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EXISTENCE, REGULARITY AND REPRESENTATION OF SOLUTIONS OF TIME FRACTIONAL WAVE EQUATIONS

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Abstract. We study the solvability of the fractional order inhomogeneous Cauchy problem

$$\mathbb{D}_t^\alpha u(t) = Au(t) + f(t), \quad t > 0, \ 1 < \alpha \leq 2,$$

where A is a closed linear operator in some Banach space X and $f : [0, \infty) \to X$ a given function. Operator families associated with this problem are defined and their regularity properties are investigated. In the case where A is a generator of a β-times integrated cosine family $(C_\beta(t))$, we derive explicit representations of mild and classical solutions of the above problem in terms of the integrated cosine family. We include applications to elliptic operators with Dirichlet, Neumann or Robin type boundary conditions on L^p-spaces and on the space of continuous functions.

1. Introduction

The classical wave equation provides the most important model for the study of oscillation phenomena in physical sciences and engineering. In the treatment of the evolutionary equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x) + f(t,x), \quad t > 0, \ x \in \Omega, \tag{1.1}$$

in function spaces over Ω, where $\Omega \subset \mathbb{R}^N$ is an open set, one needs initial conditions,

$$u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad x \in \Omega;$$

and boundary conditions. Traditionally, Dirichlet and Neumann boundary conditions are the most studied. The Robin type boundary conditions, $\nabla u \cdot \nu + \gamma u = g$ in $\partial\Omega$ (where ν denotes the outer unit normal vector at the boundary of the open set Ω), have proven important due to the fact that they arise naturally in heat conduction problems as well as in physical Geodesy. Moreover, from the Robin boundary conditions, one can recover the Dirichlet and Neumann boundary conditions (see e.g. [6, 7]). For more details and applications we refer to [6, 7, 14, 25, 43, 48, 49] and the references therein.

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For many concrete problems it has been observed that equations of fractional order in time provide a more suitable framework for their study. Typical of this are phenomena with memory effects, anomalous diffusion, problems in rheology, material science and several other areas. We refer to the monographs [39, 44, 45] and the papers [11, 12, 16, 21, 22, 23, 38, 41, 52] for more information.

We will investigate the linear inhomogeneous differential equation of fractional order:

$$\mathbb{D}_t^\alpha u(t) = Au(t) + f(t), \quad t > 0,\ 1 < \alpha \le 2, \tag{1.2}$$

in which $\mathbb{D}_t^\alpha$ is the Caputo fractional derivative. Here X is a complex Banach space and A is a closed linear operator in X. The use of the Caputo fractional derivative has the advantage (over, say, the Riemann-Liouville fractional derivative) that the initial conditions are formulated in terms of the values of the solution u and its derivative at 0. These have physically significant interpretations in concrete problems.

Our aim is to construct a basic theory for the solutions of this equation along with applications to some partial differential equations modeling phenomena from science and engineering. To study the existence, uniqueness and regularity of the solutions of Problem (1.2), in general, one needs an operator family associated with the problem [33, 34]. For example, the theory of cosine families has been developed to deal with the case $\alpha = 2$. In case A does not generate a cosine family (if $\alpha = 2$), the concept of exponentially bounded β-times integrated cosine families has been used in the treatment of Problem (1.2). In [8], an operator family called S_α has been introduced to deal with the fractional case, that is, $1 < \alpha \le 2$ and $\beta = 0$. Unfortunately, this theory does not include the case of exponentially bounded β-times integrated cosine families. Consequently, the results obtained in [8] cannot be applied to deal with the following problem in $L^p(\Omega)$, $p \ne 2$, which is the fractional version of (1.1):

$$\begin{gathered}
\mathbb{D}_t^\alpha u(t,x) - Au(t,x) = f(t,x), \quad t > 0,\ x \in \Omega,\ 1 < \alpha \le 2, \\
\frac{\partial u(t,z)}{\partial \nu_A} + \gamma(z) u(t,z) = 0, \quad t > 0,\ z \in \partial\Omega, \\
u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad x \in \Omega.
\end{gathered} \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is an open set with boundary $\partial\Omega$, A is a uniformly elliptic operator with bounded measurable coefficients formally given by

$$Au = \sum_{j=1}^N D_j \Big(\sum_{i=1}^N a_{i,j} D_i u + b_j u \Big) - \Big(\sum_{i=1}^N c_i D_i u + du \Big) \tag{1.4}$$

and

$$\frac{\partial u}{\partial \nu_A} = \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u \Big) \cdot \nu_j,$$

where ν denotes the unit outer normal vector of Ω at $\partial\Omega$ and γ is a nonnegative measurable function in $L^\infty(\partial\Omega)$ or $\gamma = \infty$.

In this paper, we introduce an appropriate operator family in a general Banach space associated with Problem (1.2) that will cover all the above mentioned cases. This family will be called an $(\alpha,1)^\beta$-resolvent family $(\mathbb{S}_\alpha^\beta(t))$ (see Definition 4.2 below) where $1 < \alpha \le 2$ and $\beta \ge 0$ is a real parameter associated with the operator A.

The case $\beta = 0$ and $\alpha = 2$ corresponds to the wave equation with A generating a cosine family. The family $\mathbb{S}_\alpha^0$ $(1 < \alpha \le 2)$ corresponds to the family S_α introduced in the reference [8] and mentioned above. The family $\mathbb{S}_\alpha^\beta$, $\beta > 0$ and $\alpha = 2$, corresponds to the theory of exponentially bounded β-times integrated cosine family. We use this framework to treat the homogeneous ($f = 0$ in (1.2)) as well as the inhomogeneous problems (under suitable conditions on the function f in (1.2)). We shall in fact consider the case where the operator A is an L^p-realization of a more general uniformly elliptic operator in divergence form (as the one in (1.4)) with various boundary conditions (Dirichlet, Neumann or Robin). We obtain a representation of mild and classical solutions in terms of the operator family $\mathbb{S}_\alpha^\beta$. Our results apply to the situation where the closed linear operator A satisfies the following condition: There exist $\omega \ge 0$ and $\gamma \ge -1$ such that

$$\|(\lambda^2 - A)^{-1}\| \le M|\lambda|^\gamma, \quad \operatorname{Re}(\lambda) > \omega. \tag{1.5}$$

In fact, several operators of interest such as the Laplace operator in $L^p(\mathbb{R}^N)$ for $N \ge 2$ and $p \ne 2$, which do not generate cosine families are generators of integrated cosine families. See e.g. [3, Chapter 8] or [17, 24]. For the case of $L^p(\Omega)$, see e.g. [30, 42]. We refer to the book of Brezis [9, Section 10.3 and p.346] for some comments about the L^p-theory of the wave equation.

The paper is organized as follows. In Section 2, we present some preliminaries on fractional derivatives, the Wright type functions and the Mittag-Leffler functions. In Section 3 we use the Laplace transform to motivate the introduction of the operator family which will be used in the sequel. Section 4 is devoted to the definition and several properties of the resolvent family $\mathbb{S}_\alpha^\beta$. In the short Section 5 we characterize the resolvent family $\mathbb{S}_\alpha^\beta$ through the regularized fractional Cauchy problem. The homogeneous (fractional) abstract Cauchy problem is solved in Section 6 . The conditions on the initial data that ensure solvability of the problem agree with the classical cases $\alpha = 2$. We take up the inhomogeneous (fractional) abstract Cauchy problem in Section 7. We are able to deal satisfactorily with this problem under natural conditions on the initial data and the inhomogeneity. The results obtained in the case $\alpha = 2$ corresponding to integrated cosine families seem to be new. In fact, we are able to deal with the full range $1 < \alpha \le 2$. In the final Section 8 we present various examples of problems that can be handled with the results obtained.

2. Preliminaries

The algebra of bounded linear operators on a Banach space X will be denoted by $\mathcal{L}(X)$, the resolvent set of a linear operator A by $\rho(A)$. We denote by g_α the function $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0, \alpha > 0$, where Γ is the usual gamma function. It will be convenient to write $g_0 := \delta_0$, the Dirac measure concentrated at 0. Note the semigroup property:

$$g_{\alpha+\beta} = g_\alpha * g_\beta, \quad \alpha, \beta \ge 0.$$

The Riemann-Liouville fractional integral of order $\alpha > 0$, of a locally integrable function $u : [0, \infty) \to X$ is given by:

$$I_t^\alpha u(t) := (g_\alpha * u)(t) := \int_0^t g_\alpha(t-s)u(s)ds.$$

The Caputo fractional derivative of order $\alpha > 0$ of a function u is defined by

$$\mathbb{D}_t^\alpha u(t) := I_t^{m-\alpha} u^{(m)}(t) = \int_0^t g_{m-\alpha}(t-s) u^{(m)}(s) ds$$

where $m := \lceil \alpha \rceil$ is the smallest integer greatest than or equal to α, $u^{(m)}$ is the m^{th}-order distributional derivative of $u(\cdot)$, under appropriate assumptions. Then, when $\alpha = n$ is a natural number, we get $\mathbb{D}_t^n := \frac{d^n}{dt^n}$. In relation to the Riemann-Liouville fractional derivative of order α, namely D_t^α, we have:

$$\mathbb{D}_t^\alpha f(t) = D_t^\alpha \Big(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t) \Big), \quad t > 0, \tag{2.1}$$

where $m := \lceil \alpha \rceil$ has been defined above, and for a locally integrable function $u : [0, \infty) \to X$,

$$D_t^\alpha u(t) := \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s) u(s)\, ds, \quad t > 0.$$

The Laplace transform of a locally integrable function $f : [0, \infty) \to X$ is defined by

$$\mathcal{L}(f)(\lambda) := \widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) dt = \lim_{R \to \infty} \int_0^R e^{-\lambda t} f(t)\, dt,$$

provided the integral converges for some $\lambda \in \mathbb{C}$. If for example f is exponentially bounded, that is, there exist $M \geq 0$ and $\omega \geq 0$ such that $\|f(t)\| \leq M e^{\omega t}$, $t \geq 0$, then the integral converges absolutely for $\mathrm{Re}(\lambda) > \omega$ and defines an analytic function there. The most general existence theorem for the Laplace transform in the vector-valued setting is given by [3, Theorem 1.4.3].

Regarding the fractional derivative, we have for $\alpha > 0$ and $m := \lceil \alpha \rceil$, the following important properties:

$$\widehat{\mathbb{D}_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{m-1} \lambda^{\alpha-k-1} f^{(k)}(0), \tag{2.2}$$

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) \lambda^{m-1-k}.$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i \arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

Next, we recall some useful properties of convolutions that will be frequently used throughout the paper. For every $f \in C([0, \infty); X)$, $k \in \mathbb{N}$, $\alpha \geq 0$ we have that for every $t \geq 0$,

$$\frac{d^k}{dt^k} [(g_{k+\alpha} * f)(t)] = (g_\alpha * f)(t). \tag{2.3}$$

Let $f \in C([0, \infty); X) \cap C^1([0, \infty); X)$. Then for every $\alpha > 0$ and $t \geq 0$,

$$\frac{d}{dt} [(g_\alpha * f)(t)] = g_\alpha(t) f(0) + (g_\alpha * f')(t). \tag{2.4}$$

Let $k \in \mathbb{N}$. If $u \in C^{k-1}([0,\infty);X)$ and $v \in C^k([0,\infty);X)$, then for every $t \geq 0$,

$$\begin{aligned}\frac{d^k}{dt^k}\left[(u*v)(t)\right] &= \sum_{j=0}^{k-1} u^{(k-1-j)}(t)v^{(j)}(0) + (u*v^{(k)})(t)\\ &= \sum_{j=0}^{k-1} \frac{d^{k-1}}{dt^{k-1}}\Big[(g_j*u)(t)v^{(j)}(0)\Big] + (u*v^{(k)})(t).\end{aligned} \tag{2.5}$$

The Mittag-Leffler function (see e.g. [22, 23, 44, 46]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\int_{Ha} e^{\mu}\frac{\mu^{\alpha-\beta}}{\mu^\alpha - z}d\mu, \quad \alpha>0,\ \beta\in\mathbb{C},\ z\in\mathbb{C}, \tag{2.6}$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counterclockwise. The Laplace transform of the Mittag-Leffler function is given by ([44]):

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{k!\lambda^{\alpha-\beta}}{(\lambda^\alpha \mp \omega)^{k+1}}, \quad \mathrm{Re}(\lambda) > |\omega|^{1/\alpha}.$$

Using this formula, we obtain for $0<\alpha\leq 2$:

$$\mathbb{D}_t^\alpha E_{\alpha,1}(zt^\alpha) = zE_{\alpha,1}(zt^\alpha), \quad t>0, z\in\mathbb{C}, \tag{2.7}$$

that is, for every $z \in \mathbb{C}$, the function $u(t) := E_{\alpha,1}(zt^\alpha)$ is a solution of the scalar valued problem

$$\mathbb{D}_t^\alpha u(t) = zu(t),\ \ t>0,\ 1<\alpha\leq 2.$$

In addition, one has the identity

$$\frac{d}{dt}E_{\alpha,1}(zt^\alpha) = zt^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha).$$

To see this, it is sufficient to write

$$\mathcal{L}\left(t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha)\right)(\lambda) = \frac{1}{\lambda^\alpha - z} = \frac{1}{z}\big[\lambda\frac{\lambda^{\alpha-1}}{\lambda^\alpha - z} - 1\big],$$

and invert the Laplace transform. Letting $v(t) := E_{\alpha,1}(zt^\alpha)x$, $t>0$, $x\in X$, we have that

$$v(t) = g_1(t)x + z(g_\alpha * v)(t). \tag{2.8}$$

By [44, Formula (1.135)] (or [8, Formula (2.9)]), if $\omega \geq 0$ is a real number, then there exist some constants $C_1, C_2 \geq 0$ such that

$$E_{\alpha,1}(\omega t^\alpha) \leq C_1 e^{t\omega^{1/\alpha}} \ \text{ and } \ E_{\alpha,\alpha}(\omega t^\alpha) \leq C_2 e^{t\omega^{1/\alpha}}, \quad t\geq 0,\ \alpha\in(0,2) \tag{2.9}$$

and the estimates in (2.9) are sharp. Recall the definition of the Wright type function [23, Formula (28)] (see also [44, 46, 50]):

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)} = \frac{1}{2\pi i}\int_\gamma \mu^{\alpha-1}e^{\mu - z\mu^\alpha}d\mu, \quad 0<\alpha<1, \tag{2.10}$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. This has sometimes also been called the Mainardi function. By [8, p.14] or [23], $\Phi_\alpha(t)$ is a probability density function, that is,

$$\Phi_\alpha(t) \geq 0, \quad t>0; \quad \int_0^\infty \Phi_\alpha(t)dt = 1,$$

and its Laplace transform is the Mittag-Leffler function in the whole complex plane. We also have that $\Phi_\alpha(0) = \frac{1}{\Gamma(1-\alpha)}$. Concerning the Laplace transform of the Wright type functions, the following identities hold:

$$e^{-\lambda^\alpha s} = \mathcal{L}\Big(\alpha \frac{s}{t^{\alpha+1}} \Phi_\alpha(st^{-\alpha})\Big)(\lambda), \quad 0 < \alpha < 1, \tag{2.11}$$

$$\lambda^{\alpha-1} e^{-\lambda^\alpha s} = \mathcal{L}\Big(\frac{1}{t^\alpha} \Phi_\alpha(st^{-\alpha})\Big)(\lambda), \quad 0 < \alpha < 1. \tag{2.12}$$

See [23, Formulas (40) and (42)] and [8, Formula (3.10)]. We notice that the Laplace transform formula (2.11) was formerly first given by Pollard and Mikusinski (see [23] and references therein).

The following formula on the moments of the Wright function will be useful:

$$\int_0^\infty x^p \Phi_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)}, \quad p+1 > 0,\ 0 < \alpha < 1. \tag{2.13}$$

The preceding formula (2.13) is derived from the representation (2.10) and can be found in [23]. For more details on the Wright type functions, we refer to the papers [8, 23, 38, 50] and the references therein. We note that the Wright functions have been used by Bochner to construct fractional powers of semigroup generators (see e.g. [51, Chapter IX]).

3. Motivation

In this section we discuss heuristically the solvability of the fractional order Cauchy problem (1.2). We proceed through the use of the Laplace transform and derive some representation formulas that will serve as motivation for the theoretical framework of the subsequent sections.

