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A WAVELET METHOD FOR SOLVING BACKWARD HEAT CONDUCTION PROBLEMS

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Abstract. In this article, we consider the backward heat conduction problem (BHCP). This classical problem is more severely ill-posed than some other problems, since the error of the data will be exponentially amplified at high frequency components. The Meyer wavelet method can eliminate the influence of the high frequency components of the noisy data. The known works on this method are limited to the *a priori* choice of the regularization parameter. In this paper, we consider also *a posteriori* choice of the regularization parameter. The Hölder type stability estimates for both *a priori* and *a posteriori* choice rules are established. Moreover several numerical examples are also provided.

1. Introduction

Wavelet theory has been widely developed in the late of the previous century. The multiscale analysis and wavelet decomposition are now still subjects of intensive development. At the same time, the investigation of mutual interactions between wavelet theory and ill-posed problems has never stopped. Some results have been applied to the analysis of some inverse problems. Wavelet methods have advantages for use in certain inverse problems: 1. They allow for the decomposition of an object into multiple resolutions (or scales). This is a particular advantage for high precision inversion of the objects; 2. The localization of wavelets in both time and frequency makes them quite useful for analyzing local features; 3. Wavelets have excellent data compression capabilities for spatially variable objects, such as signals characterized by singularities or images determined primarily by a set of edges; 4. Some methods of denoising, based on thresholding of the wavelet coefficients, have been proven to be nearly optimal for a number of tasks across a wide range of function classes [13].

We emphasized that Meyer wavelets possess specially important significance for solving many ill-posed problems. Meyer wavelets have the property that their Fourier transforms have compact supports. This means that they can be used to prevent high frequency noise from destroying the solution, i.e., by expanding the data and the solution in a basis of Meyer wavelets, high-frequency components can be filtered away. Within V_j, which is generated by the father wavelet of Meyer, the original ill-posed problem is well-posed, and we can find a regularization parameter

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J depending on the noise level of the data such that the solution in V_j is a good approximation of the original problem. Moreover, the combination of Meyer wavelets method with the fast Fourier transform (FFT) can build high-speed algorithm.

Meyer wavelet techniques have been used by Regińska et al [18, 19], Hào et al [10], Eldén [6], and Wang [24] to solve the inverse heat conduction problems (IHCP), and by Vani [23], Qiu et al [17] to solve the Cauchy problem for the Laplace equation. However, they all used the *a priori* wavelet method, and did not consider the *a posteriori* error estimate. Since the numerical results for the *a posteriori* method do not depend on the *a priori* information, the *a posteriori* wavelet method is more effective to solve practical problems than the *a priori* method. In this paper, we continue to study the *a priori* wavelet method, and then focus on the *a posteriori* wavelet method and its numerical solution.

Although the application of wavelet theory in differential equation has been mostly focused on numerical computation, the connection between the wavelet theory and differential equation is also searched all the time. Shen and Strang in [21] have introduced the concept of heatlets in order to solve the heat equation using wavelet expansions of the initial data. The heatlet is a "fundamental" solution to the heat equation, when the initial data is expanded in terms of the wavelet basis, the solution to the heat equation is then obtained from an expansion using the heatlets and the corresponding wavelet coefficients of the data. In [11] the authors combined heatlets with quasi-reversibility method to regularize the backward heat equation, and obtained some theoretical error estimates. However, there are no numerical results.

In the present paper, we consider the following backward heat equation in a strip domain by a Meyer wavelet method.

$$\begin{gathered} u_t = u_{xx}, \quad -\infty < x < \infty,\ 0 \le t < T, \\ u(x,T) = \varphi_T(x), \quad -\infty < x < \infty, \end{gathered} \tag{1.1}$$

e11

where we want to determine the temperature distribution $u(\cdot,t)$ on the interval $t \in [0,T)$ from the data $\varphi_T(x)$. Backward heat conduction problem is a classical ill-posed problem [12, 14], and is known as the most severely ill-posed problem, which has been studied by many authors by different methods [1, 2, 4, 20, 22, 25].

Let $\widehat{g}(\xi)$ denote the Fourier transform of $g(x)$ defined by

$$\widehat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x)dx, \tag{1.2}$$

e12

and $\|f\|_{H^s}$ denote the norm on the Sobolev space H^s defined by

$$\|f\|_{H^s} := \Big(\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 (1+\xi^2)^s d\xi \Big)^{1/2}. \tag{1.3}$$

e13

When $s = 0$, $\|\cdot\|_{H^0} := \|\cdot\|$ denotes the $L^2(\mathbb{R})$-norm, and $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R})$-inner product. It is easy to know that $L^2(\mathbb{R}) \subset H^s(\mathbb{R})$ for $s \le 0$.

