\mathbb{R}}(1-\lambda\partial_x^2)\Big(Q_c(y)+\frac{1-\lambda}{2(1+\lambda c)}yQ_c'(y)\Big)dy\\&=\frac{1}{1+c}[\int_{\mathbb{R}}(Q_c(y)-\lambda Q_c''(y))dy+\int_{\mathbb{R}}\Big(\frac{1-\lambda}{2(1+\lambda c)}yQ_c'(y)-\frac{\lambda(1-\lambda)}{2(1+\lambda c)}(yQ_c'(y))''\Big)dy]\\&=\frac{1}{1+c}\Big[\int_{\mathbb{R}}Q_c(y)dy-\frac{1-\lambda}{2(1+\lambda c)}\int_{\mathbb{R}}Q_c(y)dy\\&\quad-\frac{\lambda(1-\lambda)}{2(1+\lambda c)}\int_{\mathbb{R}}(yQ_c'(y))''dy\Big]\\&=[\frac{1}{1+c}-\frac{1-\lambda}{2(1+\lambda c)(1+c)}]\int_{\mathbb{R}}Q_c(y)dy,\end{aligned}$$

$$\int_{\mathbb{R}}(1-\lambda\partial_x^2)Q_c(y)dy=\int_{\mathbb{R}}Q_c(y)dy-\lambda\int_{\mathbb{R}}Q_c''(y)dy=\int_{\mathbb{R}}Q_c(y)dy,$$

$$\int_{\mathbb{R}}(yQ_c^2(y))_x\,dy=0.$$

According to

$$r\int_{\mathbb{R}}Q_c^r(y)dy=\frac{2r+1}{3(1+c)}\int_{\mathbb{R}}Q_c^{r+1}(y)dy,$$

when $r = 1$ one has

$$\int_{\mathbb{R}} Q_c^2(y)dy = (1+c)\int_{\mathbb{R}} Q_c(y)dy.$$

So we can get

$$\alpha_c = \frac{\int_{\mathbb{R}} Q_c(y)dy}{2n_0'(\varepsilon\rho)}\sqrt{\frac{1+\lambda c}{1+c}}\Big[\frac{1+2\lambda c+\lambda}{2(1+\lambda c)(1+c)}\frac{c'}{m(\varepsilon\rho)} - \frac{m'(\varepsilon\rho)c}{m(\varepsilon\rho)} + \frac{n_0'(\varepsilon\rho)(1+c)}{m(\varepsilon\rho)}\Big].$$

Step 3. From step 1, we have

$$\theta_c(t) = 2\alpha_c(t)\sqrt{\frac{1+c}{1+\lambda c}}.$$

By computing $\theta_c(t) = 2\alpha_c(t)\sqrt{\frac{1+c}{1+\lambda c}}$, we can easily assert that V_c is exponential decay as $y \to +\infty$, that is $\lim_{y\to+\infty} V_c = 0$. This completes the proof of the lemma. □

We use a method similarly to the one in [21]. According to (3.22), we obtain the estimation (3.8). In addition, from Lemma 3.6, we see that V_c satisfies the hypothesis (H1)-(H2). From (3.16) we arrive at (3.9). All these complete the proof of Theorem 3.1. □

4. Solution for $K \in L^2(R)$

The following method is similar to those introduced in [19, 21, 22]. Since $K \notin L^2(\mathbb{R})$, we introduce a cut-off function $\zeta \in C^\infty(\mathbb{R})$ satisfying the following properties

$$\begin{gathered}\zeta(y) \equiv \begin{cases} 0 & \text{for } y \le -1, \\ 1 & \text{for } y \ge 1, \end{cases} \\ 0 \le \zeta(y) \le 1, \quad 0 \le \zeta'(y) \le 1 \quad \text{for } y \in \mathbb{R}. \end{gathered} \tag{4.1}$$

Similarly, for $R(t,x)$ and $W(t,x)$ constructed in equation (3.5), we define a new approximate solution $\widetilde{K}$:

$$\widetilde{K} := \zeta_\varepsilon(y)K(t,x) = \zeta_\varepsilon(y)(R(t,x) + W(t,x)), \tag{4.2}$$

where

$$\zeta_\varepsilon(y) := \zeta(\varepsilon y + 2). \tag{4.3}$$

Theorem 4.1. *For all $0 < \varepsilon < \tau$, there exist positive constants τ and k such that: (1)*

$$\begin{gathered}\widetilde{K} = 0 \quad \text{for } y \le -\frac{3}{\varepsilon}, \\ \widetilde{K} = K(t,x) \quad \text{for } y \ge -\frac{1}{\varepsilon}. \end{gathered} \tag{4.4}$$