Let $1 < \alpha \le 2$ and suppose u satisfies (1.2) and that there exist some constants $M, \omega \ge 0$ such that $\|(g_1 * u)(t)\| \le Me^{\omega t}$, $t > 0$. We rewrite the fractional differential equation in integral form as:

$$u(t) = A(g_\alpha * u)(t) + (g_\alpha * f)(t) + u(0) + tu'(0), \quad t > 0. \tag{3.1}$$

Suppose also that $(g_1 * f)(t)$ is exponentially bounded. Taking the Laplace transform in both sides of (3.1) and assuming that $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ we have

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} u(0) + \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} u'(0) + (\lambda^\alpha - A)^{-1} \widehat{f}(\lambda), \tag{3.2}$$

for $\operatorname{Re}(\lambda) > \omega$. Now we assume that A is the generator of an exponentially bounded β-times integrated cosine family $(C_\beta(t))$ on X for some $\beta \ge 0$, and denote by $(S_\beta(t))$ the associated $(\beta+1)$-times integrated cosine family (or β-times integrated sine family), namely, $S_\beta(t)x = \int_0^t C_\beta(s)x ds$, $t \ge 0$. Then by definition there exist some constants $\omega, M \ge 0$ such that $\|C_\beta(t)x\| \le Me^{\omega t}\|x\|$, $x \in X$, $t > 0$, $\{\lambda^2 \in \mathbb{C} : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda(\lambda^2 - A)^{-1}x = \lambda^\beta \int_0^\infty e^{-\lambda t} C_\beta(t)x\, dt = \lambda^{\beta+1} \int_0^\infty e^{-\lambda t} S_\beta(t)x dt,$$

for $\mathrm{Re}(\lambda) > \omega$, $x \in X$. Substituting the above expression into (3.2) we arrive at

$$
\begin{aligned}
\widehat{u}(\lambda) =& \lambda^{\alpha-1}\lambda^{\frac{\alpha\beta}{2}-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda^{\frac{\alpha}{2}}t}C_\beta(t)u(0)\,dt \\
&+ \lambda^{\alpha-2}\lambda^{\frac{\alpha\beta}{2}-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda^{\frac{\alpha}{2}}t}C_\beta(t)u'(0)dt + \lambda^{\frac{\alpha\beta}{2}-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda^{\frac{\alpha}{2}}t}C_\beta(t)\widehat{f}(\lambda)dt \\
=& \lambda^{\frac{\alpha}{2}-1}\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})C_\beta(t)u(0)\,dsdt \\
&+ \lambda^{\frac{\alpha}{2}-2}\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}e^{-\lambda s}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})C_\beta(t)u'(0)\,dsdt \\
&+ \lambda^{\frac{\alpha\beta}{2}-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}e^{-\lambda s}\Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})C_\beta(t)ds\widehat{f}(\lambda)dt \\
=& \lambda^{\frac{\alpha}{2}-1}\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})C_\beta(t)u(0)dtds \\
&+ \lambda^{\frac{\alpha}{2}-2}\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}e^{-\lambda s}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})C_\beta(t)u'(0)dtds \\
&+ \lambda^{\frac{\alpha\beta}{2}-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}e^{-\lambda s}\Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})C_\beta(t)\widehat{f}(\lambda)dtds,
\end{aligned}
\tag{3.3}
$$

where we have used the Laplace transform formula (2.11) and Fubini's theorem. Letting

$$
R_\alpha^\beta(t)x := \int_0^\infty \frac{\alpha s}{2t^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})C_\beta(s)xds, \quad t > 0,
$$

it follows from (3.3) that

$$
\widehat{u}(\lambda) = \lambda^{\frac{\alpha\beta}{2}}(\widehat{g_{1-\frac{\alpha}{2}} * R_\alpha^\beta})(\lambda)u(0) + \lambda^{\frac{\alpha\beta}{2}}(\widehat{g_{2-\frac{\alpha}{2}} * R_\alpha^\beta})u'(0) + \lambda^{\frac{\alpha\beta}{2}}(\widehat{g_{\frac{\alpha}{2}} * R_\alpha^\beta} * f)(\lambda).
$$

If we use instead the associated "sine" function $(S_\beta(t))$, we obtain the following representation

$$
\begin{aligned}
\widehat{u}(\lambda) =& \lambda^{\frac{\alpha\beta}{2}}\lambda^{\alpha-1}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})S_\beta(t)u(0)dtds \\
&+ \lambda^{\frac{\alpha\beta}{2}}\lambda^{\alpha-2}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})S_\beta(t)u'(0)dtds \\
&+ \lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{2s^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(ts^{-\frac{\alpha}{2}})S_\beta(t)\widehat{f}(\lambda)\,dt\,ds.
\end{aligned}
\tag{3.4}
$$

From this and using the uniqueness theorem for the Laplace transform, we have the following:

$$
\begin{gathered}
(g_{\frac{\alpha}{2}} * R_\alpha^\beta)(t)x = \int_0^\infty \frac{\alpha s}{2t^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})S_\beta(s)x\,ds, \quad t > 0, \\
(g_{1-\frac{\alpha}{2}} * R_\alpha^\beta)(t)x = D_t^{\alpha-1}(g_{\frac{\alpha}{2}} * R_\alpha^\beta)(t)x, \quad t > 0, \\
(g_{2-\frac{\alpha}{2}} * R_\alpha^\beta)(t)x = (g_{2-\alpha} * g_{\frac{\alpha}{2}} * R_\alpha^\beta)(t)x, \quad t > 0.
\end{gathered}
$$

In the next section we will take inspiration from the above heuristics to define and study the regularity properties of resolvent families associated with Problem (1.2). We will also deal with the case when there is an underlying exponentially bounded integrated cosine family.

4. Resolvent families and their properties

The following two definitions are motivated by the discussion in Section 3.

Definition 4.1. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $1 < \alpha \le 2, \beta \ge 0$. We say that A is the generator of an $(\alpha,\alpha)^\beta$-resolvent family if there exists a strongly continuous function $\mathbb{P}_\alpha^\beta : [0,\infty) \to \mathcal{L}(X)$ such that $\|(g_1 * \mathbb{P}_\alpha^\beta)(t)x\| \le Me^{\omega t}\|x\|$, $x \in X$, $t \ge 0$, for some constants $M, \omega \ge 0$, $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, and

$$(\lambda^\alpha - A)^{-1}x = \lambda^{\frac{\alpha\beta}{2}} \int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^\beta(t)x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In this case, $\mathbb{P}_\alpha^\beta$ is called the $(\alpha,\alpha)^\beta$-resolvent family generated by A.

Definition 4.2. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $1 < \alpha \le 2, \beta \ge 0$. We call A the generator of an $(\alpha,1)^\beta$-resolvent family if there exists a strongly continuous function $\mathbb{S}_\alpha^\beta : [0,\infty) \to \mathcal{L}(X)$ such that $\|(g_1 * \mathbb{S}_\alpha^\beta)(t)x\| \le Me^{\omega t}\|x\|$, $x \in X$, $t \ge 0$, for some $M, \omega \ge 0$, $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{\frac{\alpha\beta}{2}} \int_0^\infty e^{-\lambda t}\mathbb{S}_\alpha^\beta(t)x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In this case, $\mathbb{S}_\alpha^\beta$ is called the $(\alpha,1)^\beta$-resolvent family generated by A.

We will say that $\mathbb{P}_\alpha^\beta$ (resp. $\mathbb{S}_\alpha^\beta$) is exponentially bounded if there exist some constants $M, \omega \ge 0$ such that $\|\mathbb{P}_\alpha^\beta(t)\| \le Me^{\omega t}$, $\forall t \ge 0$, (resp. $\|\mathbb{S}_\alpha^\beta(t)\| \le Me^{\omega t}$, $\forall t \ge 0$).

It follows from the uniqueness theorem for the Laplace transform that an operator A can generate at most one $(\alpha,1)^\beta$ (resp. $(\alpha,\alpha)^\beta$)-resolvent family for given parameters $1 < \alpha \le 2$ and $\beta \ge 0$.

We shall write $(\alpha,1)$ and (α,α) for $(\alpha,1)^0$ and $(\alpha,\alpha)^0$ respectively. Before we give some properties of the above resolvent families, we need the following preliminary result.

Lemma 4.3. *Let $f : [0,\infty) \to X$ be such that there exist some constants $M \ge 0$ and $\omega \ge 0$ such that $\|(g_1 * f)(t)\| \le Me^{\omega t}$, $t > 0$. Then for every $\alpha \ge 1$, there exist some constants $M_1 \ge 0$ and $\omega_1 \ge 0$ such that $\|(g_\alpha * f)(t)\| \le M_1 e^{\omega_1 t}$, $t > 0$.*

Proof. Assume that f satisfies the hypothesis of the lemma and let $\alpha \ge 1$. We just have to consider the case $\alpha > 1$. Then for every $t \ge 0$,

$$\begin{aligned}
\|(g_\alpha * f)(t)\| = \|(g_{\alpha-1} * g_1 * f)(t)\| &\le \int_0^t g_{\alpha-1}(s)Me^{\omega(t-s)}\, ds \\
&= Me^{\omega t}\int_0^t \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}e^{-\omega s}\, ds \\
&\le Me^{\omega t}\frac{t^{\alpha-1}}{\Gamma(\alpha)} \le M_1 e^{\omega_1 t},
\end{aligned}$$

for some constants $M_1, \omega_1 \ge 0$, and the proof is complete. $\square$

Remark 4.4. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $1 < \alpha \le 2, \beta \ge 0$.

(a) Using Lemma 4.3 (this is used to show the exponential boundedness) we have the following result. If A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$, then it generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$ given by

$$\mathbb{P}_\alpha^\beta(t)x = (g_{\alpha-1} * \mathbb{S}_\alpha^\beta)(t)x, \quad t \geq 0, \;\; x \in X. \tag{4.1}$$

(b) By the uniqueness theorem for the Laplace transform, a $(2,2)$-resolvent family corresponds to the concept of sine family, while a $(2,1)$-resolvent family corresponds to a cosine family. Furthermore, a $(2,1)^\beta$-resolvent family corresponds to the concept of exponentially bounded β-times integrated cosine family. Likewise, a $(2,2)^\beta$-resolvent family represents an exponentially bounded β-times integrated sine family. We refer to the monographs [3, 20] and the corresponding references for a study of the concepts of cosine and sine families and to [4] for an overview on the theory of integrated cosine and sine families. A systematic study in the fractional case is carried out in [8] for the case $\beta = 0$.

Some properties of $(\mathbb{P}_\alpha^\beta(t))$ and $(\mathbb{S}_\alpha^\beta(t))$ are included in the following lemmas. Their proof uses techniques from the general theory of (a,k)-regularized resolvent families [35] (see also [2, 8]). It will be of crucial use in the investigation of solutions of fractional order Cauchy problems in Sections 5, 6 and 7. The proof of the analogous results in the case of cosine families may be found in [3]. The corresponding result for the case $0 < \alpha \leq 1$ is included in [8, 28] for $\beta = 0$ and in [29] for $\beta \geq 0$. For the sake of completeness we include the full proof.

Lemma 4.5. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \leq 2, \beta \geq 0$ and assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Then the following properties hold:*

(a) $\mathbb{S}_\alpha^\beta(t)D(A) \subset D(A)$ *and* $A\mathbb{S}_\alpha^\beta(t)x = \mathbb{S}_\alpha^\beta(t)Ax$ *for all* $x \in D(A)$, $t \geq 0$.

(b) *For all* $x \in D(A)$,
$$\mathbb{S}_\alpha^\beta(t)x = g_{\frac{\alpha\beta}{2}+1}(t)x + \int_0^t g_\alpha(t-s)A\mathbb{S}_\alpha^\beta(s)x ds, \quad t \geq 0.$$

(c) *For all* $x \in X$, $(g_\alpha * \mathbb{S}_\alpha^\beta)(t)x \in D(A)$ *and*
$$\mathbb{S}_\alpha^\beta(t)x = g_{\frac{\alpha\beta}{2}+1}(t)x + A\int_0^t g_\alpha(t-s)\mathbb{S}_\alpha^\beta(s)x ds, \quad t \geq 0.$$

(d) $\mathbb{S}_\alpha^\beta(0) = g_{\frac{\alpha\beta}{2}+1}(0)$. *Thus,* $\mathbb{S}_\alpha^\beta(0) = I$ *if* $\beta = 0$ *and* $\mathbb{S}_\alpha^\beta(0) = 0$ *if* $\beta > 0$.

Proof. Let ω be as in Definition 4.2. Let $\lambda, \mu > \omega$ and $x \in D(A)$. Then $x = (I - \mu^{-\alpha}A)^{-1}y$ for some $y \in X$. Since $(I-\mu^{-\alpha}A)^{-1}$ and $(I-\lambda^{-\alpha}A)^{-1}$ are bounded and commute, and since the operator A is closed, we obtain from the definition of $\mathbb{S}_\alpha^\beta$ that,

$$\begin{aligned}
\widehat{\mathbb{S}}_\alpha^\beta(\lambda)x &= \int_0^\infty e^{-\lambda t}\mathbb{S}_\alpha^\beta(t)x\, dt \\
&= \widehat{\mathbb{S}}_\alpha^\beta(\lambda)(I-\mu^{-\alpha}A)^{-1}y \\
&= \lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}\lambda^{-\alpha}(I-\lambda^{-\alpha}A)^{-1}(I-\mu^{-\alpha}A)^{-1}y \\
&= (I-\mu^{-\alpha}A)^{-1}\lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}\lambda^{-\alpha}(I-\lambda^{-\alpha}A)^{-1}y \\
&= (I-\mu^{-\alpha}A)^{-1}\lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}y
\end{aligned}$$

$$
\begin{aligned}
&= (I-\mu^{-\alpha}A)^{-1}\widehat{\mathbb{S}}_\alpha^\beta(\lambda)y \\
&= \int_0^\infty e^{-\lambda t}(I-\mu^{-\alpha}A)^{-1}\mathbb{S}_\alpha^\beta(t)y\,dt.
\end{aligned}
$$

By the uniqueness theorem for the Laplace transform and by continuity, we obtain

$$
\mathbb{S}_\alpha^\beta(t)x = (I-\mu^{-\alpha}A)^{-1}\mathbb{S}_\alpha^\beta(t)y = (I-\mu^{-\alpha}A)^{-1}\mathbb{S}_\alpha^\beta(t)(I-\mu^{-\alpha}A)x, \quad \forall t \geq 0. \tag{4.2}
$$

It follows from (4.2) that $\mathbb{S}_\alpha^\beta(t)x \in D(A)$. Hence, $\mathbb{S}_\alpha^\beta(t)D(A) \subset D(A)$ for every $t \geq 0$. It follows also from (4.2) that $A\mathbb{S}_\alpha^\beta(t)x = \mathbb{S}_\alpha^\beta(t)Ax$ for all $x \in D(A)$ and $t \geq 0$ and we have shown the assertion (a).

Next, let $x \in D(A)$. Using the convolution theorem, we get that

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}g_{\frac{\alpha\beta}{2}+1}(t)x\,dt &= \lambda^{-\frac{\alpha\beta}{2}-1}x = \lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}(I-\lambda^{-\alpha}A)x \\
&= \widehat{\mathbb{S}}_\alpha^\beta(\lambda)(I-\lambda^{-\alpha}A)x = \widehat{\mathbb{S}}_\alpha^\beta(\lambda)x - \lambda^{-\alpha}\widehat{S}_\alpha^\beta(\lambda)Ax \\
&= \int_0^\infty e^{-\lambda t}\Big[\mathbb{S}_\alpha^\beta(t)x - \int_0^t g_\alpha(t-s)\mathbb{S}_\alpha^\beta(s)Ax\,ds\Big]\,dt.
\end{aligned}
$$

By the uniqueness theorem for the Laplace transform we obtain the assertion (b).

Next, let $\lambda \in \rho(A)$ be fixed, $x \in X$ and set $y := (\lambda - A)^{-1}x \in D(A)$. Let $z := (g_\alpha * \mathbb{S}_\alpha^\beta)(t)x$, $t \geq 0$. We have to show that $z \in D(A)$ and $Az = \mathbb{S}_\alpha^\beta(t)x - g_{\frac{\alpha\beta}{2}+1}(t)x$. Using part (b) we obtain that

$$
\begin{aligned}
z =&(\lambda - A)(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y = \lambda(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y - A(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y \\
=&\lambda(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y - (\mathbb{S}_\alpha^\beta(t)y - g_{\frac{\alpha\beta}{2}+1}(t)y) \in D(A).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Az =&\lambda A(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y - A\mathbb{S}_\alpha^\beta(t)y + g_{\frac{\alpha\beta}{2}+1}(t)Ay \\
=&\lambda(g_\alpha * A\mathbb{S}_\alpha^\beta)(t)y - \mathbb{S}_\alpha^\beta(t)Ay + g_{\frac{\alpha\beta}{2}+1}(t)(\lambda y - x) \\
=&\lambda(g_\alpha * A\mathbb{S}_\alpha^\beta)(t)y - \mathbb{S}_\alpha^\beta(t)(\lambda y - x) + g_{\frac{\alpha\beta}{2}+1}(t)(\lambda y - x) \\
=&\lambda\Big[(g_\alpha * A\mathbb{S}_\alpha^\beta)(t)y - \mathbb{S}_\alpha^\beta(t)y + g_{\frac{\alpha\beta}{2}+1}(t)y\Big] + \mathbb{S}_\alpha^\beta(t)x - g_{\frac{\alpha\beta}{2}+1}(t)x \\
=&\mathbb{S}_\alpha^\beta(t)x - g_{\frac{\alpha\beta}{2}+1}(t)x,
\end{aligned}
$$

and we have shown part (c).

Finally, it follows from the strong continuity of $\mathbb{S}_\alpha^\beta(t)$ on $[0,\infty)$ and from the assertion (c) that $\mathbb{S}_\alpha^\beta(0)x = g_{\frac{\alpha\beta}{2}+1}(0)x$ for every $x \in X$. This implies the properties in part (d) and the proof is finished. □

The corresponding result for the family $\mathbb{P}_\alpha^\beta$ is given in the following lemma. Its proof runs similar to the proof of Lemma 4.5 and we shall omit it.

Lemma 4.6. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \leq 2, \beta \geq 0$ and assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$. Then the following properties hold.*

(a) *$\mathbb{P}_\alpha^\beta(t)D(A) \subset D(A)$ and $A\mathbb{P}_\alpha^\beta(t)x = \mathbb{P}_\alpha^\beta(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*

(b) *For all $x \in D(A)$, $\mathbb{P}_\alpha^\beta(t)x = g_{\alpha(\frac{\beta}{2}+1)}(t)x + \int_0^t g_\alpha(t-s)A\mathbb{P}_\alpha^\beta(s)xds$, $t \geq 0$.*

(c) *For all $x \in X$, $(g_\alpha * \mathbb{P}_\alpha^\beta)(t)x \in D(A)$ and $\mathbb{P}_\alpha^\beta(t)x = g_{\alpha(\frac{\beta}{2}+1)}(t)x + A\int_0^t g_\alpha(t-s)\mathbb{P}_\alpha^\beta(s)xds$, $t \geq 0$.*

(d) $\mathbb{P}_\alpha(0) = g_{\alpha(\frac{\beta}{2}+1)}(0) = 0$.

Remark 4.7. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \le 2$ and $\beta \ge 0$.

(i) If A generates an $(\alpha,1)^0 = (\alpha,1)$-resolvent family $\mathbb{S}_\alpha$, then it follows from Lemma 4.5 (c) that $D(A)$ is necessarily dense in X.

(ii) We notice that if A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ and $D(A)$ is dense in X then this does not necessarily imply that $\beta = 0$. Some examples will be given in Section 8.

(iii) The examples presented below in Corollary 4.15 show that in general ($\beta > 0$) the domain of A is not necessarily dense in X.

A family $\mathbb{S}(t)$ on X is called non-degenerate if whenever we have $\mathbb{S}(t)x = 0$ for all $t \in [0,\tau]$ (for some fixed $\tau > 0$), then it follows that $x = 0$. It follows from Lemma 4.5 and Lemma 4.6 that the families $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$ are non-degenerate. We have the following description of the generator A of the resolvent family $\mathbb{S}_\alpha^\beta$. We refer to [3, Lemma 3.2.2] for related results in the case of integrated semigroups and [3, Proposition 3.14.5] in the case of cosine families. The corresponding result for the case $0 < \alpha \le 1$ and $\beta \ge 0$ is contained in [29, Proposition 6.8] which was proved by using the Laplace transform. Here, we provide an alternative proof which does not use the Laplace transform.