We assume that there exists a solution $u(x,t)$ satisfying (1.1) in the classical sense and $u(\cdot,t) \in L^2(\mathbb{R})$ for $0 < t < T$. Using the Fourier transform technique to problem (1.1) with respect to variable x, we can get the Fourier transform $\widehat{u}(\xi,t)$ of the exact solution $u(x,t)$ of problem (1.1):

$$\widehat{u}(\xi,t) = e^{\xi^2(T-t)} \widehat{\varphi}_T(\xi), \tag{1.4}$$

e14

or equivalently,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi^2 (T-t)} \widehat{\varphi}_T(\xi) d\xi. \qquad (1.5) \qquad \boxed{\texttt{e15}}$$

For any ill-posed problem some *a priori* assumptions on the exact solution are needed, otherwise the convergence of the regularization approximate solution will not be obtained or the convergence rate can be arbitrarily slow [7]. When we consider problem (1.1) in $L^2(\mathbb{R})$ for the variable x, we assume there exists an *a priori* bound for $\varphi_0(x) := u(x,0)$:

$$\|\varphi_0\| = \|u(\cdot,0)\| \le E. \qquad (1.6) \qquad \boxed{\texttt{e17}}$$

From the formula (1.5) and the Parseval formula, we know

$$\|\varphi_0\|^2 = \int_{-\infty}^{\infty} |e^{\xi^2 T} \widehat{\varphi}_T(\xi)|^2 d\xi. \qquad (1.7) \qquad \boxed{\texttt{e18}}$$

For a concrete ill-posed problem, not all regularization methods are effective. For example, for the severely ill-posed problems with growth rate of magnitude factor reaching or exceeding $O(e^{\gamma\xi^2}), \gamma > 0, \xi \to \infty$, the Mollification method with Gauss kernel suggested by Murio [15] cannot deal with them. Problem (1.1) considered in the present paper just is the case as the magnitude factor $e^{\xi^2(T-t)}$. Therefore, the Mollification method is not effective both in theory and numerical computation. In addition, the convergence rate and numerical results are also not completely the same for different regularization methods. For example, for the Modified method suggested by [16], the theoretical convergence rate for problem (1.1) is only logarithm type not Hölder type. So construction of specific regularization methods for different ill-posed problems is significant.

The main goal of this paper is to provide a Meyer wavelet method for solving the backward heat equation (1.1). Our method of proving stability estimates is constructive: We construct a stable solution to the problem for both *a priori* and *a posteriori* choice rules.

The outline of this paper is as follows. In Section 2, a brief survey on some fundamental properties of Meyer wavelet is presented. On this basis, we give the Meyer wavelet regularization method to solve the problem (1.1) for both *a priori* and *a posteriori* parameter choice rules, and obtain the error estimates of the *a priori* and *a posteriori* situations respectively. In Section 3, four numerical examples are provided, and the comparison of numerical effects for *a posteriori* wavelet method with other methods for Examples 3.1 and 3.2 are also taken into account.

2. Wavelet regularization and error estimates

Let $\varphi(x), \psi(x)$ be Meyer scaling and wavelet functions respectively. Then from [3] we know

$$\operatorname{supp} \widehat{\varphi} = [-\frac{4}{3}\pi, \frac{4}{3}\pi], \quad \operatorname{supp} \widehat{\psi} = [-\frac{8}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi],$$

and $\psi_{jk}(x) = 2^{\frac{j}{2}} \psi(2^j x - k), j,k \in \mathbb{Z}$ constitute an orthonormal basis of $L^2(\mathbb{R})$ and

$$\operatorname{supp} \widehat{\psi}_{jk}(\xi) = [-\frac{8}{3}\pi 2^j, -\frac{2}{3}\pi 2^j] \cup [\frac{2}{3}\pi 2^j, \frac{8}{3}\pi 2^j], \quad k \in \mathbb{Z}. \qquad (2.1) \qquad \boxed{\texttt{e21}}$$