For any t in a given interval, $W(t,x) \in H^1(\mathbb{R})$ satisfies

$$\|W(t,x)\|_{H^1(\mathbb{R})} \le k\varepsilon^{1/2} e^{-\gamma\varepsilon|\rho(t)|}. \tag{4.5}$$

(2) The error term associated to the new solution $\widetilde{K}$ satisfies

$$\begin{aligned} S[\widetilde{K}] = (1-\lambda\partial_x^2)[c'(\partial_c\widetilde{K} + O_{H^1(\mathbb{R})}(\varepsilon n_0')) \\ - c\big(\partial_y\widetilde{K} + O_{H^1(\mathbb{R})}(\varepsilon n_0')\big)] + S_0[\widetilde{K}], \end{aligned} \tag{4.6}$$

where

$$\|S_0[\widetilde{K}]\|_{H^1(\mathbb{R})} \le k\varepsilon^{3/2} e^{-\gamma\varepsilon|\rho(t)|}. \tag{4.7}$$

Proof. (1) It follows from (4.5) that

$$\|\zeta_\varepsilon(y)W(t,x)\|_{H^1(\mathbb{R})} \le k\|W(t,x)\|_{H^1(y\ge -\frac{3}{\varepsilon})}.$$

From (H1)-(H2), we have

$$\|\varepsilon n_0'(\varepsilon\rho(t))V_{c(t)}(y)\|_{H^1(y\ge -\frac{3}{\varepsilon})} \le k\varepsilon^{1/2} e^{-\gamma\varepsilon|\rho(t)|}.$$

(2) We make a simple calculation as follows

$$\begin{aligned} S[\widetilde{K}] &= S[\zeta_\varepsilon(y)K] = S[\zeta_\varepsilon(y)(R+W)] \\ &= \zeta_\varepsilon S[K] + (\zeta_\varepsilon)_t(1-\lambda\partial_x^2)K + 2\varepsilon{\zeta_\varepsilon}'K_{xx} + 3\varepsilon^2{\zeta_\varepsilon}''K_x + \varepsilon^3{\zeta_\varepsilon}'''K. \end{aligned}$$

and

$$2\varepsilon{\zeta_\varepsilon}'K_{xx} + 3\varepsilon^2{\zeta_\varepsilon}''K_x + \varepsilon^3{\zeta_\varepsilon}'''K = O_{H^1(\mathbb{R})}(\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|}) + O_{H^1(\mathbb{R})}(\varepsilon^{30}).$$

Similarity, we have

$$(\zeta_\varepsilon)_t(1-\lambda\partial_x^2)K = O_{H^1(\mathbb{R})}(\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|}) + O_{H^1(\mathbb{R})}(\varepsilon^{30}),$$

so we obtain

$$S[\zeta_\varepsilon(y)K] = \zeta_\varepsilon S[K] + O_{H^1(\mathbb{R})}(\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|}) + O_{H^1(\mathbb{R})}(\varepsilon^{30}).$$

Finally, from (H1)-(H2), and (3.5) and (3.7), we have

$$\|\zeta_\varepsilon S_0[K]\|_{H^1(\mathbb{R})} \le k\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|} + \varepsilon^3,$$

$$\begin{aligned} &\zeta_\varepsilon(1-\lambda\partial_x^2)(c'\partial_c K - c\partial_y K) \\ &= (1-\lambda\partial_x^2)c'\partial_c(\zeta_\varepsilon K) - (1-\lambda\partial_x^2)c\partial_y(\zeta_\varepsilon K) - \varepsilon(1-\lambda\partial_x^2)c{\zeta_\varepsilon}'K. \end{aligned}$$

Note that

$$\varepsilon(1-\lambda\partial_x^2){\zeta_\varepsilon}'K = O_{H^1(\mathbb{R})}(\varepsilon^{3/2}e^{-\gamma\varepsilon|\rho(t)|}).$$

Consequently, we obtain the desired results. □

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