Proposition 4.8. *Let A be a closed linear operator on a Banach space X with domain $D(A)$. Let $1 < \alpha \le 2$, $\beta \ge 0$ and assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Then*

$$A = \{(x,y) \in X \times X,\ \ \mathbb{S}_\alpha^\beta(t)x = g_{\frac{\alpha\beta}{2}+1}(t)x + (g_\alpha * \mathbb{S}_\alpha^\beta)(t)y,\ \forall t > 0\}. \tag{4.3}$$

Proof. First we notice that since the $(\alpha,1)^\beta$- resolvent family $\mathbb{S}_\alpha^\beta$ is non-degenerate, the right hand side of (4.3) defines a single-valued operator. Next, let $x, y \in X$. We have to show that $x \in D(A)$ and $Ax = y$ if and only if

$$\mathbb{S}_\alpha^\beta(t)x = g_{\frac{\alpha\beta}{2}+1}(t)x + (g_\alpha * \mathbb{S}_\alpha^\beta)(t)y,\ \forall t > 0. \tag{4.4}$$

Indeed, let $x \in D(A)$ and assume that $Ax = y$. Since A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ and $Ax = y$, then (4.4) follows from Lemma 4.5. Conversely, let $x, y \in X$ and assume that (4.4) holds. Let $\lambda \in \rho(A)$. It follows from (4.4) and Lemma 4.5 that for all $t \in [0,\tau]$,

$$\begin{aligned}(\lambda - A)^{-1}(g_\alpha * \mathbb{S}_\alpha^\beta)(t)y &= (\lambda - A)^{-1}A(g_\alpha * \mathbb{S}_\alpha^\beta)(t)x \\ &= -(g_\alpha * \mathbb{S}_\alpha^\beta)(t)x + \lambda(\lambda - A)^{-1}(g_\alpha * \mathbb{S}_\alpha^\beta)(t)x.\end{aligned}$$

This implies

$$(g_\alpha * \mathbb{S}_\alpha^\beta)(t)\left[(\lambda - A)^{-1}y + x - \lambda(\lambda - A)^{-1}x\right] = 0.$$

Since $\mathbb{S}_\alpha^\beta$ is non-degenerate, we have that $(\lambda - A)^{-1}y + x - \lambda(\lambda - A)^{-1}x = 0$ and this implies that $x \in D(A)$ and $Ax = y$. The proof is finished. □

Lemma 4.9. *Let A be a closed linear operator on a Banach space X and let $1 < \alpha \le 2, \beta \ge 0$. Assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Then for every $x \in D(A)$ the mapping $t \mapsto \mathbb{S}_\alpha^\beta(t)x$ is differentiable on $(0,\infty)$ and*

$$(\mathbb{S}_\alpha^\beta)'(t)x = g_{\frac{\alpha\beta}{2}}(t)x + \mathbb{P}_\alpha^\beta(t)Ax, \quad t > 0. \tag{4.5}$$

Proof. Let $x \in D(A)$. Then it is clear that the right-hand side of (4.5) belongs to $C((0,\infty),\mathcal{L}(X))$. Taking the Laplace transform and using the fact that $\mathbb{S}_\alpha^\beta(0)=0$, we get that for $\mathrm{Re}(\lambda)>\omega$ (where ω is the real number from the definition of $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$),

$$\widehat{(\mathbb{S}_\alpha^\beta)'}(\lambda)x=\lambda\widehat{\mathbb{S}_\alpha^\beta}(\lambda)x=\lambda\lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}x=\lambda^{-\frac{\alpha\beta}{2}}\lambda^\alpha(\lambda^\alpha-A)^{-1}x.$$

On the other hand, for $\mathrm{Re}(\lambda)>\omega$,

$$\begin{aligned}\widehat{g_{\frac{\alpha\beta}{2}}}(\lambda)x+\widehat{\mathbb{P}_\alpha^\beta}(\lambda)Ax&=\lambda^{-\frac{\alpha\beta}{2}}x+\lambda^{-\frac{\alpha\beta}{2}}(\lambda^\alpha-A)^{-1}Ax\\&=\lambda^{-\frac{\alpha\beta}{2}}x-\lambda^{-\frac{\alpha\beta}{2}}(\lambda^\alpha-A)^{-1}(\lambda^\alpha-A-\lambda^\alpha)x\\&=\lambda^{-\frac{\alpha\beta}{2}}x-\lambda^{-\frac{\alpha\beta}{2}}x+\lambda^{-\frac{\alpha\beta}{2}}\lambda^\alpha(\lambda^\alpha-A)^{-1}x\\&=\lambda^{-\frac{\alpha\beta}{2}}\lambda^\alpha(\lambda^\alpha-A)^{-1}x.\end{aligned}$$

By the uniqueness theorem for the Laplace transform and continuity of the right-hand side of (4.5), we conclude that the identity (4.5) holds. $\square$

Next, we give the principle of extrapolation of the families $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$ in terms of the parameter β.

Proposition 4.10. *Let A be a closed linear operator on a Banach space X and let $1<\alpha\le 2,\beta\ge 0$. Then the following assertions hold.*

(a) *If A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$, then it generates an $(\alpha,\alpha)^{\beta'}$-resolvent family $\mathbb{P}_\alpha^{\beta'}$ for every $\beta'\ge\beta$ and for every $x\in X$,*
$$\mathbb{P}_\alpha^{\beta'}(t)x=(g_{\alpha\frac{\beta'-\beta}{2}}*\mathbb{P}_\alpha^\beta)(t)x,\quad\forall t\ge 0. \tag{4.6}$$

(b) *If A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$, then it generates an $(\alpha,1)^{\beta'}$-resolvent family $\mathbb{S}_\alpha^{\beta'}$ for every $\beta'\ge\beta$ and for every $x\in X$,*
$$\mathbb{S}_\alpha^{\beta'}(t)x=(g_{\alpha\frac{\beta'-\beta}{2}}*\mathbb{S}_\alpha^\beta)(t)x,\quad\forall t\ge 0. \tag{4.7}$$

Proof. Let A be a closed linear operator on a Banach space X and let $1<\alpha\le 2,\beta\ge 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$. Then, by definition, there exists $\omega\ge 0$ such that $\{\lambda^\alpha:\mathrm{Re}(\lambda)>\omega\}\subset\rho(A)$ and

$$(\lambda^\alpha-A)^{-1}x=\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^\beta(t)xdt,\quad \mathrm{Re}(\lambda)>\omega,\quad x\in X. \tag{4.8}$$

Let $\beta'\ge\beta$ and let $\mathbb{P}_\alpha^{\beta'}$ be given in (4.6). Using Lemma 4.6 we have that for every $x\in X$ and $t\ge 0$,

$$\begin{aligned}\mathbb{P}_\alpha^{\beta'}(t)x=&(g_{\alpha\frac{\beta'-\beta}{2}}*\mathbb{P}_\alpha^\beta)(t)x=(g_{\alpha\frac{\beta'-\beta}{2}}*g_{\alpha(\frac{\beta}{2}+1)})(t)x+A\left(g_{\alpha\frac{\beta'-\beta}{2}}*g_\alpha*\mathbb{P}_\alpha^\beta\right)(t)x\\=&g_{\alpha(\frac{\beta'}{2}+1)}(t)x+A\left(g_{\alpha(\frac{\beta'-\beta}{2}+1)}*\mathbb{P}_\alpha^\beta\right)(t)x.\end{aligned}$$

Hence, $\mathbb{P}_\alpha^{\beta'}$ is strongly continuous from $[0,\infty)$ into $\mathcal{L}(X)$. By (4.6), we have that for every $x\in X$ and $t\ge 0$,

$$(g_1*\mathbb{P}_\alpha^{\beta'})(t)x=(g_{\alpha\frac{\beta'-\beta}{2}+1}*\mathbb{P}_\alpha^\beta)(t)x,$$

and since by hypothesis $\|(g_1*\mathbb{P}_\alpha^\beta)(t)x\|\le Me^{\omega t}\|x\|$, $x\in X$, $t\ge 0$, for some constants $M,\omega\ge 0$, it follows from Lemma 4.3 that there exist some constants

$M', \omega' \geq 0$ such that $\|(g_1 * P_\alpha^{\beta'})(t)x\| \leq M' e^{\omega' t}\|x\|$, $x \in X$, $t \geq 0$. Next, using (4.8), we have that for $\mathrm{Re}(\lambda) > \omega$, $x \in X$ and $\beta' \geq \beta$,

$$\begin{aligned}
(\lambda^\alpha - A)^{-1}x &= \lambda^{\frac{\alpha\beta}{2}} \int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^\beta(t)x dt = \lambda^{\frac{\alpha\beta'}{2}} \lambda^{\alpha\frac{\beta-\beta'}{2}} \int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^\beta(t)x dt \\
&= \lambda^{\frac{\alpha\beta'}{2}} \int_0^\infty e^{-\lambda s} g_{\alpha\frac{\beta-\beta'}{2}}(s)\, ds \int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^\beta(t)x dt \\
&= \lambda^{\frac{\alpha\beta'}{2}} \int_0^\infty e^{-\lambda t}(g_{\alpha\frac{\beta-\beta'}{2}} * \mathbb{P}_\alpha^\beta)(t)x dt = \lambda^{\frac{\alpha\beta'}{2}} \int_0^\infty e^{-\lambda t}\mathbb{P}_\alpha^{\beta'}(t)x dt.
\end{aligned}$$

Hence, A generates an $(\alpha, \alpha)^{\beta'}$-resolvent family $\mathbb{P}_\alpha^{\beta'}$ given by (4.6) and we have shown the assertion (a).

(b) The proof of this part follows the lines of the proof of part (a) where now we use Lemma 4.5. $\square$

The following example shows that a generation of an $(\alpha, 1)^\beta$ or $(\alpha, \alpha)^\beta$-resolvent family does not imply a generation of an $(\alpha', 1)^\beta$ or $(\alpha', \alpha')^\beta$-resolvent family for $\alpha' > \alpha > 1$. That is, an extrapolation property in terms of the parameter α does not always hold.

Example 4.11. Let $1 \leq p < \infty$ and let Δ_p be the realization of the Laplacian in $L^p(\mathbb{R}^N)$. It is well-known that Δ_p generates an analytic C_0-semigroup of contractions of angle $\pi/2$. Hence, for every $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\|(\lambda - \Delta_p)^{-1}\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\pi-\varepsilon}. \tag{4.9}$$

where for $0 < \gamma < \pi$, $\Sigma_\gamma := \{z \in \mathbb{C} : 0 < |arg(z)| < \gamma\}$. Let $\theta \in [0, \pi)$ and let the operator A_p on $L^p(\mathbb{R}^N)$ be given by $A_p := e^{i\theta}\Delta_p$. It follows from (4.9) that

$$\begin{aligned}
\|(\lambda - A_p)^{-1}\| &= \|(\lambda - e^{i\theta}\Delta_p)^{-1}\| = \|(\lambda e^{-i\theta} - \Delta_p)^{-1}\| \\
&\leq \frac{C}{|\lambda|}, \quad \lambda e^{-i\theta} \in \Sigma_{\pi-\varepsilon}.
\end{aligned} \tag{4.10}$$

Now, let $1 < \alpha < 2$. It follows from (4.10) that, if $\frac{\pi}{2} < \theta < \left(1 - \frac{\alpha}{4}\right)\pi$, then $\rho(A_p) \supset \Sigma_{\frac{\alpha\pi}{2}}$ and

$$\|(\lambda - A_p)^{-1}\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\frac{\alpha\pi}{2}}. \tag{4.11}$$

By [8, Proposition 3.8], the estimate (4.11) implies that A_p generates an $(\alpha, 1) = (\alpha, 1)^0$-resolvent family on $L^p(\mathbb{R}^N)$. Hence, by Proposition 4.10 (c), A_p generates an $(\alpha, 1)^\beta$-resolvent family on $L^p(\mathbb{R}^N)$ for any $\beta \geq 0$. But one can verify by inspection of the resolvent set of A_p that it does not generate an $(2, 1)^\beta$-resolvent family, that is a β-times integrated cosine family on $L^p(\mathbb{R}^N)$ for any $\beta \geq 0$. However, A_p does generates a bounded analytic semigroup.

Remark 4.12. In view of the asymptotic expansion of the Wright function (see e.g. [23, 50]), for a locally integrable function $f : [0, \infty) \to X$ which is exponentially bounded at infinity, and for any $0 < \sigma < 1$, the integral $\int_0^\infty \Phi_\sigma(\tau)f(\tau)\, d\tau$ converges. This property will be frequently used in the remainder of the article without any mention.

Concerning subordination of resolvent families we have the following preliminary result.

Lemma 4.13. *Let A be a closed linear operator on a Banach space X. Let $1 < \alpha \le 2$, $\beta \ge 0$. Then the following assertions hold.*

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}^\beta_\alpha$. Let $1 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and set

$$P(t)x := \sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)\mathbb{P}^\beta_\alpha(st^\sigma)xds, \quad t>0,\ x\in X. \tag{4.12}$$

Then $(g_1 * P)(t)x$ is exponentially bounded. Moreover, $(g_1*P)(t)x = \mathbb{P}(t)x$ where

$$\mathbb{P}(t)x := \int_0^\infty \frac{\sigma s}{t^{\sigma+1}}\Phi_\sigma(st^{-\sigma})(g_{\frac{1}{\sigma}} * \mathbb{P}^\beta_\alpha)(s)xds,\ t>0,\ x\in X. \tag{4.13}$$

(b) Assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}^\beta_\alpha$. Let $1<\alpha'<\alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and set

$$\mathbb{S}(t)x := \int_0^\infty \frac{1}{t^\sigma}\Phi_\alpha(st^{-\sigma})(g_{\frac{1}{\sigma}} * \mathbb{S}^\beta_\alpha)(s)xds, \quad t>0,\ x\in X. \tag{4.14}$$

Then $\mathbb{S}$ is exponentially bounded. Moreover, $\mathbb{S}(t)x = (g_1 * S)(t)x$ where

$$S(t)x = \int_0^\infty \Phi_\sigma(s)\mathbb{S}^\beta_\alpha(st^\sigma)x\,ds, \quad \forall t\ge 0,\ x\in X. \tag{4.15}$$

Proof. Let A, α and β be as in the statement of the lemma.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent famlily $\mathbb{P}^\beta_\alpha$ and let $1<\alpha'<\alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and $x\in X$. Let $P(t)$ be given by (4.12). By hypothesis, there exist $M,\omega \ge 0$ such that $\|(g_1 * \mathbb{P}^\beta_\alpha)(t)x\| \le Me^{\omega t}\|x\|$ for every $x\in X$, $t\ge 0$. We show that there exist some constants $M_1,\omega_1 \ge 0$ such that for every $x \in X$, $\|(g_1*P)(t)x\| \le M_1 e^{\omega_1 t}\|x\|$, $t\ge 0$. Using (4.12), Fubini's theorem, (2.13), (2.6) and (2.9), we get that for every $t\ge 0$ and $x\in X$,

$$\begin{aligned}
\|\int_0^t P(\tau)x\,d\tau\| &\le \int_0^\infty s\Phi_\sigma(s)\|\int_0^t \sigma\tau^{\sigma-1}\mathbb{P}^\beta_\alpha(s\tau^\sigma)x\,d\tau\|\,ds\\
&= \int_0^\infty \Phi_\sigma(s)\|\int_0^{st^\sigma}\mathbb{P}^\beta_\alpha(\tau)x\,d\tau\|\,ds\\
&\le M\|x\|\int_0^\infty \Phi_\sigma(s)e^{\omega st^\sigma}\,ds\\
&= M\|x\|\sum_{n=0}^\infty \frac{(\omega t^\sigma)^n}{n!}\int_0^\infty \Phi_\sigma(s)s^n\,ds\\
&\le M\|x\|\sum_{n=0}^\infty \frac{(\omega t^\sigma)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(\sigma n+1)}\\
&= M\|x\|\sum_{n=0}^\infty \frac{(\omega t^\sigma)^n}{\Gamma(\sigma n+1)} = M\|x\|E_{\sigma,1}(\omega t^\sigma)\\
&\le M_1 e^{t\omega^{\frac{1}{\sigma}}}\|x\|,
\end{aligned}$$

for some constant $M_1 \geq 0$. Taking the Laplace transform of (4.13) by using (2.11) and Fubini's theorem, we have that for Re $> \omega$ and $x \in X$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\mathbb{P}(t)x\,dt &= \int_0^\infty e^{-\lambda t}\int_0^\infty \frac{\sigma s}{t^{\sigma+1}}\Phi_\sigma(st^{-\sigma})(g_{\frac{1}{\sigma}} * \mathbb{P}_\alpha^\beta)(s)x ds\,dt\\ &= \int_0^\infty e^{-\lambda^\sigma s}(g_{\frac{1}{\sigma}} * \mathbb{P}_\alpha^\beta)(s)x ds = \lambda^{-1}\lambda^{-\frac{\alpha'\beta}{2}}(\lambda^{\alpha'} - A)^{-1}x.\end{aligned}$$

Similarly, we have that for Re $> \omega$ and $x \in X$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}(g_1 * P)(t)x\,dt =&\lambda^{-1}\int_0^\infty e^{-\lambda t}P(t)x\,dt\\ =&\lambda^{-1}\int_0^\infty e^{-\lambda t}\sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)\mathbb{P}_\alpha^\beta(st^\sigma)x\,ds\,dt\\ =&\lambda^{-1}\int_0^\infty \mathbb{P}_\alpha^\beta(\tau)x\int_0^\infty e^{-\lambda t}\frac{\sigma\tau}{t^{\sigma+1}}\Phi_\sigma(\tau t^{-\sigma})\,dt\,d\tau\\ =&\lambda^{-1}\int_0^\infty e^{-\tau\lambda^\sigma}\mathbb{P}_\alpha^\beta(\tau)x\,d\tau\\ =&\lambda^{-1}\lambda^{-\frac{\alpha'\beta}{2}}(\lambda^{\alpha'} - A)^{-1}x.\end{aligned}$$

By the uniqueness theorem for the Laplace transform and by continuity, we have that $(g_1 * P)(t)x = \mathbb{P}(t)x$ for all $t \geq 0$ and $x \in X$ and this completes the proof of part (a).

(b) Assume that A generates an $(\alpha, 1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ and let $1 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and $x \in X$. Then there exist some constants $M, \omega \geq 0$ such that $\|(g_1 * \mathbb{S}_\alpha^\beta)(t)x\| \leq Me^{\omega t}\|x\|$, $t \geq 0$. Since $\frac{1}{\sigma} > 1$, it follows from Lemma 4.3 that there exist some constants $M_1, \omega_1 \geq 0$ such that for every $t \geq 0$ and $x \in X$,

$$\|(g_{\frac{1}{\sigma}} * \mathbb{S}_\alpha^\beta)(t)x\| \leq M_1 e^{\omega_1 t}\|x\|. \tag{4.16}$$

Using (4.14), (2.13), (4.16), (2.6) and (2.9), we have that for every $x \in X$, $t > 0$,

$$\begin{aligned}\|S(t)x\| \leq& M_1\|x\|\int_0^\infty \frac{1}{t^\sigma}\Phi_\sigma(st^{-\sigma})e^{\omega_1 s}\,ds = M_1\|x\|\int_0^\infty \Phi_\sigma(\tau)e^{\omega_1\tau t^\sigma}\,d\tau\\ \leq& M_1\|x\|\sum_{n=0}^\infty \frac{(\omega_1 t^\sigma)^n}{n!}\int_0^\infty \Phi_\sigma(\tau)\tau^n\,d\tau\\ =& M_1\|x\|\sum_{n=0}^\infty \frac{(\omega_1 t^\sigma)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(\sigma n+1)}\\ \leq& M_1\|x\|\sum_{n=0}^\infty \frac{(\omega_1 t^\sigma)^n}{\Gamma(\sigma n+1)} = M_1 E_{\sigma,1}(\omega_1 t^\sigma)\|x\|\\ \leq& Me^{t\omega_1^{\frac{1}{\sigma}}}\|x\|,\end{aligned}$$

for some constant $M \geq 0$. This completes the proof. □

Next, we present the principle of subordination of the families $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$ in terms of the parameter α.