The multiresolution analysis (MRA) $\{V_j\}_{j\in\mathbb{Z}}$ of Meyer wavelet is generated by

$$\begin{gathered} V_j = \overline{\{\varphi_{jk} : k \in \mathbb{Z}\}}, \quad \varphi_{jk} := 2^{\frac{j}{2}}\varphi(2^j x - k), \quad j, k \in \mathbb{Z}, \\ \operatorname{supp}\widehat{\varphi}_{jk}(\xi) = [-\frac{4}{3}\pi 2^j, \frac{4}{3}\pi 2^j], \quad k \in \mathbb{Z}. \end{gathered} \tag{2.2}$$

e22

The orthogonal projection of a function $g \in L^2(\mathbb{R})$ on space V_J is given by $P_J g := \sum_{k\in\mathbb{Z}}(g, \varphi_{Jk})\varphi_{Jk}$, while $Q_J g := \sum_{k\in\mathbb{Z}}(g, \psi_{Jk})\psi_{Jk}$ denotes the orthogonal projection on wavelet space W_J with $V_{J+1} = V_J \bigoplus W_J$. It is easy to see from (2.2) and (2.1) that

$$\widehat{P_J g}(\xi) = 0, \quad \text{for } |\xi| \geq \frac{4}{3}\pi 2^J, \tag{2.3}$$

e23

$$\widehat{Q_j g}(\xi) = 0, \quad \text{for } j > J \text{ and } |\xi| < \frac{4}{3}\pi 2^J. \tag{2.4}$$

e24

Since $(I - P_J)g = \sum_{j\geq J} Q_j g$ and from (2.4), we know

$$((I - P_J)g)\widehat{\,}(\xi) = \widehat{Q_J g}(\xi), \quad \text{for } |\xi| < \frac{4}{3}\pi 2^J. \tag{2.5}$$

e25

lem2.1 **Lemma 2.1** ([10]). *Let $\{V_j\}_{j\in\mathbb{Z}}$ be Meyer's MRA and suppose $J \in \mathbb{N}, r \in \mathbb{R}$. Then for all $g \in V_J$, it holds the estimate*

$$\|D^k g\|_{H^r} \leq C 2^{(J-1)k}\|g\|_{H^r}, \quad k \in \mathbb{N}, \tag{2.6}$$

e26

where C is a positive constant and $D^k = \frac{d^k}{dx^k}$.

Define an operator $A_t : \varphi_T(x) \mapsto u(x,t)$ by (1.4), i.e.,

$$A_t \varphi_T = u(x,t), \quad 0 \leq t < T,$$

or

$$\widehat{A_t \varphi_T}(\xi) = e^{\xi^2(T-t)}\widehat{\varphi}_T(\xi), \quad 0 \leq t < T. \tag{2.7}$$

e27

Then we have the following lemma.

lem2.2 **Lemma 2.2.** *Let $\{V_j\}_{j\in\mathbb{Z}}$ be Meyer's MRA and suppose $J \in \mathbb{N}, r \in \mathbb{R}, 0 \leq t < T$. Then for all $g \in V_J$ we have*

$$\|A_t g\|_{H^r} \leq 2C \exp\big\{2^{2(J-1)}(T-t)\big\}\|g\|_{H^r}, \tag{2.8}$$

e28

where constant C is the same as in (2.6).

Proof. From (2.6) we know that

$$\begin{aligned} \|A_t g\|_{H^r} &= \Big(\int_{-\infty}^{\infty} |\widehat{A_t g}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2} \\ &= \Big(\int_{-\infty}^{\infty} |e^{\xi^2(T-t)}\widehat{g}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2} \\ &\leq \Big(\int_{-\infty}^{\infty} |2\cosh(\xi^2(T-t))\widehat{g}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2} \\ &= 2\Big(\int_{-\infty}^{\infty} |\sum_{k=0}^{\infty}\frac{(T-t)^{2k}}{(2k)!}\xi^{4k}\widehat{g}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2} \\ &= 2\Big(\int_{-\infty}^{\infty} |\sum_{k=0}^{\infty}\frac{(T-t)^{2k}}{(2k)!}(i\xi)^{4k}\widehat{g}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\le 2\sum_{k=0}^{\infty}\frac{(T-t)^{2k}}{(2k)!}\|D^{4k}g\|_{H^r}\\
&\le 2C\sum_{k=0}^{\infty}\frac{(T-t)^{2k}}{(2k)!}2^{(J-1)4k}\|g\|_{H^r}\\
&= 2C\cosh\big(2^{2(J-1)}(T-t)\big)\|g\|_{H^r}\\
&\le 2C\exp\big\{2^{2(J-1)}(T-t)\big\}\|g\|_{H^r}.
\end{aligned}
$$