Theorem 4.14. *Let A be a closed linear operator on a Banach space X and let $1 < \alpha \leq 2, \beta \geq 0$. Then the following assertions hold.*

(a) *If A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}^\beta_\alpha$, then it generates an $(\alpha',\alpha')^\beta$-resolvent family $\mathbb{P}^\beta_{\alpha'}$ for every $1<\alpha'<\alpha$ and*

$$\mathbb{P}^\beta_{\alpha'}(t)x=\sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)\mathbb{P}^\beta_\alpha(st^\sigma)x\,ds,\ \ \forall t>0,\ x\in X, \tag{4.17}$$

where $\sigma:=\frac{\alpha'}{\alpha}$.

(b) *If A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}^\beta_\alpha$, then it generates an $(\alpha',1)^\beta$-resolvent family $\mathbb{S}^\beta_{\alpha'}$ for every $1<\alpha'<\alpha$ and*

$$\mathbb{S}^\beta_{\alpha'}(t)x=\int_0^\infty \Phi_\sigma(s)\mathbb{S}^\beta_\alpha(st^\sigma)x\,ds,\ \ \forall t\ge 0,\ x\in X,\ \ \text{where}\ \ \sigma:=\frac{\alpha'}{\alpha}. \tag{4.18}$$

Proof. Let A be a closed linear operator on a Banach space X and let $1<\alpha\le 2,\beta\ge 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}^\beta_\alpha$. Let $1<\alpha'<\alpha$ and let $\mathbb{P}^\beta_{\alpha'}$ be given by (4.17). Then it is clear that $\mathbb{P}^\beta_{\alpha'}$ is strongly continuous from $(0,\infty)$ into $\mathcal{L}(X)$. We show that $\mathbb{P}^\beta_{\alpha'}(t)$ is strongly continuous at 0. Since $\mathbb{P}^\beta_\alpha(t)\simeq g_{\alpha(\frac{\beta}{2}+1)}(t)=\frac{t^{\alpha(\frac{\beta}{2}+1)-1}}{\Gamma(\alpha(\frac{\beta}{2})+1}$ as $t\to 0$, we get from (4.17) that

$$\mathbb{P}^\beta_{\alpha'}(t)\simeq t^{\frac{\alpha'}{\alpha}-1}t^{\frac{\alpha'}{\alpha}\alpha(\frac{\beta}{2}+1)-\frac{\alpha'}{\alpha}}=t^{\alpha'(\frac{\beta}{2}+1)-1}\quad \text{as } t\to 0.$$

We have shown that $\mathbb{P}^\beta_{\alpha'}(t)$ is strongly continuous at 0. By Lemma 4.13(a), there exist some constants $M_1,\omega_1\ge 0$ such that $\|(g_1*\mathbb{P}^\beta_{\alpha'})(t)x\|\le M_1e^{\omega_1 t}$, $x\in X$, $t\ge 0$. Now, it follows from (4.8) and (2.11) that $\{\lambda^{\alpha'}:\ \operatorname{Re}(\lambda)>\omega\}\subset\rho(A)$ and for $\operatorname{Re}(\lambda)>\omega$, $x\in X$,

$$\begin{aligned}
(\lambda^{\alpha'}-A)^{-1}x=&\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda^\sigma t}\mathbb{P}^\beta_\alpha(t)xdt\\
=&\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\int_0^\infty \sigma\frac{s}{t^{\sigma+1}}\Phi_\sigma(st^{-\sigma})\mathbb{P}^\beta_\alpha(s)xds\,dt\\
=&\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)\mathbb{P}^\beta_\alpha(st^\sigma)xds\,dt\\
=&\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\mathbb{P}^\beta_{\alpha'}(t)x\,dt.
\end{aligned}$$

Hence, A generates an $(\alpha',\alpha')^\beta$-resolvent family $\mathbb{P}^{\beta'}_{\alpha'}$ given by (4.17) and we have shown part (a).

(b) Now assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}^\beta_\alpha$. Then by definition, there exists $\omega\ge 0$ such that $\{\lambda^\alpha:\ \operatorname{Re}(\lambda)>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}x=\lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda t}\mathbb{S}^\beta_\alpha(t)x\,dt,\ \operatorname{Re}(\lambda)>\omega,\ \forall x\in X. \tag{4.19}$$

Let $1<\alpha'<\alpha$ and let $\mathbb{S}^\beta_{\alpha'}$ be given by (4.18). Then it is clear that $\mathbb{S}^\beta_{\alpha'}$ is strongly continuous from $[0,\infty)$ into $\mathcal{L}(X)$. By Lemma 4.13(b), there exist some constants $M_1,\omega_1\ge 0$ such that for every $x\in X$, $\|(g_1*\mathbb{S}^\beta_{\alpha'})(t)x\|\le M_1e^{\omega_1 t}\|x\|$, $t\ge 0$. It follows from (4.19) and (2.12) that $\{\lambda^{\alpha'}:\ \operatorname{Re}(\lambda)>\omega\}\subset\rho(A)$ and for $\operatorname{Re}(\lambda)>\omega$, $x\in X$,

$$\lambda^{\alpha'-1}(\lambda^{\alpha'}-A)^{-1}x=\lambda^{\frac{\alpha'\beta}{2}}\lambda^{\sigma-1}\int_0^\infty e^{-\lambda^\sigma t}\mathbb{S}^\beta_\alpha(t)xdt$$

$$
\begin{aligned}
&=\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\int_0^\infty \frac{1}{t^\sigma}\Phi_\sigma(st^{-\sigma})\mathbb{S}_\alpha^\beta(s)xds\,dt\\
&=\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\int_0^\infty \Phi_\sigma(s)\mathbb{S}_\alpha^\beta(st^\sigma)x\,ds\,dt\\
&=\lambda^{\frac{\alpha'\beta}{2}}\int_0^\infty e^{-\lambda t}\mathbb{S}_{\alpha'}^\beta(t)x\,dt.
\end{aligned}
$$

Hence, A generates an $(\alpha',1)^\beta$-resolvent family $\mathbb{S}_{\alpha'}^{\beta'}$ given by (4.18). The proof of the theorem is finished. □

We have the following result as a corollary of the preceding theorem.

Corollary 4.15. *Let $1<\alpha\le 2$, $\beta\ge 0$ and let A be a closed linear operator on a Banach space X. If A generates an exponentially bounded $\beta-$times integrated cosine family $(C_\beta(t))$, then A generates an exponentially bounded $(\alpha,1)^\beta$-resolvent family $(\mathbb{S}_\alpha^\beta(t))$ given by*

$$
\mathbb{S}_\alpha^\beta(t)x=\int_0^\infty t^{-\frac{\alpha}{2}}\Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})C_\beta(s)xds=\int_0^\infty \Phi_{\frac{\alpha}{2}}(\tau)C_\beta(\tau t^{\frac{\alpha}{2}})xd\tau, \tag{4.20}
$$

for $t>0$, $x\in X$. In particular, it follows from the first representation formula in (4.20) that $(\mathbb{S}_\alpha^\beta(t))$ is analytic for $t>0$, and, from the second one, that $\mathbb{S}_\alpha^\beta(0)=C_\beta(0)$.

Let $(\mathbb{P}_\alpha^\beta(t))$ be the associated $(\alpha,\alpha)^\beta$-resolvent family generated by A which exists by Remark 4.4 (b). Then for every $x\in X$,

$$
\begin{aligned}
\mathbb{P}_\alpha^\beta(t)x&=\frac{\alpha}{2}\int_0^\infty \frac{s}{t^{\frac{\alpha}{2}+1}}\Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})C_\beta(s)xds\\
&=\frac{\alpha}{2}\int_0^\infty \frac{\tau}{t^{1-\frac{\alpha}{2}}}\Phi_{\frac{\alpha}{2}}(\tau)C_\beta(\tau t^{\frac{\alpha}{2}})xd\tau,
\end{aligned}
\tag{4.21}
$$

for $t>0$.

Proof. Let α, β and A be as in the statement of the theorem. The fact that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ and an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$ is a direct consequence of Theorem 4.14 since by hypothesis A generates a β-times integrated cosine family, that is a $(2,1)^\beta$-resolvent family, and a β-times integrated sine family, that is a $(2,2)^\beta$-resolvent family. The formulas (4.20) and (4.21) are the corresponding formulas (4.18) and (4.17), respectively, in Theorem 4.14. It remains to show that $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$ are exponentially bounded. By hypothesis, $(C_\beta(t))$ is exponentially bounded, that is, there exist some constants $M,\omega\ge 0$ such that $\|C_\beta(t)x\|\le Me^{\omega t}\|x\|$ for all $t\ge 0$, $x\in X$. Using (4.20), (2.13), (2.6) and (2.9), we have that for every $x\in X$, $t\ge 0$,

$$
\begin{aligned}
\|\mathbb{S}_\alpha^\beta(t)x\|\le&\int_0^\infty \Phi_{\frac{\alpha}{2}}(\tau)\|C_\beta(\tau t^{\frac{\alpha}{2}})x\|d\tau\le M\|x\|\int_0^\infty \Phi_{\frac{\alpha}{2}}(\tau)e^{\omega\tau t^{\frac{\alpha}{2}}}\,d\tau\\
\le& M\|x\|\sum_{n=0}^\infty\frac{(\omega t^{\frac{\alpha}{2}})^n}{n!}\int_0^\infty \Phi_{\frac{\alpha}{2}}(\tau)\tau^n\,d\tau=M\|x\|\sum_{n=0}^\infty\frac{(\omega t^{\frac{\alpha}{2}})^n}{n!}\frac{\Gamma(n+1)}{\Gamma(\frac{\alpha}{2}n+1)}\\
\le& M\|x\|\sum_{n=0}^\infty\frac{(\omega t^{\frac{\alpha}{2}})^n}{\Gamma(\frac{\alpha}{2}n+1)}=M\|x\|E_{\frac{\alpha}{2},1}(\omega t^{\frac{\alpha}{2}})\le M_1e^{t\omega^{\frac{2}{\alpha}}}\|x\|,
\end{aligned}
$$

for some constant $M_1\ge 0$ and we have shown that $\mathbb{S}_\alpha^\beta$ is exponentially bounded.

We note that $\mathbb{P}_\alpha^\beta$ is bounded in a neighborhood of $t=0^+$ by strong continuity on $[0,\infty)$. We show that $\mathbb{P}_\alpha^\beta$ is exponentially bounded away from 0. Indeed, using (4.21), (2.13), (2.6) and (2.9), for a fixed $\varepsilon>0$, we have that for every $t\ge\varepsilon$ and $x\in X$,

$$\begin{aligned}
\|\mathbb{P}_\alpha^\beta(t)x\| \le& M\int_0^\infty \frac{\tau}{t^{1-\frac{\alpha}{2}}}\Phi_{\frac{\alpha}{2}}(\tau)e^{\omega\tau t^{\frac{\alpha}{2}}}\|x\|d\tau \le M\|x\|\int_0^\infty \tau\Phi_{\frac{\alpha}{2}}(\tau)e^{\omega\tau t^{\frac{\alpha}{2}}}d\tau\\
\le& M\|x\|\sum_{n=0}^\infty \frac{(\omega t^{\frac{\alpha}{2}})^n}{n!}\int_0^\infty \Phi_{\frac{\alpha}{2}}(\tau)\tau^{n+1}d\tau\\
=& M\|x\|\sum_{n=0}^\infty \frac{(\omega t^{\frac{\alpha}{2}})^n}{n!}\frac{\Gamma(n+2)}{\Gamma(\frac{\alpha}{2}(n+1)+1)}\\
\le& M\|x\|\sum_{n=0}^\infty \frac{(\omega t^{\frac{\alpha}{2}})^n}{\Gamma(\frac{\alpha}{2}(n+1))} = M\|x\|E_{\frac{\alpha}{2},\frac{\alpha}{2}}(\omega t^{\frac{\alpha}{2}}) \le M_1 e^{t\omega^{\frac{2}{\alpha}}}\|x\|,
\end{aligned}$$

for some constant $M_1\ge 0$, and this completes the proof. $\square$

We notice that alternatively, one can also show that $\mathbb{P}_\alpha^\beta$ is exponentially bounded on $[0,\infty)$ by using the fact that $\mathbb{S}_\alpha^\beta$ is exponentially bounded and that $\mathbb{P}_\alpha^\beta(t)x = (g_{\alpha-1}*S_\alpha^\beta)(t)x$, $x\in X$, $t\ge 0$ (by Remark 4.4(a)).

If B generates an exponentially bounded β-times integrated group $(U_\beta(t))$, then $A=B^2$ generates an exponentially bounded β-times integrated cosine family $(C_\beta(t))$ given by $C_\beta(t)=\frac{U_\beta(t)+U_\beta(-t)}{2}$. Moreover, operators that satisfy the estimate (1.5) are generators of exponentially bounded integrated cosine families (see [32, Theorem 2.2.4] or [40]). The corresponding situation for integrated semigroups is treated in [3, Theorem 3.2.8].

Next, we show that we have a double subordination principle for the families $\mathbb{S}_\alpha^\beta$ and $\mathbb{P}_\alpha^\beta$ in terms of the parameters α and β.

Corollary 4.16. *Let A be a closed linear operator on a Banach space X and let $1<\alpha\le 2, \beta\ge 0$. Then the following assertions hold.*

(a) *If A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$, then it generates a $\frac{\beta}{2}$-times integrated semigroup $(T_{\frac{\beta}{2}}(t))$ such that $(g_1*T_{\frac{\beta}{2}})(t)$ is exponentially bounded and for every $x\in X$, and $t>0$,*
$$T_{\frac{\beta}{2}}(t)x = \sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)\mathbb{P}_\alpha^\beta(st^\sigma)x\,ds, \quad \textit{where} \quad \sigma := \frac{1}{\alpha}.$$

(b) *If A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$, then it generates a $\frac{\beta}{2}$-times integrated semigroup $(T_{\frac{\beta}{2}}(t))$ such that $(g_1*T_{\frac{\beta}{2}})(t)$ is exponentially bounded and for every $x\in X$, and $t\ge 0$,*
$$T_{\frac{\beta}{2}}(t)x = \int_0^\infty \Phi_\sigma(s)\mathbb{S}_\alpha^\beta(st^\sigma)x\,ds, \quad \forall t\ge 0, \quad \textit{where} \quad \sigma := \frac{1}{\alpha}.$$

The proof of Corollary 4.16 is a simple combination of the proofs of Proposition 4.10, Theorem 4.14 and Corollary 4.15.

Remark 4.17. (i) It follows from Theorem 4.14 and Corollary 4.16 that we have the following more general situation. Let $1<\alpha\le 2$ and $\beta\ge 0$ be given. If A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$, then A also generates the $(\alpha',1)^{\frac{\beta}{2}}$-resolvent family $S_{\alpha'}^{\frac{\beta}{2}}$ introduced in [8, 28, 29] for any $0<\alpha'\le 1$. More precisely, in [29, Definition

4.2], for $0 < \alpha' \leq 1$ and $\beta \geq 0$, an $(\alpha',1)^\beta$-resolvent family associated to a closed linear operator A defined on a Banach space X, has been defined to be a strongly continuous function $S_{\alpha'}^\beta : [0,\infty) \to \mathcal{L}(X)$ such that, $\|(g_1 * S_{\alpha'}^\beta)(t)x\| \leq Me^{\omega t}\|x\|$, $x \in X$, $t \geq 0$, for some constants $M, \omega \geq 0$, $\{\lambda^{\alpha'} : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, and

$$\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x = \lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda t}S_{\alpha'}^\beta(t)xdt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In the same direction, we observe that a generator of an $(\alpha,1)$-resolvent family for $1 < \alpha \leq 2$ is already the generator of an analytic strongly continuous semigroup.

(ii) We mention the following remarkable result obtained in [8, Section 3]. Let A be a closed linear operator on a Banach space X. If A generates a bounded analytic strongly continuous semigroup $(T(t))$ of angle $\pi/2$, then A generates an $(\alpha,1)^0 = (\alpha,1)$-resolvent family $\mathbb{S}_\alpha$ on X for every $1 < \alpha < 2$, and hence, generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ on X for every $1 < \alpha < 2$ and $\beta \geq 0$. But unfortunately, there is no explicit representation of $\mathbb{S}_\alpha^\beta(t)$ in terms of $T(t)$.

(iii) In general, generators of resolvent families even in the case $\beta = 0$ are not stable under bounded perturbations. In the case $\beta = 0$, an example in [8, Example 2.24] shows that they need not be stable by perturbations by multiple of the identity operator. Therefore the resolvent families obtained through Corollary 4.15 are of special interest since they are stable under perturbations by multiple of the identities. Other admissible perturbations have been studied, see e.g. [3, 32] and the references therein.

From Lemma 4.13, Theorem 4.14 and Corollary 4.16 we derive the following result.

Lemma 4.18. *Let A be a closed linear operator on a Banach space X. Let $1 < \alpha \leq 2$, $\beta \geq 0$ and $\mu > 0$. Then the following assertions hold.*

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$. Let $1 \leq \alpha' < \alpha$ and let $\mathbb{P}_{\alpha'}^\beta$ be the $(\alpha',\alpha')^\beta$-resolvent family generated by A, or the $\frac{\beta}{2}$-times integrated semigroup $(T_{\frac{\beta}{2}}(t))$ generated by A. Then for every $x \in X$ and $t > 0$,

$$\int_0^\infty \frac{\sigma s}{t^{\sigma+1}}\Phi_\sigma(st^{-\sigma})(g_\mu * \mathbb{P}_\alpha^\beta)(s)xds = (g_{\mu\sigma} * \mathbb{P}_{\alpha'}^\beta)(t)x, \quad \sigma = \frac{\alpha'}{\alpha}, \tag{4.22}$$

$$\int_0^\infty \frac{\sigma s}{t^{\sigma+1}}\Phi_\sigma(st^{-\sigma})(g_\mu * \mathbb{P}_\alpha^\beta)(s)xds = (g_{\mu\sigma} * T_{\frac{\beta}{2}}(t)x, \quad \sigma = \frac{1}{\alpha}. \tag{4.23}$$

(b) Assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Let $1 \leq \alpha' < \alpha$, and let $\mathbb{S}_{\alpha'}^\beta$ be the $(\alpha',1)^\beta$-resolvent family generated by A, or the $\frac{\beta}{2}$-times integrated semigroup $(T_{\frac{\beta}{2}}(t))$ generated by A. Then for every $x \in X$ and $t > 0$,

$$\int_0^\infty \frac{1}{t^\sigma}\Phi_\alpha(st^{-\sigma})(g_\mu * \mathbb{S}_\alpha^\beta)(s)xds = (g_{\mu\sigma} * \mathbb{S}_{\alpha'}^\beta)(t)x, \quad \sigma := \frac{\alpha'}{\alpha}, \tag{4.24}$$

$$\int_0^\infty \frac{1}{t^\sigma}\Phi_\alpha(st^{-\sigma})(g_\mu * \mathbb{S}_\alpha^\beta)(s)xds = (g_{\mu\sigma} * T_{\frac{\beta}{2}}(t)x, \quad \sigma := \frac{1}{\alpha}. \tag{4.25}$$

Proof. Let A, α, β be as in the statement of the lemma and let $x \in X$ and $\mu > 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_\alpha^\beta$. Let ω be the real number from the definition of $\mathbb{P}_\alpha^\beta$. Let $1 \leq \alpha' < \alpha$. Using the Laplace transform, we have that for $\mathrm{Re}(\lambda) > \omega$,

$$(\widehat{g_{\mu\sigma} * \mathbb{P}_{\alpha'}^\beta})(\lambda)x = \lambda^{-\mu\sigma}\lambda^{-\frac{\alpha'\beta}{2}}(\lambda^{\alpha'} - A)^{-1}x = \lambda^{-\mu\sigma-\frac{\alpha'\beta}{2}}(\lambda^{\alpha'} - A)^{-1}x. \tag{4.26}$$

On the other hand, using (2.11) and Fubini's theorem, we obtain that for $\mathrm{Re}(\lambda) > \omega$,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \int_0^\infty \frac{\sigma s}{t^{\sigma+1}} \Phi_\sigma(st^{-\sigma})(g_\mu * \mathbb{P}_\alpha^\beta)(s)x ds\, dt \\
&= \int_0^\infty e^{-\lambda^\sigma s}(g_\mu * \mathbb{P}_\alpha^\beta)(s)x ds \\
&= \lambda^{-\sigma(\mu+\frac{\alpha\beta}{2})}(\lambda^{\alpha\sigma} - A)^{-1}x \\
&= \lambda^{-\sigma\mu-\frac{\alpha'\beta}{2}}(\lambda^{\alpha'} - A)^{-1}x.
\end{aligned} \tag{4.27}$$

In view of (4.26) and (4.27), the equality (4.22) follows from the uniqueness theorem for the Laplace transform and by continuity. The proof of (4.23) follows the lines of the proof of (4.22).

(b) Similarly, for $\mathrm{Re}(\lambda) > \omega$ (here ω be the real number from the definition of $\mathbb{S}_\alpha^\beta$),

$$\begin{aligned}
(\widehat{g_{\sigma\mu} * \mathbb{S}_{\alpha'}^\beta})(\lambda)x &= \lambda^{-\sigma\mu}\lambda^{-\frac{\alpha'\beta}{2}}\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x \\
&= \lambda^{-\sigma\mu-\frac{\alpha'\beta}{2}}\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x.
\end{aligned} \tag{4.28}$$

Using (2.12) and Fubini's theorem, we obtain for $\mathrm{Re}(\lambda) > \omega$,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \int_0^\infty \frac{1}{t^\sigma} \Phi_\alpha(st^{-\sigma})(g_\mu * \mathbb{S}_\alpha^\beta)(s)x ds\, dt \\
&= \lambda^{\sigma-1}\int_0^\infty e^{-\lambda^\sigma t}(g_{\alpha\mu} * \mathbb{S}_\alpha^\beta)(s)x ds \\
&= \lambda^{\sigma-1}\lambda^{-\mu\sigma-\sigma\frac{\alpha\beta}{2})}\lambda^{\sigma(\alpha-1)}(\lambda^{\sigma\alpha} - A)^{-1}x \\
&= \lambda^{\sigma-1}\lambda^{-\mu\sigma-\frac{\alpha'\beta}{2})}\lambda^{\alpha'-\sigma}(\lambda^{\alpha'} - A)^{-1}x \\
&= \lambda^{-\sigma\mu-\frac{\alpha'\beta}{2}}\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x.
\end{aligned} \tag{4.29}$$

Using (4.28) and (4.29), the equality (4.24) also follows from the uniqueness theorem for the Laplace transform and by continuity. The proof of (4.25) also follows the lines of the proof of (4.24). □

The following result on the regularity properties of the family $\mathbb{S}_\alpha^\beta$ is crucial and will be used several times in the subsequent sections to obtain our main results.

Lemma 4.19. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \le 2, \beta \ge 0$, $k := \lceil\frac{\alpha\beta}{2}\rceil$, $n := \lceil\beta\rceil$ and assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Then the following properties hold.*

(a) Let $m \in \mathbb{N}\cup\{0\}$. Then for every $x \in D(A^{m+1})$ and $t \ge 0$,

$$\mathbb{S}_\alpha^\beta(t)x = \sum_{j=0}^m g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^jx + \int_0^t g_{\alpha(m+1)}(t-s)\mathbb{S}_\alpha^\beta(s)A^{m+1}x\, ds. \tag{4.30}$$

(b) For every $x \in D(A^{n+1})$, the map $t \mapsto (g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x$ belongs to the space $C^k([0,\infty); D(A)) \cap C^{k+1}([0,\infty); X)$ and for every $t \ge 0$,

$$\frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x\Big] = \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^jx + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^nx, \tag{4.31}$$

$$\frac{d^{k+1}}{dt^{k+1}}\Big[(g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta)(t)x\Big]=\sum_{j=1}^{n}g_{\alpha j}(t)A^jx+(g_{\alpha(n-\frac{\beta}{2})+\alpha-1}*\mathbb{S}_\alpha^\beta)(t)A^{n+1}x. \quad (4.32)$$

In particular,

$$\frac{d^j}{dt^j}\Big[g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta\Big](0)x=0,\ j=0,1,\dots,k-1,\quad \frac{d^k}{dt^k}\Big[g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta\Big](0)x=x, \quad (4.33)$$

$$\frac{d^{k+1}}{dt^{k+1}}\Big[g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta\Big](0)x=0,\quad \frac{d^{k+1}}{dt^{k+1}}\Big[g_1*g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta\Big](0)x=x. \quad (4.34)$$

(c) In general, for every $x\in D(A^{n+1-i})$, $i=0,1,\dots,n$, *the mapping* $t\mapsto(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha i}*\mathbb{S}_\alpha^\beta)(t)x$ *belongs to* $C^k([0,\infty);D(A))$ *and for every* $t\ge0$,

$$\begin{aligned}&\frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha i}*\mathbb{S}_\alpha^\beta)(t)x\Big]\\&=\sum_{j=0}^{n-i}g_{\alpha j+1+\alpha i}(t)A^jx+(g_{\alpha(n-\frac{\beta}{2})}*g_\alpha*\mathbb{S}_\alpha^\beta)(t)A^{n+1-i}x.\end{aligned} \quad (4.35)$$

(d) For every $x\in D(A^n)$, *the mapping* $t\mapsto(g_{k-\frac{\alpha\beta}{2}}*\mathbb{S}_\alpha^\beta)(t)x$ *belongs to the class* $C^k([0,\infty);X)$ *and the equalities* (4.31) *and* (4.33) *hold.*

(e) In general, for every $x\in D(A^{n-i})$, $i=0,1,\dots,n$, *the mapping* $t\mapsto(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha i}*\mathbb{S}_\alpha^\beta)(t)x$ *belongs to* $C^k([0,\infty);X)$ *and for every* $t\ge0$,

$$\begin{aligned}&\frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha i}*\mathbb{S}_\alpha^\beta)(t)x\Big]\\&=\sum_{j=0}^{n-i}g_{\alpha j+1+\alpha i}(t)A^jx+A(g_{\alpha(n-\frac{\beta}{2})}*g_\alpha*\mathbb{S}_\alpha^\beta)(t)A^{n-i}x.\end{aligned} \quad (4.36)$$

Proof. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1<\alpha\le2,\beta\ge0$ and set $k:=\lceil\frac{\alpha\beta}{2}\rceil$, $n:=\lceil\beta\rceil$. Note that $k\le n$. Assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$.

(a) We prove (4.30) by induction. If $m=0$, then for every $x\in D(A)$, the equality (4.30) reads

$$\mathbb{S}_\alpha^\beta(t)x=g_{\frac{\alpha\beta}{2}+1}(t)x+\int_0^t g_\alpha(t-s)\mathbb{S}_\alpha^\beta(s)Ax\,ds,\quad \forall t\ge0$$

which is given by Lemma 4.5(b). Assume that (4.30) holds for $m-1$ for some $m\in\mathbb{N}$. Now, let $x\in D(A^{m+1})\subset D(A^m)$. Then using Lemma 4.5(b), we have that for every $t\ge0$,

$$\begin{aligned}\mathbb{S}_\alpha^\beta(t)x&=\sum_{j=0}^{m-1}g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^jx+(g_{\alpha m}*\mathbb{S}_\alpha^\beta)(t)A^mx\\&=\sum_{j=0}^{m-1}g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^jx+A^m(g_{\alpha m}*\mathbb{S}_\alpha^\beta)(t)x\\&=\sum_{j=0}^{m-1}g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^jx+A^mg_{\alpha m}*\Big(g_{\frac{\alpha\beta}{2}+1}x+g_\alpha*\mathbb{S}_\alpha^\beta Ax\Big)(t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-1} g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^j x + g_{\alpha(\frac{\beta}{2}+m)+1}(t)A^m x + (g_{\alpha(m+1)} * \mathbb{S}_{\alpha\beta})(t)A^{m+1}x \\
&= \sum_{j=0}^{m} g_{\alpha(\frac{\beta}{2}+j)+1}(t)A^j x + (g_{\alpha(m+1)} * \mathbb{S}_{\alpha}^{\beta})(t)A^{m+1}x.
\end{aligned}
$$

We conclude that the equality (4.30) holds and this completes the proof of part (a).
(b) Let $x \in D(A^{n+1})$. Then using (4.30) with $m = n$ we get that for every $t \geq 0$,

$$
(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x = \sum_{j=0}^{n} g_{k+\alpha j+1}(t)A^j x + (g_{\alpha(n+1)+k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)A^{n+1}x.
$$

Therefore, using Lemma 4.5(b), we have that for all $t \geq 0$,

$$
\begin{aligned}
\frac{d^k}{dt^k}\big[(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x\big] &= \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n+1)-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)A^{n+1}x \\
&= \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + \Big(g_{\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_{\alpha}^{\beta}\Big)(t)A^{n+1}x \\
&= \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * (\mathbb{S}_{\alpha}^{\beta} - g_{\frac{\alpha\beta}{2}+1}))(t)A^{n}x \\
&= \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_{\alpha}^{\beta})(t)A^{n}x - g_{\alpha n+1}(t)A^n x \\
&= \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_{\alpha}^{\beta})(t)A^{n}x,
\end{aligned}
$$

and we have shown (4.31). Since $A^n x \in D(A)$, it follows from (4.31) and Lemma 4.5 that $\frac{d^k}{dt^k}[(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x] \in C([0,\infty); D(A))$. Hence, $(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x \in C^k([0,\infty); D(A))$. Since $g_1(0^+) = 1$ and $g_{\alpha j+1}(0^+) = 0$ for every $j = 1, 2, \dots, n-1$, the equalities in (4.33) follow from (4.31).

By Remark 4.4 and Lemma 4.9, A generates an $(\alpha,\alpha)^\beta$-resolvent family $\mathbb{P}_{\alpha}^{\beta}$ and for every $x \in D(A)$, $\mathbb{S}_{\alpha}^{\beta}(t)x \in C([0,\infty); D(A)) \cap C^1((0,\infty); X)$. Now, let $x \in D(A^{n+1})$. We have to show that $(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x \in C^{k+1}([0,\infty); X)$ and (4.34) holds. It follows from (4.31), (4.5) and the fact that $\mathbb{S}_{\alpha}^{\beta}(t)A^n x \in C([0,\infty); D(A)) \cap C^1((0,\infty); X)$, that for every $t \geq 0$,

$$
\begin{aligned}
&\frac{d^{k+1}}{dt^{k+1}}(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_{\alpha}^{\beta})(t)x \\
&= \sum_{j=1}^{n-1} g_{\alpha j}(t)A^j x + \big[(g_{\alpha(n-\frac{\beta}{2})} * (\mathbb{S}_{\alpha}^{\beta})')(t)A^n x\big] \\
&= \sum_{j=1}^{n-1} g_{\alpha j}(t)A^j x + \Big[g_{\alpha(n-\frac{\beta}{2})} * \Big(g_{\frac{\alpha\beta}{2}} A^n x + \mathbb{P}_{\alpha}^{\beta} A^{n+1}x\Big)(t)\Big] \\
&= \sum_{j=1}^{n} g_{\alpha j}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{P}_{\alpha}^{\beta})(t)A^{n+1}x
\end{aligned}
$$

$$= \sum_{j=1}^{n} g_{\alpha j}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})+\alpha-1} * \mathbb{S}_\alpha^\beta)(t) A^{n+1} x \in C([0,\infty); X),$$

and we have shown (4.32). Therefore, $(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x \in C^{k+1}([0,\infty); X)$. It also follows from (4.32) that

$$\frac{d^{k+1}}{dt^{k+1}} \left[g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta \right](0)x = 0.$$

Now, using (4.33) we get that

$$\frac{d^{k+1}}{dt^{k+1}} \left[g_1 * g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta \right](0)x = \frac{d^k}{dt^k} \left[g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta \right](0)x = x\,.$$

This completes the proof of part (b).

(c) Let $x \in D(A^{n+1-i})$, $i = 0, 1, \dots, n$. Proceeding as in the proof of part (b) we get that for every $t \geq 0$,

$$(g_{\alpha i} * g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x = \sum_{j=0}^{n-i} g_{k+\alpha j+\alpha i+1}(t) A^j x + (g_{k+\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_\alpha^\beta)(t) A^{n+1-i} x.$$

This implies that for every $t \geq 0$,

$$\begin{aligned} &\frac{d^k}{dt^k} \left[(g_{\alpha i} * g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x \right] \\ &= \sum_{j=0}^{n-i} g_{\alpha j+\alpha i+1}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_\alpha^\beta)(t) A^{n+1-i} x. \end{aligned}$$

Using Lemma 4.5, the preceding equality shows that $(g_{\alpha i} * g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x \in C^k([0,\infty); D(A))$ and one has the equality (4.35).

(d) Let $x \in D(A^n)$. Proceeding as in part (b), we also get the equality (4.31) and this implies that $(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x \in C^k([0,\infty); X)$ and (4.33) holds.

(e) Let $x \in D(A^{n-i}), i = 0, 1, \dots, n$. Proceeding as in part (c), we get that $(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha i} * \mathbb{S}_\alpha^\beta)(t)x \in C^k([0,\infty); X)$ and the equality (4.35) holds. The proof of the lemma is complete. □

5. Resolvent families and the regularized abstract Cauchy problem

In this section we show that the above defined resolvent family $\mathbb{S}_\alpha^\beta$ is necessary and sufficient to solve the regularized abstract Cauchy problem

$$\begin{gathered} \mathbb{D}_t^\alpha v(t) = Av(t) + g_{\frac{\alpha\beta}{2}+1}(t)x, \quad t > 0,\ 1 < \alpha \leq 2,\ \beta \geq 0, \\ v(0) = v'(0) = 0, \end{gathered} \tag{5.1}$$

where A is a closed linear operator with domain $D(A)$ defined on a Banach space X. By a classical solution of (5.1) we mean a function $v \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ such that $(g_{2-\alpha} * v) \in C^2([0,\infty); X)$ and (5.1) is satisfied.

The following is the main result of this section.

Theorem 5.1. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \leq 2$ and $\beta \geq 0$. Then the following assertions are equivalent.*

(i) *The operator A generates an $(\alpha, 1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ on X.*

(ii) *For all $x \in X$, there exists a unique classical solution v of Problem* (5.1) *such that $(g_{2-\alpha} * v')(t)$ is exponentially bounded.*

Proof. Let A, α and β be as the statement of the theorem.

(i) $\Rightarrow$ (ii): Assume that A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ on X and let $x \in X$. Define

$$v(t) := (g_\alpha * \mathbb{S}_\alpha^\beta)(t)x = \int_0^t g_\alpha(t-s)\mathbb{S}_\alpha^\beta(s)x\,ds, \quad t \ge 0.$$

Then $v(0) = 0$ and by Lemma 4.5 we have that $v \in C([0,\infty); D(A))$. Since $v'(t) = (g_{\alpha-1} * \mathbb{S}_\alpha^\beta)(t)x$, we have that $v \in C^1([0,\infty); X)$ and $v'(0) = 0$. Since for every $t \ge 0$,

$$(g_{2-\alpha} * v)(t) = (g_{2-\alpha} * g_\alpha * \mathbb{S}_\alpha^\beta)(t)x = (g_2 * \mathbb{S}_\alpha^\beta)(t)x,$$

it follows that $(g_{2-\alpha} * v) \in C^2([0,\infty); X)$. Since $v(0) = v'(0) = 0$, it follows from (2.1) and (2.3) that for every $t \ge 0$,

$$\begin{aligned}
\mathbb{D}_t^\alpha v(t) =& (g_{2-\alpha} * v'')(t) = \frac{d^2}{dt^2}\left[(g_{2-\alpha} * v)(t)\right] \\
=& \frac{d^2}{dt^2}\left[(g_2 * \mathbb{S}_\alpha^\beta)(t)x\right] = \mathbb{S}_\alpha^\beta(t)x \\
=& A(g_\alpha * \mathbb{S}_\alpha^\beta)(t)x + g_{\frac{\alpha\beta}{2}+1}(t)x = Av(t) + g_{\frac{\alpha\beta}{2}+1}(t)x.
\end{aligned}$$

Hence, v is a classical solution of (5.1). Since $(g_1 * \mathbb{S}_\alpha^\beta)(t)$ is exponentially bounded and for every $x \in X$, $t \ge 0$,

$$(g_{2-\alpha} * v')(t) = (g_{2-\alpha} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta)(t)x = (g_1 * \mathbb{S}_\alpha^\beta)(t)x,$$

it follows that $(g_{2-\alpha} * v')(t)$ is exponentially bounded. Assume that (5.1) has two classical solutions v_1 and v_2 and set $V := v_1 - v_2$. Then $V \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$, $V(0) = V'(0) = 0$, $(g_{2-\alpha} * V) \in C^2([0,\infty); X)$, $(g_{2-\alpha} * V)(t)$ is exponentially bounded and $\mathbb{D}_t^\alpha V(t) = AV(t)$ for every $t > 0$. Taking the Laplace transform, we get that for $\mathrm{Re}(\lambda) > \omega$ (where ω is the real number from the above mentioned exponential boundedness), $(\lambda^\alpha - A)\widehat{V}(\lambda) = 0$. Since $(\lambda^\alpha - A)$ is invertible, we have that $\widehat{V}(\lambda) = 0$. By the uniqueness theorem for the Laplace transform and by continuity, we get that $V(t) = 0$ for every $t \ge 0$. We have shown uniqueness of solutions and this completes the proof of part (ii).