□

Let $\varphi_T(x), \varphi_T^\delta(x)$ be exact and measured data, respectively, which satisfy

$$
\|\varphi_T-\varphi_T^\delta\|_{H^r}<\delta,\quad \text{for some } r\le 0. \tag{2.9}
$$

e29

Since $\varphi_T^\delta(x)$ belongs, in general, to $L^2(\mathbb{R})\subset H^r(\mathbb{R})$ for $r\le 0$, r should not be positive. In general, L^2 *a priori* bound for exact solution as (1.6) can only lead to a Hölder stability estimate for the regularization solution, but this *a priori* assumption can not ensure the convergence of the regularization solution at $t=0$ for problem (1.1). To obtain a more sharp convergence for the regularization solution, here we assume, for some $s\ge r$, there exists an *a priori* bound:

$$
\|\varphi_0\|_{H^s}\le E. \tag{2.10}
$$

e210

Denote operator $A_{t,J}:=A_tP_J$, we can show it approximates A_t in a stable way for an appropriate choice of $J\in\mathbb{N}$ depending on δ and E. In fact, we have

$$
\|A_t\varphi_T-A_{t,J}\varphi_T^\delta\|_{H^r}\le\|A_t\varphi_T-A_{t,J}\varphi_T\|_{H^r}+\|A_{t,J}\varphi_T-A_{t,J}\varphi_T^\delta\|_{H^r}. \tag{2.11}
$$

e211

From Lemma 2.2 and condition (2.9), we can see that the second term of the right-hand side of (2.11) satisfies

$$
\begin{aligned}
\|A_{t,J}\varphi_T-A_{t,J}\varphi_T^\delta\|_{H^r} &= \|A_tP_J(\varphi_T-\varphi_T^\delta)\|_{H^r}\\
&\le 2C\exp\big\{2^{2(J-1)}(T-t)\big\}\|P_J(\varphi_T-\varphi_T^\delta)\|_{H^r}\\
&\le 2C\exp\big\{2^{2(J-1)}(T-t)\big\}\delta.
\end{aligned} \tag{2.12}
$$

e212

For the first term of the right-hand side of (2.11), from (2.3) we have

$$
\begin{aligned}
&\|A_t\varphi_T-A_{t,J}\varphi_T\|_{H^r}\\
&=\|A_t(I-P_J)\varphi_T\|_{H^r}\\
&=\Big(\int_{-\infty}^{\infty}|e^{\xi^2(T-t)}\widehat{\big((I-P_J)\varphi_T\big)}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&=\Big(\int_{|\xi|\ge\frac43\pi 2^J}|e^{\xi^2(T-t)}\widehat{\varphi}_T(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&\quad+\Big(\int_{|\xi|<\frac43\pi 2^J}|e^{\xi^2(T-t)}\widehat{\big((I-P_J)\varphi_T\big)}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}:=I_1+I_2.
\end{aligned} \tag{2.13}
$$

e213

Noting (1.4) and (2.10), we have

$$
\begin{aligned}
I_1 &= \Big(\int_{|\xi|\ge\frac{4}{3}\pi 2^J} |e^{\xi^2(T-t)}\widehat{\varphi}_T(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&= \Big(\int_{|\xi|\ge\frac{4}{3}\pi 2^J} |e^{-t\xi^2}\widehat{\varphi}_0(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&\le \sup_{|\xi|\ge\frac{4}{3}\pi 2^J} e^{-t\xi^2}\frac{1}{(1+\xi^2)^{\frac{s-r}{2}}}\Big(\int_{|\xi|\ge\frac{4}{3}\pi 2^J} |\widehat{\varphi}_0(\xi)|^2(1+\xi^2)^s d\xi\Big)^{1/2}\\
&\le \sup_{|\xi|\ge\frac{4}{3}\pi 2^J} e^{-t\xi^2}\frac{1}{(1+\xi^2)^{\frac{s-r}{2}}}\|\varphi_0\|_{H^s}\\
&\le e^{-t(\frac{4}{3}\pi 2^J)^2}\frac{E}{(\frac{4}{3}\pi 2^J)^{s-r}} \le e^{-t2^{2(J+2)}}\frac{E}{2^{(J+2)(s-r)}}\\
&= 2^{-(J+2)(s-r)}\exp\{-t2^{2(J+2)}\}E.
\end{aligned}
\tag{2.14}
$$

e214

From (2.5), Lemma 2.2, and noting that $Q_J\varphi_T \in W_J \subset V_{J+1}$, it is easy to see that I_2 satisfies:

$$
\begin{aligned}
I_2 &= \Big(\int_{|\xi|<\frac{4}{3}\pi 2^J} |e^{\xi^2(T-t)}\,((I-P_J)\varphi_T)\widehat{\ }(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&= \Big(\int_{|\xi|<\frac{4}{3}\pi 2^J} |e^{\xi^2(T-t)}\widehat{Q_J\varphi_T}(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&\le \|A_tQ_J\varphi_T\|_{H^r}\\
&\le 2C\exp\{2^{\,2J}(T-t)\}\|Q_J\varphi_T\|_{H^r}.
\end{aligned}
$$