(ii) $\Rightarrow$ (i): For $x \in X$, we let $S_{\alpha,\beta}(t)x := \mathbb{D}_t^\alpha v(t,x)$ where $v(t,x)$ is the unique classical solution of (5.1). Using (2.1) and the fact that $v(0) = 0 = v'(0)$ we get that for every $t \ge 0$,

$$(g_\alpha * S_{\alpha,\beta})(t)x = (g_\alpha * \mathbb{D}_t^\alpha v)(t) = v(t,x) - v(0,x) - v'(0,x)t = v(t,x).$$

Hence, $(g_\alpha * S_{\alpha,\beta})(t)x \in D(A)$ for every $x \in X$, $t \ge 0$, and one has the equality

$$A(g_\alpha * S_{\alpha,\beta})(t)x + g_{\frac{\alpha\beta}{2}+1}(t)x = Av(t,x) + g_{\frac{\alpha\beta}{2}+1}(t)x = S_{\alpha,\beta}(t)x. \tag{5.2}$$

By the closed graph theorem we also have that $S_{\alpha,\beta}(t) \in \mathcal{L}(X)$ for $t \ge 0$ and we note that $S_{\alpha,\beta}(t)$ is strongly continuous on $[0,\infty)$. Since by hypothesis $(g_{2-\alpha} * v')(t)$ is exponentially bounded and given that for every $x \in X$, $t \ge 0$,

$$(g_1 * S_{\alpha,\beta})(t)x = (g_1 * g_{2-\alpha} * v'')(t) = (g_{2-\alpha} * v')(t),$$

we have that $(g_1 * S_{\alpha,\beta})(t)x$ is exponentially bounded. By the uniform exponential boundedness principle [3, Lemma 3.2.14], we have that there exist some constants $M, \omega \geq 0$ such that

$$\|(g_{2-\alpha} * v')(t)\| = \|(g_1 * S_{\alpha,\beta})(t)x\| \leq Me^{\omega t}, \quad t \geq 0,\ x \in X. \tag{5.3}$$

Taking the Laplace transform on both sides of the equality (5.2) we get that for $\mathrm{Re}(\lambda) > \omega$ (where ω is the real number from the above mentioned exponential boundedness),

$$A\lambda^{-\alpha}\widehat{S}_{\alpha,\beta}(\lambda)x - \widehat{S}_{\alpha,\beta}(\lambda)x = -\lambda^{-\frac{\alpha\beta}{2}-1}x.$$

Multiplying the preceding equality by λ^α we get that

$$(\lambda^\alpha - A)\widehat{S}_{\alpha,\beta}(\lambda)x = \lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}x.$$

The above equality shows that $(\lambda^\alpha - A)$ is surjective. To prove injectivity, suppose that $(\lambda^\alpha - A)x = 0$ for some $x \in D(A)$ and $\mathrm{Re}(\lambda) > \omega$, that is, $Ax = \lambda^\alpha x$ for $\mathrm{Re}(\lambda) > \omega$. It is enough to consider that $Ax = \lambda^\alpha x$ for λ real and $\lambda > \omega$. Then setting $v(t) = (g_{\frac{\alpha\beta}{2}+\alpha} * \widetilde{E})(t)x$ where $\widetilde{E}(t)x = E_{\alpha,1}(\lambda^\alpha t^\alpha)x$, we prove that v is a solution of Equation (5.1). Obviously $v \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ and $(g_{2-\alpha} * v) \in C^2([0,\infty); X)$. Using (2.8), we have that for every $t > 0$,

$$\begin{aligned}
\mathbb{D}_t^\alpha v(t) &= g_{2-\alpha} * \frac{d^2}{dt^2}\Big[(g_{\frac{\alpha\beta}{2}+\alpha} * \widetilde{E})(t)x\Big] \\
&= (g_{\frac{\alpha\beta}{2}} * \widetilde{E})(t)x = g_{\frac{\alpha\beta}{2}} * (g_1 + \lambda^\alpha g_\alpha * \widetilde{E}))(t)x \\
&= g_{\frac{\alpha\beta}{2}+1}(t)x + (g_{\frac{\alpha\beta}{2}+\alpha} * \widetilde{E})(t)\lambda^\alpha x = g_{\frac{\alpha\beta}{2}+1}(t)x + (g_{\frac{\alpha\beta}{2}+\alpha} * \tilde{E})(t)Ax \\
&= g_{\frac{\alpha\beta}{2}+1}(t)x + A(g_{\frac{\alpha\beta}{2}+\alpha} * \widetilde{E})(t)x = g_{\frac{\alpha\beta}{2}+1}(t)x + Av(t).
\end{aligned}$$

We have shown that v is a solution of Equation (5.1). Since all the solutions v of Equation (5.1) satisfy the estimate (5.3), we must have this estimate for the solution $v(t) = (g_{\frac{\alpha\beta}{2}+\alpha} * \widetilde{E})(t)x$ just found. But using (2.6) we have that

$$\widetilde{E}(t) = \sum_{n=0}^{\infty}\frac{\lambda^{\alpha n}t^{\alpha n}}{\Gamma(\alpha n+1)}$$

which gives

$$\begin{aligned}
(g_{2-\alpha} * v')(t) = (g_{\frac{\alpha\beta}{2}+1} * \tilde{E})(t)x &= t^{\frac{\alpha\beta}{2}+1}\sum_{n=0}^{\infty}\frac{\lambda^{\alpha n}t^{\alpha n}}{\Gamma(\alpha n + \frac{\alpha\beta}{2} + 2)} \\
&= t^{\frac{\alpha\beta}{2}+1}E_{\alpha,\frac{\alpha\beta}{2}+2}(\lambda^\alpha t^\alpha)x,
\end{aligned}$$

and hence by (2.9), $\|(g_{2-\alpha}*v')(t)\| \leq Me^{\lambda t}\|x\|$ and this estimate is sharp. Therefore we can only have the estimate (5.3) if $x = 0$. We have shown that $(\lambda^\alpha - A)$ is injective, hence is invertible and

$$\widehat{S}_{\alpha,\beta}(\lambda)x = \lambda^{-\frac{\alpha\beta}{2}}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x,$$

that is, for $\mathrm{Re}(\lambda) > \omega$, and $x \in X$,

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{\frac{\alpha\beta}{2}}\int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)x\,dt.$$

Hence, A generates an $(\alpha,1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$ and by the uniqueness theorem for the Laplace transform and by continuity we have that $\mathbb{S}_\alpha^\beta(t)x = S_{\alpha,\beta}(t)x$ for every $x \in X$, $t \ge 0$. We have shown the assertion (i) and the proof is finished. $\square$

Remark 5.2. (a) We note that in Theorem 5.1, the assertion that $(g_{2-\alpha} * v')(t)$ is exponentially bounded agrees with the limiting cases $\alpha = 1$ in which the conclusion reads $(g_1 * v')(t) = v(t)$ is exponentially bounded (see e.g. [3, Theorem 3.2.13]), and $\alpha = 2$, in which we have that $v'(t)$ is exponentially bounded. An example showing that the exponential boundedness assumption cannot be omitted is included in [3, Remark 3.2.15(b)] for the limiting case $\alpha = 1$.

(b) We mention that if the family $\mathbb{S}_\alpha^\beta$ is exponentially bounded, then the solution v in Theorem 5.1 is exponentially bounded as well.

6. Resolvent families and the homogeneous abstract Cauchy problem

In this section we use the above defined resolvent families to investigate the existence, regularity and the representation of solutions of the homogeneous abstract Cauchy problem

$$\begin{gathered} \mathbb{D}_t^\alpha u(t) = Au(t), \quad t > 0,\ 1 < \alpha \le 2, \\ u(0) = x, \quad u'(0) = y, \end{gathered} \tag{6.1}$$

where A is a closed linear operator with domain $D(A)$ defined on a Banach space X and x, y are given vectors in X.

Definition 6.1. A function $u \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ is said to be a classical solution of Problem (6.1) if $g_{2-\alpha} * (u - u(0) - u'(0)g_2) \in C^2([0,\infty); X)$ and (6.1) is satisfied.

We adopt the following definition of mild solutions.

Definition 6.2. A function $u \in C([0,\infty); X)$ is said to be a mild solution of (6.1) if $I_t^\alpha u(t) := (g_\alpha * u)(t) \in D(A)$ for every $t \ge 0$, and

$$u(t) = x + ty + A\int_0^t g_\alpha(t-s)u(s)\, ds, \quad \forall t \ge 0.$$

We have the following uniqueness result.

Proposition 6.3. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and let $1 < \alpha \le 2$. Then the following assertions hold.*

- (a) *If u is a classical solution of (6.1), then it is a mild solution of (6.1).*
- (b) *If $(\lambda^\alpha - A)$ is invertible for $\mathrm{Re}(\lambda)$ large enough, and if a mild solution u exists and $(g_1 * u)(t)$ is exponentially bounded, then it is unique.*

Proof. Let $1 < \alpha \le 2$ and let A be a closed linear operator with domain $D(A)$ defined on a Banach space X.

(a) Let u be a classical solution of (6.1). Since $u \in C([0,\infty); D(A))$, we have that $(g_\alpha * u)(t) \in C([0,\infty); D(A))$. Since $\mathbb{D}_t^\alpha u(t) = Au(t)$, that is, $(g_{2-\alpha} * u'')(t) = Au(t)$, we have that $(g_\alpha * g_{2-\alpha} * u'')(t) = A(g_\alpha * u)(t)$, i.e., $(g_2 * u'')(t) = A(g_\alpha * u)(t)$. Hence, $u(t) - u(0) - tu'(0) = A(g_\alpha * u)(t)$ for every $t \ge 0$ and we have shown that u is a mild solution of (6.1).

(b) Assume that (6.1) has two mild solutions u and v and set $U := u - v$. Then $U \in C([0,\infty); X)$, $(g_\alpha * U)(t) \in D(A)$ for every $t \ge 0$ and $U(t) = A(g_\alpha * U)(t)$. Taking the Laplace transform, we get that $(I - \lambda^{-\alpha}A)\widehat{U}(\lambda) = 0$ for $\mathrm{Re}(\lambda) > \omega$

(where $\omega \geq 0$ is the real number from the exponential boundedness of $(g_1 * u)(t)$). Since $(I - \lambda^{-\alpha} A)$ is invertible, we have that $\widehat{U}(\lambda) = 0$. By the uniqueness theorem for the Laplace transform and by continuity, we get that $U(t) = 0$ for every $t \geq 0$. Hence, $u(t) = v(t)$ for every $t \geq 0$. The proof is finished. □

Remark 6.4. We mention that to prove the existence of solutions of Problem (6.1), we proceed by direct construction and make minimal use of the Laplace transform.

The following result is the main result of this section.

Theorem 6.5. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \leq 2, \beta \geq 0$ and set $n := \lceil \beta \rceil$, $k := \lceil \frac{\alpha\beta}{2} \rceil$. Assume that A generates an $(\alpha, 1)^{\beta}$-resolvent family $\mathbb{S}_{\alpha}^{\beta}$. Then the following assertions hold.*

(a) *For every $x, y \in D(A^{n+1})$, the function $u(t) := D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_{\alpha}^{\beta}(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_{\alpha}^{\beta})(t)y$ is the unique classical solution of* (6.1).

(b) *For every $x, y \in D(A^n)$, the function $u(t) := D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_{\alpha}^{\beta}(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_{\alpha}^{\beta})(t)y$ is the unique mild solution of* (6.1).

Proof. Let A, α, β, $n := \lceil \beta \rceil$ and $k := \lceil \frac{\alpha\beta}{2} \rceil$ be as in the statement of the theorem. First we prove existence of classical and mild solutions.

(a) Let $x, y \in D(A^{n+1})$ and set $u(t) := D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_{\alpha}^{\beta}(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_{\alpha}^{\beta})(t)y$. It follows from Lemma 4.19 that $u \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$, $u(0) = x$ and $u'(0) = y$. Since $u(0) = x$, $u'(0) = y$, using Lemma 4.19 and Lemma 4.5, we have that for every $t \geq 0$,

$$
\begin{aligned}
& g_{2-\alpha} * (u - u(0) - u'(0) g_2)(t) \\
&= g_{2-\alpha} * \Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_{\alpha}^{\beta})(t) A^n x - x \Big] \\
&\quad + g_{2-\alpha} * \Big[\sum_{j=0}^{n-1} g_{\alpha j+2}(t) A^j y + (g_{\alpha(n-\frac{\beta}{2})+1} * \mathbb{S}_{\alpha}^{\beta})(t) A^n y - ty \Big] \\
&= \sum_{j=1}^{n-1} g_{\alpha j+3-\alpha}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})+2-\alpha} * \mathbb{S}_{\alpha}^{\beta})(t) A^n x \\
&\quad + \sum_{j=1}^{n-1} g_{\alpha j+4-\alpha}(t) A^j y + (g_{\alpha(n-\frac{\beta}{2})+3-\alpha} * \mathbb{S}_{\alpha}^{\beta})(t) A^n y \\
&= \sum_{j=1}^{n} g_{\alpha j+3-\alpha}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})+2} * \mathbb{S}_{\alpha}^{\beta})(t) A^{n+1} x \\
&\quad + \sum_{j=1}^{n} g_{\alpha j+4-\alpha}(t) A^j y + (g_{\alpha(n-\frac{\beta}{2})+3} * \mathbb{S}_{\alpha}^{\beta})(t) A^{n+1} y.
\end{aligned}
\tag{6.2}
$$

Using (6.2) and Lemma 4.19 we get that for every $t \geq 0$,

$$
\begin{aligned}
& \frac{d^2}{dt^2} \Big[g_{2-\alpha} * (u - u(0) - u'(0) g_2) \Big](t) \\
&= \sum_{j=1}^{n} g_{\alpha j+1-\alpha}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_{\alpha}^{\beta})(t) A^{n+1} x
\end{aligned}
$$

$$+\sum_{j=1}^{n} g_{\alpha j+2-\alpha}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^{n+1}y \in C([0,\infty);X).$$

Hence, $g_{2-\alpha} * (u - u(0) - u'(0)g_2) \in C^2([0,\infty);X)$. We have to show that u satisfies (6.1). Using (4.32) in Lemma 4.19, we get that for every $t \geq 0$,

$$\begin{aligned}
\mathbb{D}_t^\alpha u(t) =& \mathbb{D}_t^\alpha D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + \mathbb{D}_t^\alpha D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y \\
=& g_{2-\alpha} * \Big[\frac{d^{k+2}}{dt^{k+2}} \Big(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta\Big)(t)x + \frac{d^{k+2}}{dt^{k+2}} \Big(g_{k-\frac{\alpha\beta}{2}} * g_1 * \mathbb{S}_\alpha^\beta\Big)(t)y\Big] \\
=& g_{2-\alpha} * \frac{d}{dt}\Big[\sum_{j=1}^{n} g_{\alpha j}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})+\alpha-1} * \mathbb{S}_\alpha^\beta)(t)A^{n+1}x\Big] \\
&+ g_{2-\alpha} * \Big[\sum_{j=1}^{n} g_{\alpha j}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})+\alpha-1} * \mathbb{S}_\alpha^\beta)(t)A^{n+1}y\Big] \\
=& \sum_{j=1}^{n} g_{\alpha j+1-\alpha}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^{n+1}x \\
&+ \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^{j+1} y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^{n+1}y \\
=& \sum_{j=0}^{n} g_{\alpha j+1}(t)A^{j+1} x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^{n+1}x \\
&+ \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^{j+1} y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^{n+1}y \\
=& A\Big[\sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^n x\Big] \\
&+ A\Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^n y\Big] \\
=& A\Big[D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y\Big] = Au(t)
\end{aligned}$$

and this completes the proof of the existence part in the assertion (a).

(b) Let $x, y \in D(A^n)$ and set $u(t) := D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} g_1 * \mathbb{S}_\alpha^\beta(t)y$. Using (4.31) in the proof of Lemma 4.19 we get that for every $t \geq 0$,

$$\begin{aligned}
u(t) =& \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t)x\Big] + \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_1 * \mathbb{S}_\alpha^\beta)(t)y\Big] \\
=& \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^n x \\
&+ \sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^n y.
\end{aligned} \tag{6.3}$$

It follows from (6.3) and Lemma 4.19 that $u \in C([0,\infty);X)$. Using (6.3) we get that for every $t \ge 0$,

$$\begin{aligned} I_t^\alpha u(t) :=& (g_\alpha * u)(t) \\ =& \sum_{j=0}^{n-1} g_{\alpha j+1+\alpha}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n x \\ &+ \sum_{j=0}^{n-1} g_{\alpha j+2+\alpha}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n y. \end{aligned} \tag{6.4}$$

It follows from (6.4) and Lemma 4.5 that $I_t^\alpha u(t) \in D(A)$ for every $t \ge 0$. Using Lemma 4.19, Lemma 4.5 and (4.35), we have that for every $t \ge 0$,

$$\begin{aligned} u(t) =& \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^n x \\ &+ \sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^n y \\ =& x + ty + \sum_{j=1}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^n x \\ &+ \sum_{j=1}^{n-1} g_{\alpha j+2}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^n y \\ =& x + ty + A\Big[\sum_{j=1}^{n-1} g_{\alpha j+1}(t)A^{j-1} x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^{n-1} x\Big] \\ &+ A\Big[\sum_{j=1}^{n-1} g_{\alpha j+2}(t)A^{j-1} y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^{n-1} y\Big] \\ =& x + ty + A\Big[\sum_{j=1}^{n} g_{\alpha j+1}(t)A^{j-1} x + (g_{\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n x\Big] \\ &+ A\Big[\sum_{j=1}^{n} g_{\alpha j+2}(t)A^{j-1} y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n y\Big] \\ =& x + ty + A\Big[\sum_{j=0}^{n-1} g_{\alpha j+1+\alpha}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n x\Big] \\ &+ A\Big[\sum_{j=0}^{n-1} g_{\alpha j+2+\alpha}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * g_\alpha * \mathbb{S}_\alpha^\beta)(t)A^n y\Big] \\ =& x + ty + Ag_\alpha * \Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})} * \mathbb{S}_\alpha^\beta)(t)A^n x\Big] \\ &+ Ag_\alpha * \Big[\sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_1 * \mathbb{S}_\alpha^\beta)(t)A^n y\Big] \end{aligned}$$

$$=x + ty + A(g_\alpha * u)(t). \tag{6.5}$$

Hence, u is a mild solution of (6.1) and this completes the proof of the existence part in the assertion (b).

It remains to show the uniqueness of solutions. Let $x, y \in D(A^n)$ and let u be a mild solution. We just have to show that $(g_1 * u)(t)$ is exponentially bounded. Using the first equality in (6.5), we have that for every $t \geq 0$,

$$\begin{aligned}(g_1 * u)(t) = &\sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j x + (g_{\alpha(n-\frac{\beta}{2})+1} * \mathbb{S}_\alpha^\beta)(t)A^n x \\ &+ \sum_{j=0}^{n-1} g_{\alpha j+3}(t)A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_2 * \mathbb{S}_\alpha^\beta)(t)A^n y.\end{aligned}$$

Using Lemma 4.3 we get from the preceding equality that there exist some constants $M, \omega \geq 0$ such that for every $t \geq 0$,

$$\|(g_1 * u)(t)\| \leq Me^{\omega t} \sum_{j=0}^{n} (\|A^j x\| + \|A^j y\|).$$

We have shown that $(g_1 * u)(t)$ is exponentially bounded. Now, Proposition 6.3 implies the uniqueness of mild and classical solutions. The proof of the theorem is finished. $\square$

Remark 6.6. We observe that although in (6.1) we have the Caputo fractional derivative $\mathbb{D}_t^\alpha$, the solution is given by the Riemann-Liouville derivative $D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y$. If $\frac{\alpha\beta}{2}$ is not an integer, then the function $\mathbb{D}_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + \mathbb{D}_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y$ is not a solution of (6.1), unless $x = y = 0$.