Denote χ_J as the characteristic function of the interval $[-\frac{2}{3}\pi 2^J, \frac{2}{3}\pi 2^J]$. We introduce an operator M_J defined by

$$
\widehat{M_Jg} = (1-\chi_J)\widehat{g}, \quad g \in L^2(\mathbb{R}).
$$

Noting that $\varphi_T \in L^2(\mathbb{R})$, so from Parseval formula and (2.1) we have

$$
\begin{aligned}
Q_J\varphi_T &= \sum_{k\in\mathbb{Z}}(\varphi_T,\psi_{Jk})\psi_{Jk} = \sum_{k\in\mathbb{Z}}(\widehat{\varphi}_T,\widehat{\psi}_{Jk})\psi_{Jk}\\
&= \sum_{k\in\mathbb{Z}}((1-\chi_J)\widehat{\varphi}_T,\widehat{\psi}_{Jk})\psi_{Jk}\\
&= \sum_{k\in\mathbb{Z}}(M_J\varphi_T,\psi_{Jk})\psi_{Jk}\\
&= Q_JM_J\varphi_T.
\end{aligned}
$$

So, from (1.4),

$$\begin{aligned}
\|Q_J\varphi_T\|_{H^r} &= \|Q_J M_J \varphi_T\|_{H^r} \le \|M_J\varphi_T\|_{H^r}\\
&= \Big(\int_{|\xi|\ge\frac{2}{3}\pi 2^J} |\widehat{\varphi}_T(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&= \Big(\int_{|\xi|\ge\frac{2}{3}\pi 2^J} |e^{-\xi^2T}\widehat{\varphi}_0(\xi)|^2(1+\xi^2)^r d\xi\Big)^{1/2}\\
&= \Big(\int_{|\xi|\ge\frac{2}{3}\pi 2^J} |e^{-\xi^2T}\frac{1}{(1+\xi^2)^{\frac{s-r}{2}}}|^2|\widehat{\varphi}_0(\xi)|^2(1+\xi^2)^s d\xi\Big)^{1/2}\\
&\le \sup_{|\xi|\ge\frac{2}{3}\pi 2^J}\frac{e^{-\xi^2T}}{|\xi|^{s-r}}\|\varphi_0\|_{H^s}\\
&\le 2^{-(J+1)(s-r)}\exp\big\{-T2^{2(J+1)}\big\}E.
\end{aligned} \tag{2.15}$$

e2*

Therefore,

$$\begin{aligned}
I_2 &\le 2C\exp\big\{2^{2J}(T-t)\big\}\,2^{-(J+1)(s-r)}\exp\big\{-T2^{2(J+1)}\big\}E\\
&\le 2C2^{-(J+1)(s-r)}\exp\big\{2^{2(J+1)}(T-t)-T2^{2(J+1)}\big\}E\\
&= 2C2^{-(J+1)(s-r)}\exp\big\{-t2^{2(J+1)}\big\}E.
\end{aligned} \tag{2.16}$$

e215

From (2.14), (2.16) and (2.13), we obtain

$$\begin{aligned}
&\|A_t\varphi_T - A_{t,J}\varphi_T\|_{H^r}\\
&\le 2^{-(J+2)(s-r)}\exp\{-t2^{2(J+2)}\}E + 2C2^{-(J+1)(s-r)}\exp\big\{-t2^{2(J+1)}\big\}E\\
&\le (1+2C)2^{-(J+1)(s-r)}\exp\big\{-t2^{2(J+1)}\big\}E.
\end{aligned} \tag{2.17}$$

e216

Combining (2.17), (2.12) with (2.11), we have

$$\begin{aligned}
&\|A_t\varphi_T - A_{t,J}\varphi_T^\delta\|_{H^r}\\
&\le 2C\exp\big\{2^{2(J-1)}(T-t)\big\}\delta + (1+2C)2^{-(J+1)(s-r)}\exp\big\{-t2^{2(J+1)}\big\}E\\
&\le 2C\exp\big\{2^{2J}(T-t)\big\}\,\delta + (1+2C)2^{-(J+1)(s-r)}\exp\big\{-t2^{2(J+1)}\big\}E.
\end{aligned} \tag{2.18}$$

e217

Based on the above results, we will give the estimates for the *a priori* and *a posteriori* parameter choice rules, respectively.

2.1. ***A-priori* parameter choice.** Now we can firstly give an error estimate between the wavelet regularization solution $A_{t,J}\varphi_T^\delta$ and the exact solution $u(x,t) = A_t\varphi_T$ in $L^2(\mathbb{R})$.

thm2.1 **Theorem 2.3.** *Suppose that* $\varphi_0 \in L^2(\mathbb{R})$ *and* (2.9), (2.10) *hold for* $r = s = 0$. *The problem of calculating* $A_{t,J}\varphi_T^\delta$ *is stable. Furthermore, taking*

$$J^* := \Big[\frac{1}{2}\log_2\Big(\ln\big(\frac{E}{\delta}\big)^{1/T}\Big)\Big], \tag{2.19}$$

e218

where $[a]$ *denotes the largest integer less than or equal to* $a \in \mathbb{R}$, *then the following stability estimate holds:*

$$\|A_t\varphi_T - A_{t,J^*}\varphi_T^\delta\| \le (4C+1)E^{1-\frac{t}{T}}\delta^{t/T}, \tag{2.20}$$

e219

where C *is the same as in* (2.6).

Proof. Note that

$$
\begin{aligned}
&\exp\big\{2^{2J^*}(T-t)\big\}\delta \\
&\le \exp\big\{(T-t)\ln\big(\frac{E}{\delta}\big)^{1/T}\big\}\delta = \big(\frac{E}{\delta}\big)^{\frac{T-t}{T}}\delta = E^{1-\frac{t}{T}}\delta^{t/T},
\end{aligned}
$$

$$
\exp\big\{-t2^{2(J^*+1)}\big\}E \le \exp\big\{-t\ln\big(\frac{E}{\delta}\big)^{1/T}\big\}E = E^{1-\frac{t}{T}}\delta^{t/T}.
$$

and from estimate (2.18) we have

$$
\begin{aligned}
\|A_t\varphi_T - A_{t,J^*}\varphi_T^\delta\|_{H^0} &= \|A_t\varphi_T - A_{t,J^*}\varphi_T^\delta\| \\
&\le 2CE^{1-\frac{t}{T}}\delta^{t/T} + (1+2C)E^{1-\frac{t}{T}}\delta^{t/T} \\
&= (4C+1)E^{1-\frac{t}{T}}\delta^{t/T}.
\end{aligned}
$$

The proof is complete. □

rmk2.1 **Remark 2.4.** From the result of reference [22] we know estimate (2.20) is an order optimal Hölder stability estimate in $L^2(\mathbb{R})$. This suggests that wavelet method must be useful for solving the considered ill-posed problem. However, from (2.20) we know when $t \to 0^+$, the accuracy of the regularized solution becomes progressively lower. At $t = 0$, it merely implies that the error is bounded by $4C+1$, i.e., the convergence of the regularized solution at $t = 0$ is not proved theoretically. This defect is remedied by the following result.

thm2.2 **Theorem 2.5.** *Suppose that $\varphi_0 \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$ and (2.9) holds for $r \le \min\{0, s\}$. Take*

$$
J^{**} := \Big[\frac{1}{2}\log_2\Big(\ln\Big(\big(\frac{E}{\delta}\big)^{1/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}\Big)\Big)\Big], \tag{2.21}
$$ e220

where the bracket $[a]$ is the same as in (2.19). Then

$$
\begin{aligned}
&\|A_t\varphi_T - A_{t,J^{**}}\varphi_T^\delta\|_{H^r} \\
&\le \Big(2C + (1+2C)\Big(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta} + \ln\big(\ln\frac{E}{\delta}\big)^{-(s-r)/(2T)}}\Big)^{\frac{s-r}{2}}\Big) \\
&\quad \times E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}}.
\end{aligned} \tag{2.22}
$$ e221