7. Resolvent families and the inhomogeneous Cauchy problem

In this section we study the solvability and the representation of solutions of the inhomogeneous fractional order abstract Cauchy problem

$$\begin{gathered}\mathbb{D}_t^\alpha u(t) = Au(t) + f(t), \quad t > 0,\ 1 < \alpha \leq 2, \\ u(0) = x, \quad u'(0) = y,\end{gathered} \tag{7.1}$$

where A is a closed linear operator with domain $D(A)$ defined in a Banach space, $f:\ [0, \infty) \to X$ is a given function and x, y are given vectors in X.

Definition 7.1. A function $u \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$ is said to be a classical solution of Problem (7.1) if $g_{2-\alpha} * (u - u(0) - u'(0)t) \in C^2([0, \infty); X)$ and (7.1) is satisfied.

We adopt the following definition of mild solutions.

Definition 7.2. A function $u \in C([0, \infty); X)$ is said to be a mild solution of Problem (7.1) if $I_t^\alpha u(t) := (g_\alpha * u)(t) \in D(A)$ for every $t \geq 0$, and

$$u(t) = x + ty + A \int_0^t g_\alpha(t-s)u(s)\, ds + \int_0^t g_\alpha(t-s)f(s)\, ds, \quad \forall t \geq 0.$$

We have the following uniqueness result.

Proposition 7.3. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $1 < \alpha \leq 2$. Then the following assertions hold.*

(a) *If u is a classical solution of* (7.1)*, then it is a mild solution of* (7.1)*.*
(b) *If $(\lambda^\alpha - A)$ is invertible for* $\mathrm{Re}(\lambda)$ *large enough, and if a mild solution u exists and $(g_1 * u)(t)$ is exponentially bounded, then it is unique.*

Proof. Let $1 < \alpha \le 2$ and let A be a closed linear operator with domain $D(A)$ defined on a Banach space X.

(a) Let u be a classical solution of (7.1). Since $u \in C([0,\infty); D(A))$, we have that $(g_\alpha * u)(t) \in C([0,\infty); D(A))$. Since $\mathbb{D}_t^\alpha u(t) = Au(t) + f(t)$, that is, $(g_{2-\alpha} * u'')(t) = Au(t) + f(t)$, we have that $(g_\alpha * g_{2-\alpha} * u'')(t) = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$, i.e., $(g_2 * u'')(t) = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$. Hence, $u(t) - u(0) - tu'(0) = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$ for every $t \ge 0$ and we have shown that u is a mild solution of (7.1).

(b) Assume that (7.1) has two mild solutions u and v and set $U := u - v$. Then $U \in C([0,\infty); X)$, $(g_\alpha * U)(t) \in D(A)$ for every $t \ge 0$ and $U(t) = A(g_\alpha * U)(t)$. Taking the Laplace transform, we get that $(I - \lambda^{-\alpha} A)\widehat{U}(\lambda) = 0$ for $\mathrm{Re}(\lambda) > \omega$ (where $\omega \ge 0$ is the real number from the exponential boundedness of $(g_1 * u)(t)$). Since $(I - \lambda^{-\alpha} A)$ is invertible, we have that $\widehat{U}(\lambda) = 0$. By the uniqueness theorem for the Laplace transform and by continuity, we get that $U(t) = 0$ for every $t \ge 0$. Hence, $u(t) = v(t)$ for every $t \ge 0$. The proof is finished. $\square$

Remark 7.4. As for the homogeneous equation in Section 6, to prove the existence of mild and classical solutions of Problem (7.1), we proceed by a direct method with minimal use of the Laplace transform.

We have the following result of existence and representation of classical and mild solutions which is the main result of this section.

Theorem 7.5. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \le 2, \beta > 0$ and set $n := \lceil \beta \rceil$, $k := \lceil \frac{\alpha\beta}{2} \rceil$. Assume that A generates an $(\alpha, 1)^\beta$-resolvent family $\mathbb{S}_\alpha^\beta$. Let $\mathbb{P}_\alpha^\beta$ be the $(\alpha,\alpha)^\beta$-resolvent family generated by A. Then the following assertions hold.*

(a) For every $f \in C^k([0,\infty); D(A)) \cap C^{k+1}([0,\infty); X)$, $f^{(2i)}(0), f^{(2i+1)}(0) \in D(A^{n+1-i})$, $i = 0,1,\dots,\frac{k-1}{2}$, if k is odd, $f^{(2i)}(0) \in D(A^{n+1-i})$, $i = 0,1,\dots,\frac{k}{2}$, $f^{(2i+1)}(0) \in D(A^{n+1-i})$, $i = 0,\dots,\frac{k}{2}-1$, if k is even, $\mathbb{D}_t^{\frac{\alpha\beta}{2}} f(t) := (g_{k-\frac{\alpha\beta}{2}} * f^{(k)})(t)$ is exponentially bounded, and for every $x, y \in D(A^{n+1})$, Problem (7.1) *has a unique classical solution u given by*

$$u(t) = D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y + D_t^{\frac{\alpha\beta}{2}} (\mathbb{P}_\alpha^\beta * f)(t), \quad t \ge 0. \tag{7.2}$$

(b) For every $f \in C^k([0,\infty); X)$, $f^{(2i)}(0)$, $i = 0,1,\dots,\frac{k-1}{2}$, $f^{(2i+1)}(0) \in D(A^{n-i})$, $i = 0,1,\dots,\frac{k-1}{2}-1$, if k is odd, $f^{(2i)}(0), f^{(2i+1)}(0) \in D(A^{n-i})$, $i = 0,1,\dots,\frac{k}{2}-1$, if k is even, $\mathbb{D}_t^{\frac{\alpha\beta}{2}} f(t) := (g_{k-\frac{\alpha\beta}{2}} * f^{(k)})(t)$ is exponentially bounded, and for every $x, y \in D(A^n)$, Problem (7.1) *has a unique mild solution u given by* (7.2)*.*

Proof. Let A, α, β, n and k be as in the statement of the theorem. First we prove existence of classical and mild solutions.

(a) Let $x, y \in D(A^{n+1})$. It follows from the proof of Theorem 6.5(a) that $D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$. Moreover,

$$D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(0)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(0)y = x,$$

$$\frac{d}{dt}\big[D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta\big](0)x + \frac{d}{dt}\big[D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)\big](0)y = y.$$

Now, assume that f satisfies the assumptions in the statement of part (a) of the theorem. Using Remark 4.4 and (2.5), we get that for every $t \ge 0$,

$$
\begin{aligned}
&D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)(t)\\
&= D_t^{\frac{\alpha\beta}{2}}(g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t) = \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t)\Big]\\
&= \frac{d^{k-1}}{dt^{k-1}}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta)(t)f(0)\Big] + (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f')(t)\\
&= \sum_{i=0}^{k-1}\frac{d^{k-1-i}}{dt^{k-1-i}}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta)(t)f^{(i)}(0)\Big] + (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f^{(k)})(t)\\
&= \sum_{i=0}^{k-1}\frac{d^{k}}{dt^{k}}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * g_{i+1} * \mathbb{S}_\alpha^\beta)(t)f^{(i)}(0)\Big] + (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f^{(k)})(t)\\
&= \sum_{i=0}^{k-1} g_\alpha * \frac{d^{k}}{dt^{k}}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{i} * \mathbb{S}_\alpha^\beta)(t)f^{(i)}(0)\Big] + (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f^{(k)})(t).
\end{aligned}
\tag{7.3}
$$

If k is odd, then using (7.3) we have that for every $t \ge 0$,

$$
\begin{aligned}
D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)(t) &= \sum_{i=0}^{\frac{k-1}{2}} g_\alpha * \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{2i} * \mathbb{S}_\alpha^\beta)(t)f^{(2i)}(0)\Big]\\
&\quad + \sum_{i=0}^{\frac{k-1}{2}-1} g_\alpha * \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{2i+1} * \mathbb{S}_\alpha^\beta)(t)f^{(2i+1)}(0)\Big]\\
&\quad + (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f^{(k)})(t)\\
&= \sum_{i=0}^{\frac{k-1}{2}} g_{\alpha+i(2-\alpha)} * \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha i} * \mathbb{S}_\alpha^\beta)(t)f^{(2i)}(0)\Big]\\
&\quad + \sum_{i=0}^{\frac{k-1}{2}-1} g_{\alpha+(2-\alpha)i+1} * \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha i} * \mathbb{S}_\alpha^\beta)(t)f^{(2i+1)}(0)\Big]\\
&\quad + (g_{k-\alpha\beta} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f^{(k)})(t).
\end{aligned}
$$

Using the preceding equality, Lemma 4.19(c) and Lemma 4.5, we get that for every $t \ge 0$,

$$
\begin{aligned}
D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)(t) &= \sum_{i=0}^{\frac{k-1}{2}}\sum_{j=0}^{n-1-i} g_{\alpha(i+j)+(2-\alpha)i+\alpha+1}(t)A^j f^{(2i)}(0)\\
&\quad + \sum_{i=0}^{\frac{k-1}{2}} (g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+\alpha} * \mathbb{S}_\alpha^\beta)(t)A^{n-i} f^{(2i)}(0)\\
&\quad + \sum_{i=0}^{\frac{k-1}{2}-1}\sum_{j=0}^{n-1-i} g_{\alpha(i+j)+(2-\alpha)i+\alpha+2}(t)A^j f^{(2i+1)}(0)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{\frac{k-1}{2}-1}(g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+\alpha+1}*\mathbb{S}_{\alpha}^{\beta})(t)A^{n-i}f^{(2i+1)}(0)\\
&+(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha-1}*\mathbb{S}_{\alpha}^{\beta}*f^{(k)})(t).
\end{aligned}\tag{7.4}
$$

Using the above equality we get that for every $t\ge 0$,

$$
\begin{aligned}
&\frac{d}{dt}\big[D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_{\alpha}^{\beta}*f)(t)\big]\\
&=\sum_{i=0}^{\frac{k-1}{2}}\sum_{j=0}^{n-1-i}g_{\alpha(i+j)+(2-\alpha)i+\alpha}(t)A^{j}f^{(2i)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}}(g_{\alpha(n-\beta)+(2-\alpha)i+\alpha-1}*\mathbb{S}_{\alpha}^{\beta})(t)A^{n-i}f^{(2i)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}-1}\sum_{j=0}^{n-1-i}g_{\alpha(i+j)+(2-\alpha)i+\alpha+1}(t)A^{j}f^{(2i+1)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}-1}(g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+\alpha}*\mathbb{S}_{\alpha}^{\beta})(t)A^{n-i}f^{(2i+1)}(0)\\
&+(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha-1}*\mathbb{S}_{\alpha}^{\beta})(t)f^{(k)}(0)+(g_{k-\frac{\alpha\beta}{2}}*g_{\alpha-1}*\mathbb{S}_{\alpha}^{\beta}*f^{(k+1)})(t).
\end{aligned}\tag{7.5}
$$

Now, it follows from (7.4), (7.5), Lemma 4.19, Lemma 4.5 and the hypothesis, that $D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_{\alpha}^{\beta}*f)(t)\in C([0,\infty);D(A))\cap C^1([0,\infty);X)$.

If k is even, proceeding as for the case k odd and using Lemma 4.19, Lemma 4.5 and the hypothesis, we also get that $D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_{\alpha}^{\beta}*f)(t)\in C([0,\infty);D(A))\cap C^1([0,\infty);X)$.

From (7.4) and (7.5), it is clear that $D_t^{\alpha\beta}(\mathbb{P}_{\alpha}^{\beta}*f)(0)=\frac{d}{dt}\big[D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_{\alpha}^{\beta}*f)\big](0)=0$. We have shown that $u\in C([0,\infty);D(A))\cap C^1([0,\infty);X)$, $u(0)=x$ and $u'(0)=y$. By the proof of Theorem 6.5(a) we have that

$$
g_{2-\alpha}*\Big[D_t^{\frac{\alpha\beta}{2}}\mathbb{S}_{\alpha}^{\beta}(t)x-x+D_t^{\frac{\alpha\beta}{2}}(g_1*\mathbb{S}_{\alpha}^{\beta})(t)y-ty\Big]\in C^2([0,\infty);X).
$$

Using (7.4), we have that if k is odd, then for every $t\ge 0$,

$$
\begin{aligned}
\frac{d^2}{dt^2}\Big[g_{2-\alpha}*D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_{\alpha}^{\beta}*f)(t)\Big]&=\sum_{i=0}^{\frac{k-1}{2}}\sum_{j=0}^{n-1-i}g_{\alpha(i+j)+(2-\alpha)i+1}(t)A^{j}f^{(2i)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}}(g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i}*\mathbb{S}_{\alpha}^{\beta})(t)A^{n-i}f^{(2i)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}-1}\sum_{j=0}^{n-1-i}g_{\alpha(i+j)+(2-\alpha)i+2}(t)A^{j}f^{(2i+1)}(0)\\
&+\sum_{i=0}^{\frac{k-1}{2}-1}(g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+1}*\mathbb{S}_{\alpha}^{\beta})(t)A^{n-i}f^{(2i+1)}(0)
\end{aligned}
$$

$$+ (g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta)(t) f^{(k)}(0) + (g_{k-\frac{\alpha\beta}{2}} * \mathbb{S}_\alpha^\beta * f^{(k+1)})(t).$$

We get a similar formula if k is even. Therefore, $(g_{2-\alpha} * D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)) \in C^2([0,\infty);X)$ and hence, $(g_{2-\alpha} * (u - u(0) - u'(0)g_2) \in C^2([0,\infty);X)$. It also follows from the proof of Theorem 6.5(a) that for every $t \geq 0$,

$$\mathbb{D}_t^\alpha \Big[D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}}(g_1 * \mathbb{S}_\alpha^\beta)(t)y \Big] = A\Big[D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}}(g_1 * \mathbb{S}_\alpha^\beta)(t)y \Big]. \tag{7.6}$$

Using Lemma 4.5, we get that for every $t \geq 0$,

$$\begin{aligned}
\mathbb{D}_t^\alpha D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)(t) =& \mathbb{D}_t^\alpha D_t^{\frac{\alpha\beta}{2}}(g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t) \\
=& \Big(g_{2-\alpha} * \frac{d^2}{dt^2}\big[D_t^{\frac{\alpha\beta}{2}}(g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)\big] \Big)(t) \\
=& \frac{d^{k+2}}{dt^{k+2}} \Big[(g_{k+2-\frac{\alpha\beta}{2}-\alpha} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t) \Big] \\
=& \frac{d^{k+2}}{dt^{k+2}} \Big[(g_{k+2} * f)(t) + (g_{k+2-\frac{\alpha\beta}{2}} * A g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t) \Big] \\
=& f(t) + A \frac{d^k}{dt^k}\Big[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t) \Big] \\
=& f(t) + A D_t^{\frac{\alpha\beta}{2}}(\mathbb{P}_\alpha^\beta * f)(t).
\end{aligned} \tag{7.7}$$

It follows from (7.6) and (7.7) that $\mathbb{D}_t^\alpha u(t) = Au(t) + f(t)$ for every $t \geq 0$. Hence, u is a classical solution of (7.1) and this completes the proof of the existence part in the assertion (a).

(b) Let $x, y \in D(A^n)$. From the proof of Theorem 6.5(b) we have $D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}}(g_1 * \mathbb{S}_\alpha^\beta)(t)y \in C([0,\infty);X)$ and that $I_t^\alpha\big[D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}}(g_1 * \mathbb{S}_\alpha^\beta)(t)y \big] \in D(A)$ for all $t \geq 0$. Assume that f satisfies the hypothesis in the statement of part (b) of the theorem. Using (7.4), Lemma 4.6, Lemma 4.5 and Lemma 4.19, we have that if k is odd, then for every $t \geq 0$,

$$\begin{aligned}
I_t^\alpha D_t^{\frac{\alpha\beta}{2}} \left(\mathbb{P}_\alpha^\beta * f \right)(t) =& g_\alpha * D_t^{\frac{\alpha\beta}{2}} \left(g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f \right)(t) \\
=& \sum_{i=0}^{\frac{k-1}{2}-1} \sum_{j=0}^{n-2-i} g_{\alpha(i+j)+(2-\alpha)i+\alpha+1}(t) A^j f^{(2i)}(0) \\
&+ \sum_{i=0}^{\frac{k-1}{2}-1} (g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+\alpha} * \mathbb{S}_\alpha^\beta)(t) A^{n-1-i} f^{(2i)}(0) \\
&+ \sum_{i=0}^{\frac{k-1}{2}-2} \sum_{j=0}^{n-1-i} g_{\alpha(i+j)+(2-\alpha)i+\alpha+2}(t) A^j f^{(2i+1)}(0) \\
&+ \sum_{i=0}^{\frac{k-1}{2}-2} (g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+\alpha+1} * \mathbb{S}_\alpha^\beta)(t) A^{n-1-i} f^{(2i+1)}(0) \\
&+ (g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * g_\alpha * \mathbb{S}_\alpha^\beta * f^{(k)})(t) \in D(A).
\end{aligned} \tag{7.8}$$

We get a similar formula if k is even. Hence, for every $t \geq 0$,

$$I_t^\alpha u(t) = I_t^\alpha D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + I_t^\alpha D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y + I_t^\alpha D_t^{\frac{\alpha\beta}{2}} \left(\mathbb{P}_\alpha^\beta \; astf\right)(t) \in D(A).$$

It follows from (6.5) in the proof of Theorem 6.5 that for every $t \geq 0$,

$$\begin{aligned} &D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y \\ &= x + ty + A\Big[(g_\alpha * D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta)(t)x + (g_\alpha * D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta))(t)y\Big]. \end{aligned} \tag{7.9}$$

Proceeding as in (7.3) and using Lemma 4.5 and (2.3), we have that for every $t \geq 0$,

$$\begin{aligned} D_t^{\frac{\alpha\beta}{2}} \left(\mathbb{P}_\alpha^\beta * f\right)(t) =& \frac{d^k}{dt^k}\left[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t)\right] \\ =& \frac{d^k}{dt^k}\left[(g_k * g_\alpha * f)(t) + A(g_\alpha * g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t)\right] \\ =& (g_\alpha * f)(t) + A g_\alpha * \frac{d^k}{dt^k}\left[(g_{k-\frac{\alpha\beta}{2}} * g_{\alpha-1} * \mathbb{S}_\alpha^\beta * f)(t)\right] \\ =& (g_\alpha * f)(t) + A(g_\alpha * D_t^{\frac{\alpha\beta}{2}} \left(\mathbb{P}_\alpha^\beta * f\right))(t). \end{aligned} \tag{7.10}$$

Combining (7.9) and (7.10), we get that for every $t \geq 0$,

$$\begin{aligned} u(t) =& D_t^{\frac{\alpha\beta}{2}} \mathbb{S}_\alpha^\beta(t)x + D_t^{\frac{\alpha\beta}{2}} (g_1 * \mathbb{S}_\alpha^\beta)(t)y + D_t^{\frac{\alpha\beta}{2}} \left(\mathbb{P}_\alpha^\beta * f\right)(t) \\ =& x + ty + A(g_\alpha * u)(t) + (g_\alpha * f)(t). \end{aligned}$$

Hence, u is a mild solution of Problem (7.1). This completes the proof of the existence part in the assertion (b).

It remains to show the uniqueness of solutions. Let $x, y \in D(A^n)$ and let f satisfy the assumptions in part (b) of the theorem. Let u be a mild solution. Using (6.3) and proceeding as in (7.8) we get that, if k is odd, then for every $t \geq 0$,

$$\begin{aligned} (g_1 * u)(t) =& \sum_{j=0}^{n-1} g_{\alpha j+2}(t) A^j x + (g_{\alpha(n-\frac{\beta}{2})+1} * \mathbb{S}_\alpha^\beta)(t) A^n x \\ &+ \sum_{j=0}^{n-1} g_{\alpha j+3}(t) A^j y + (g_{\alpha(n-\frac{\beta}{2})} * g_2 * \mathbb{S}_\alpha^\beta)(t) A^n y \\ &+ \sum_{i=0}^{\frac{k-1}{2}-1} \sum_{j=0}^{n-2-i} g_{\alpha(i+j)+(2-\alpha)i+2}(t) A^j f^{(2i)}(0) \\ &+ \sum_{i=0}^{\frac{k-1}{2}-1} (g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+1} * \mathbb{S}_\alpha^\beta)(t) A^{n-1-i} f^{(2i)}(0) \\ &+ \sum_{i=0}^{\frac{k-1}{2}-2} \sum_{j=0}^{n-1-i} g_{\alpha(i+j)+(2-\alpha)i+3}(t) A^j f^{(2i+1)}(0) \\ &+ \sum_{i=0}^{\frac{k-1}{2}-2} (g_{\alpha(n-\frac{\beta}{2})+(2-\alpha)i+2} * \mathbb{S}_\alpha^\beta)(t) A^{n-1-i} f^{(2i+1)}(0) \\ &+ (g_{k-\frac{\alpha\beta}{2}} * g_\alpha * \mathbb{S}_\alpha^\beta * f^{(k)})(t). \end{aligned} \tag{7.11}$$

We get a similar equality if k is even. Since by assumption $(g_1 * \mathbb{S}_\alpha^\beta)(t)$ is exponentially bounded, and that there exist some constants $M_1, \omega_1 \geq 0$ such that $\|(g_{k-\frac{\alpha\beta}{2}} * f^{(k)})(t)\| \leq M_1 e^{\omega_1 t}$, $t \geq 0$, it follows from (7.11) that if k is odd, then there exist some constants $M, \omega \geq 0$ such that for every $t \geq 0$,

$$
\begin{aligned}
&\|(g_1 * u)(t)\| \\
&\leq Me^{\omega t}\Big[\sum_{j=0}^{n}(\|A^j x\| + \|A^j y\|) + \sum_{i=0}^{\frac{k-1}{2}-1}\sum_{j=0}^{n-2-i}\|A^j f^{(2i)}(0)\| \\
&\quad + \sum_{i=0}^{\frac{k-1}{2}-1}\|A^{n-1-i} f^{(2i)}(0)\|\Big] + Me^{\omega t}\Big[\sum_{i=0}^{\frac{k-1}{2}-2}\sum_{j=0}^{n-1-i}\|A^j f^{(2i+1)}(0)\| \\
&\quad + \sum_{i=0}^{\frac{k-1}{2}-2}\|A^{n-1-i} f^{(2i+1)}(0)\| + M_1 e^{\omega_1 t}\Big].
\end{aligned}
$$

We get a similar estimate if k is even. We have shown that $(g_1 * u)(t)$ is exponentially bounded. Now, the uniqueness of mild and classical solutions follows from Proposition 7.3 and this completes the proof. □

Remark 7.6. Theorem 7.5 holds in the case $\beta = 0$ as follows: Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $1 < \alpha \leq 2$. Assume that A generates an $(\alpha, 1)$-resolvent family $\mathbb{S}_\alpha$. Let $\mathbb{P}_\alpha$ be the (α, α)-resolvent family generated by A. Then the following assertions hold.

(a) For every $f \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ exponentially bounded, and for every $x, y \in D(A)$, Problem (7.1) has a unique classical solution

$$
u(t) = \mathbb{S}_\alpha(t)x + (g_1 * \mathbb{S}_\alpha)(t)y + (\mathbb{P}_\alpha * f)(t), \quad t \geq 0.
$$

(b) For every $f \in C([0,\infty); X)$ exponentially bounded, and for every $x, y \in X$, Problem (7.1) has a unique mild solution u given by (7.2).

8. Applications

In this section we give some examples where the situations of the previous sections are applied. Throughout this section $\Omega \subset \mathbb{R}^N$ denotes an open set with Lipschitz continuous boundary $\partial\Omega$. Let the real valued coefficients satisfy $a_{ij} \in L^\infty(\Omega)$, $b_j, c_j, d \in L^\infty(\Omega)$, $i, j = 1, 2, \dots, N$. We assume also that there exists a constant $\mu > 0$ such that

$$
\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N,
$$

for a.e. $x \in \Omega$. Let A be the elliptic operator formally given by

$$
Au = \sum_{j=1}^{N} D_j\Big(\sum_{i=1}^{N} a_{i,j} D_i u + b_j u\Big) - \Big(\sum_{i=1}^{N} c_i D_i u + du\Big). \tag{8.1}
$$

Next we have an example for Dirichlet, Neumann and Robin boundary conditions on L^2-spaces.

Example 8.1. For $1 < \alpha \le 2$, we consider the fractional order Cauchy problem

$$\begin{gathered}\mathbb{D}_t^\alpha u(t,x) - Au(t,x) = f(t,x), \quad t>0,\ x\in\Omega,\\ \frac{\partial u(t,z)}{\partial \nu_A} + \gamma(z)u(t,z) = 0, \quad t>0,\ z\in\partial\Omega,\\ u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad x\in\Omega.\end{gathered} \tag{8.2}$$

Here, $u_0, u_1 \in L^2(\Omega)$, $f \in C([0,\infty); L^2(\Omega))$, A is the operator given in (8.1),

$$\frac{\partial u}{\partial \nu_A} = \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u\Big)\cdot \nu_j,$$

where ν denotes the outer normal vector of Ω at the boundary $\partial\Omega$ and $\gamma \ge 0$ belongs to $L^\infty(\partial\Omega)$ of $\gamma = \infty$. If $\gamma = \infty$, then the boundary conditions in (8.2) become the Dirichlet boundary conditions $u(t,z) = 0$, $t>0$ and $z\in\partial\Omega$ (see e.g. [6, 7]).

We consider the first order Sobolev spaces

$$H^1(\Omega) := \{u\in L^2(\Omega),\ \int_\Omega |\nabla u|^2\,dx < \infty\}$$

endowed with the norm

$$\|u\|_{H^1(\Omega)} := \Big(\int_\Omega |u|^2\,dx + \int_\Omega |\nabla u|^2\,dx\Big)^{1/2},$$

and $H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$ where $\mathcal{D}(\Omega)$ denotes the space of test functions on Ω.

Let $\mathcal{A}_\gamma$ be the bilinear form on $L^2(\Omega)$ with domain $H^1(\Omega)$ and given for $u,v\in H^1(\Omega)$ by

$$\begin{aligned}\mathcal{A}_\gamma(u,v) := &\int_\Omega \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u\Big) D_j v\,dx\\ &+ \int_\Omega \Big(\sum_{j=1}^N c_j D_j u + du\Big) v\,dx + \int_{\partial\Omega} \gamma u v\,d\sigma,\end{aligned}$$

where σ denotes the usual Lebesgue surface measure on the boundary $\partial\Omega$, and let $\mathcal{A}_D$ be the bilinear form on $L^2(\Omega)$ with domain $H_0^1(\Omega)$ and given for $u,v\in H_0^1(\Omega)$ by

$$\mathcal{A}_D(u,v) := \int_\Omega \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u\Big) D_j v\,dx + \int_\Omega \Big(\sum_{j=1}^N c_j D_j u + du\Big) v\,dx.$$

It is easy to see that the bilinear forms $\mathcal{A}_\gamma$ and $\mathcal{A}_D$ are closed in $L^2(\Omega)$. Let $A_{2,\gamma}$ and $A_{2,D}$ be the closed linear operators in $L^2(\Omega)$ associated with the form $\mathcal{A}_\gamma$ and $\mathcal{A}_D$, respectively. That is,

$$\begin{gathered}D(A_{2,\gamma}) := \{u\in H^1(\Omega),\ \exists\, v\in L^2(\Omega),\ \mathcal{A}_\gamma(u,\varphi) = (v,\varphi)_{L^2(\Omega)},\ \forall \varphi\in H^1(\Omega)\}\\ A_{2,\gamma} u = v\end{gathered}$$

and

$$\begin{gathered}D(A_{2,D}) := \{u\in H_0^1(\Omega),\ \exists\, v\in L^2(\Omega),\ \mathcal{A}_D(u,\varphi) = (v,\varphi)_{L^2(\Omega)},\ \forall \varphi\in H_0^1(\Omega)\}\\ A_{2,D} u = v.\end{gathered}$$

One has the following more explicit description of the operators $A_{2,\gamma}$ and $A_{2,D}$ on $L^2(\Omega)$.

$$D(A_{2,\gamma}) = \{u \in H^1(\Omega),\ Au \in L^2(\Omega),\ \frac{\partial u}{\partial \nu_A} + \gamma u = 0\}, \quad A_{2,\gamma}u = Au,$$
$$D(A_{2,D}) = \{u \in H_0^1(\Omega) :\ Au \in L^2(\Omega)\}, \quad A_{2,D}u = Au.$$

The operator $A_{2,\beta}$ (resp. $A_{2,D}$) is a realization of the operator A in $L^2(\Omega)$ with Robin boundary conditions and Neumann boundary conditions if $\gamma = 0$ (resp. with Dirichlet boundary conditions). With this setting Problem (8.2) can be rewritten as an abstract Cauchy problem in the Hilbert space $L^2(\Omega)$,

$$\mathbb{D}_t^\alpha u(t) = \widetilde{A}u(t) + f(t),\ t \ge 0,\ 1 < \alpha \le 2,$$
$$u(0) = u_0, \quad u_t(0), = u_1,$$

with $\widetilde{A} = A_{2,\gamma}$ or $A_{2,D}$. It is well-known (see e.g. [3]) that the operators $A_{2,\beta}$ and $A_{2,D}$ generate cosine families on $L^2(\Omega)$ and hence generate $(\alpha,1)$-resolvent families $\mathbb{S}_\alpha$ for every $1 < \alpha \le 2$. Therefore all the results in Theorem 7.5 hold for Problem (8.2) with $n = k = 0$.

Next, we consider the one-dimensional case.

Example 8.2 (Elliptic operators in one-dimension)**.** Let $a \in W^{1,\infty}(0,1)$ satisfy $a(x) \ge \mu_0 > 0$ for some constant μ_0. Let $b, c \in L^\infty(0,1)$, $1 \le p < \infty$ and let α_j, β_j $(j = 0,1)$ be complex numbers such that $(\alpha_j, \beta_j) \ne (0,0)$. For $1 < \alpha \le 2$, we consider the fractional order Cauchy problem

$$\begin{gathered}\mathbb{D}_t^\alpha u(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t,x) + c(x)u(t,x) + f(t,x), \quad t > 0,\ x \in (0,1),\\ \alpha_j u_x(t,j) + \beta_j u(t,j) = 0, \quad j = 0,1,\ t > 0,\\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad x \in (0,1).\end{gathered} \tag{8.3}$$

Let $\widetilde{A}_p$ be the operator defined on $L^p(0,1)$ by

$$D(\widetilde{A}_p) := \{u \in W^{2,p}(0,1) : \alpha_j u'(j) + \beta_j u(j) = 0,\ j = 0,1\},$$
$$A_p u = a(x)u'' + b(x)u' + c(x)u.$$

The operator $\widetilde{A}_p$ is a realization of A (given by $Au = a(x)u'' + b(x)u' + c(x)u$) on $L^p(0,1)$ with Dirichlet boundary conditions if $\alpha_j = 0, \beta_j \ne 0$ $(j = 0,1)$, with Neumann boundary conditions if $\alpha_j \ne 0, \beta_j = 0$ $(j = 0,1)$ and Robin boundary conditions if $\alpha_j \ne 0, \beta_j \ne 0$ $(j = 0,1)$. With the same assumption on α_j, β_j, a realization $\widetilde{A}_\infty$ of A with Dirichlet boundary condition on $C_0(0,1) := \{u \in C[0,1] : u(0) = u(1) = 0\}$ or with Neumann and Robin boundary conditions on $C[0,1]$ is given by

$$D(\widetilde{A}_\infty) := \{u \in C^2[0,1] : \alpha_j u'(j) + \beta_j u(j) = 0,\ j = 0,1\},$$
$$A_\infty u = a(x)u'' + b(x)u' + c(x)u.$$

By [10, 30] the operator $\widetilde{A}_p$ generates a cosine family on $L^p(0,1)$ and $\widetilde{A}_\infty$ generates a cosine family on $C[0,1]$ (on $C_0(0,1)$ if it is Dirichlet boundary condition). The case of Wentzell (or dynamical) boundary conditions on $L^p(0,1) \times \mathbb{C}$ and on $C[0,1]$ has been investigated in [1, 31]. Therefore, one has the same results for Problem (8.3) as the ones given in Example 8.1. More precisely, letting $X_p := L^p(0,1)$

(or $L^p(0,1)\times\mathbb{C}$ in the case of Wentzell boundary conditions) if $1\le p<\infty$ and $X_\infty = C[0,1]$ (or $C_0(0,1)$ in the case of Dirichlet boundary condition), then all the results in Theorem 7.5 hold for Problem (8.3) with $n=k=0$.

Example 8.3 (Elliptic operators on general L^p spaces)**.** For simplicity we assume that $\Omega\subset\mathbb{R}^N$ ($N\ge 2$) is bounded. For $1<\alpha\le 2$, we consider the fractional order Cauchy problem

$$\begin{gathered}
\mathbb{D}_t^\alpha u(t,x) = Au(t,x) + f(t,x), \quad t>0,\ x\in\Omega,\\
\frac{\partial u(t,z)}{\partial \nu_A} + \gamma(z)u(t,z) = 0, \quad t>0,\ z\in\partial\Omega,\\
u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad x\in\Omega.
\end{gathered} \tag{8.4}$$

Here, $u_0,u_1\in L^p(\Omega)$, $f\in C([0,\infty);L^p(\Omega))$, for some $p\in[1,\infty)$ ($p\ne 2$), or $u_0,u_1\in C(\overline{\Omega})$, $f\in C([0,\infty];C(\overline{\Omega}))$ are given functions, and the operator A is given in (8.1). Let $\widetilde{A}$ be the closed linear operator in $L^2(\Omega)$ introduced in Example 8.1. Recall that $\widetilde{A}=A_{2,\gamma}$ or $\widetilde{A}=A_{2,D}$. For $2\le p<\infty$, we let A_p denote the part of the operator $A_{2,\gamma}$ in $L^p(\Omega)$ and for $1\le p<2$, we let A_p be the closure in $L^p(\Omega)$ of the operator B defined by

$$D(B) = \{u\in D(A_{2,\gamma})\cap L^p(\Omega),\ Au\in L^p(\Omega)\}, \quad Bu = A_{2,\gamma}u = Au.$$

The operator A_p is a realization of the operator A in $L^p(\Omega)$ with Robin boundary conditions, Neumann boundary conditions if $\gamma=0$ and Dirichlet boundary conditions if $\gamma=\infty$. By [30, 42], the operator A_p generates a β-times integrated cosine family $(C_\beta(t))$ on $L^p(\Omega)$ with $\beta := N|\frac12-\frac1p|$. Hence, all the results in Theorem 7.5 hold for Problem (8.4) with $n:=\lceil\beta\rceil$ and $k:=\lceil\frac{\alpha\beta}{2}\rceil$.

Letting A_∞ be a realization of the operator A with Robin, Neumann or Dirichlet boundary conditions on $L^\infty(\Omega)$, we have that A_∞ generates a β-times integrated cosine family on $L^\infty(\Omega)$ with $\beta>\frac{N}{2}$ and one can also apply Theorem 7.5. We notice that $D(A_\infty)$ is not dense in $L^\infty(\Omega)$.

Next, we consider the case of the Laplace operator on some special open subsets of $\mathbb{R}^N$.

Example 8.4 (Laplace operator on some special open sets)**.** Let $\Omega:=\mathbb{R}^N$ or $\Omega:=(0,1)^N\subset\mathbb{R}^N$ and let $\widetilde{A}_p$ be a realization of the Laplace operator on $L^p(\Omega)$ ($p\ne 2$) with Dirichlet, Neumann or Robin boundary conditions defined above. By [17, 24, 30] the operator $\widetilde{A}_p$ generates a β-times integrated cosine family on $L^p(\Omega)$ with $\beta=(N-1)|\frac12-\frac1p|$. Therefore, one has the same results as in Example 8.3 with here $\beta=(N-1)|\frac12-\frac1p|$.

As in the previous example, here also, letting A_∞ be a realization of the Laplace operator with Robin, Neumann or Dirichlet boundary conditions on $L^\infty(\Omega)$, we have that A_∞ generates a β-times integrated cosine family on $L^\infty(\Omega)$ with $\beta=\frac{N-1}{2}$ and one can also apply Theorem 7.5. We also notice that $D(A_\infty)$ is not dense in $L^\infty(\Omega)$.

We conclude the paper with an example involving a Schrödinger like operator.

Example 8.5. We consider the fractional order Schrödinger like equation

$$\begin{gathered}\mathbb{D}_t^\alpha u(t,x) = e^{i\theta}\Delta_p u(t,x) + f(t,x), \quad t>0,\ x\in\mathbb{R}^N,\ 1<\alpha<2,\\ u(0,x)=u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad x\in\mathbb{R}^N.\end{gathered} \tag{8.5}$$

Here, the operator Δ_p is a realization of the Laplace operator on $L^p(\mathbb{R}^N)$, $1\le p<\infty$, the angle θ satisfies $\frac{\pi}{2}<\theta<\left(1-\frac{\alpha}{4}\right)\pi$. Let $A_p := e^{i\theta}\Delta_p$. Then $D(A_p) = W^{2,p}(\mathbb{R}^N)$. We have shown in Example 4.11 that A_p generates an $(\alpha,1)=(\alpha,1)^0$-resolvent family $\mathbb{S}_\alpha$ on $L^p(\mathbb{R}^N)$. Using Theorem 7.5 we get the following results on existence of solutions to Problem (8.5).

- For every $f\in C([0,\infty);W^{2,p}(\mathbb{R}^N))\cap C^1([0,\infty);L^p(\mathbb{R}^N))$ and $u_0,u_1\in W^{2,p}(\mathbb{R}^N)$, Problem (8.5) has a classical solution u.
- For every $f\in C([0,\infty);L^p(\mathbb{R}^N))$ and $u_0,u_1\in L^p(\mathbb{R}^N)$, Problem (8.5) has a mild solution u.

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