Proof. Note that

$$
\begin{aligned}
\exp\big\{2^{2J^{**}}(T-t)\big\}\delta &\le \exp\Big\{(T-t)\ln\Big(\big(\frac{E}{\delta}\big)^{1/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}\Big)\Big\}\delta \\
&= \big(\frac{E}{\delta}\big)^{\frac{T-t}{T}}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}}\delta \\
&= E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}},
\end{aligned}
$$

$$
\begin{aligned}
\exp\big\{-t2^{2(J^{**}+1)}\big\}E &\le \exp\Big\{-t\ln\Big(\big(\frac{E}{\delta}\big)^{1/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}\Big)\Big\}E \\
&= \big(\frac{E}{\delta}\big)^{-\frac{t}{T}}\big(\ln\frac{E}{\delta}\big)^{\frac{t(s-r)}{2T}}E \\
&= E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{\frac{t(s-r)}{2T}},
\end{aligned}
$$

$$
\begin{aligned}
2^{-(J^{**}+1)(s-r)} &= 2^{2(J^{**}+1)(-\frac{s-r}{2})} \\
&\le \Big(\ln\Big(\big(\frac{E}{\delta}\big)^{1/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}\Big)\Big)^{-\frac{s-r}{2}} \\
&= \Big(\frac{1}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}}\Big)^{\frac{s-r}{2}} \\
&= \Big(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}}\Big)^{\frac{s-r}{2}}\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2}},
\end{aligned}
$$

and from (2.18) we know

$$
\begin{aligned}
&\|A_t\varphi_T - A_{t,J^{**}}\varphi_T^\delta\|_{H^r} \\
&\le 2CE^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}} + (1+2C)\Big(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2T}}}\Big)^{\frac{s-r}{2}} \\
&\quad\times\big(\ln\frac{E}{\delta}\big)^{-\frac{T(s-r)}{2T}}E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{\frac{t(s-r)}{2T}} \\
&\le \Big(2C+(1+2C)\Big(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{s-r}{2T}}}\Big)^{\frac{s-r}{2}}\Big)E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}},
\end{aligned}
$$

where the factor

$$
\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{s-r}{2T}}}
$$

is bounded as $\delta\to 0^+$. So, the proof of estimate (2.22) is complete. $\square$

rmk2.2 **Remark 2.6.** When $s=r=0$, estimate (2.21) becomes estimate (2.19) and the convergence speed given by (2.22) is faster than the one given by (2.20) for $s>r$. Especially, when $t=0$, estimate (2.22) becomes

$$
\begin{aligned}
&\|A_0\varphi_T - A_{0,J^{**}}\varphi_T^\delta\|_{H^r} \\
&\le \Big(2C+(1+2C)\Big(\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{s-r}{2T}}}\Big)^{\frac{s-r}{2}}\Big)\big(\ln\frac{E}{\delta}\big)^{-\frac{(s-r)}{2}}E\to 0,
\end{aligned}
$$

when $\delta\to 0^+$ and $s>r$. This is a logarithmical stability estimate and an important improvement to estimate (2.20).

rmk2.3 **Remark 2.7.** Noting that

$$
\lim_{\delta\to 0}\frac{\ln\frac{E}{\delta}}{\frac{1}{T}\ln\frac{E}{\delta}+\ln\big(\ln\frac{E}{\delta}\big)^{-\frac{s-r}{2T}}} = T,
$$

estimate (2.22) also can be rewritten in the asymptotic form

$$
\begin{aligned}
&\|A_t\varphi_T - A_{t,J^{**}}\varphi_T^\delta\|_{H^r} \\
&\le \Big(2C+(1+2C)T^{\frac{s-r}{2}}+o(1)\Big)E^{1-\frac{t}{T}}\delta^{t/T}\big(\ln\frac{E}{\delta}\big)^{-\frac{(T-t)(s-r)}{2T}},
\end{aligned}
$$

as $\delta\to 0$.

2.2. ***A-posteriori* parameter choice.** In this subsection, we consider the *a posteriori* regularization parameter choice in the Morozov's discrepancy principle. This principle has been used by Feng et al [8] to solve the numerical analytic continuation, however the backward heat conduction problem is more severely ill-posed than the numerical analytic continuation.

lem2.3 **Lemma 2.8.** *Assume conditions* (2.9) *for* $r = 0$ *and* (2.10) *for* $s = 0$ *hold. If* J *is chosen as the solution of the inequalities*

$$\|P_J\varphi_T^\delta - \varphi_T^\delta\| \le \tau\delta \le \|P_{J-1}\varphi_T^\delta - \varphi_T^\delta\|,\ \tau > 1, \tag{2.23}$$ e227

then it holds

$$\exp(2^{2J}T) \le \frac{2E}{(\tau-1)\delta}. \tag{2.24}$$ e228

Proof. From Equations (2.3) and (2.5), we know

$$\begin{aligned} &\|P_{J-1}\varphi_T - \varphi_T\| \\ &= \Big(\int_{-\infty}^{\infty} |((I-P_{J-1})\varphi_T)\widehat{\ }(\xi)|^2 d\xi\Big)^{1/2} \\ &\le \Big(\int_{|\xi|\ge\frac{4}{3}\pi 2^{J-1}} |\hat{\varphi_T}(\xi)|^2 d\xi\Big)^{1/2} + \Big(\int_{|\xi|<\frac{4}{3}\pi 2^{J-1}} |(Q_{J-1}\varphi_T)\widehat{\ }(\xi)|^2 d\xi\Big)^{1/2} \\ &=: I_3 + I_4. \end{aligned} \tag{2.25}$$ e229

From (1.4),

$$I_3 \le \exp(-(\frac{4}{3}\pi 2^{J-1})^2 T)E \le \exp(-2^{2J}T)E. \tag{2.26}$$ e230

From (2.15),

$$I_4 \le \|(Q_{J-1}\varphi_T)\widehat{\ }(\xi)\| \le 2^{-Js}\exp(-T2^{2J})E \le \exp(-2^{2J}T)E, \quad s \ge 0 \tag{2.27}$$ e231

Combining (2.25) with (2.26) and (2.27), we obtain

$$\|P_{J-1}\varphi_T - \varphi_T\| \le 2\exp(-2^{2J}T)E. \tag{2.28}$$ e232

On the other hand, by (2.9) for $r = 0$, (2.23), and the triangle inequality give

$$\|P_{J-1}\varphi_T - \varphi_T\| \ge \|(I-P_{J-1})\varphi_T^\delta\| - \|(I-P_{J-1})(\varphi_T - \varphi_T^\delta)\| \ge (\tau-1)\delta. \tag{2.29}$$ e233

From (2.28) and (2.29), estimate (2.24) is proved. □

thm2.3 **Theorem 2.9.** *Assume that conditions* (2.9) *for* $r = 0$ *and* (2.10) *for* $s = 0$ *hold. If the regularization parameter* J *is chosen as the solution of inequalities* (2.23), *then*

$$\|A_t\varphi_T - A_{t,J}\varphi_T^\delta\| \le \widetilde{C}E^{1-\frac{t}{T}}\delta^{t/T}, \tag{2.30}$$ e234

where $\widetilde{C} = (2C(\frac{2}{\tau-1})^{1-\frac{t}{T}} + (\tau+1)^{t/T})$, *and the constant* C *is the same as in* (2.6).

Proof. By the Parseval formula and the triangle inequality,

$$\|A_t\varphi_T - A_{t,J}\varphi_T^\delta\| \le \|\widehat{A_{t,J}\varphi_T^\delta} - \widehat{A_{t,J}\varphi_T}\| + \|\widehat{A_{t,J}\varphi_T} - \widehat{A_t\varphi_T}\| =: I_5 + I_6. \tag{2.31}$$ e235

From Equation (2.12) and Lemma 2.8,

$$\begin{aligned} I_5 &\le 2C\exp(2^{2(J-1)}(T-t))\delta \\ &= 2C(\exp(2^{2(J-1)}T))^{\frac{T-t}{T}}\delta \\ &\le 2C\Big(\frac{2E}{(\tau-1)\delta}\Big)^{\frac{T-t}{T}}\delta. \end{aligned} \tag{2.32}$$ e236

Moreover,

$$
\begin{aligned}
&I_6^2\\
&= \int_{-\infty}^{\infty} |e^{\xi^2(T-t)}((I-P_J)\varphi_T)\widehat{\ }(\xi)|^2 d\xi\\
&= \int_{-\infty}^{\infty} |e^{\xi^2 T}((I-P_J)\varphi_T)\widehat{\ }(\xi)|^{2\frac{T-t}{T}} |((I-P_J)\varphi_T)\widehat{\ }(\xi)|^{2\frac{t}{T}} d\xi\\
&\le \Big(\int_{-\infty}^{\infty} |e^{\xi^2 T}((I-P_J)\varphi_T)\widehat{\ }(\xi)|^2 d\xi\Big)^{\frac{T-t}{T}} \Big(\int_{-\infty}^{\infty} |((I-P_J)\varphi_T)\widehat{\ }(\xi)|^2 d\xi\Big)^{t/T}.
\end{aligned}
\tag{2.33}
$$

e237

By combining (1.6) with (2.23) and (2.33), it holds

$$
I_6^2 \le E^{2(1-\frac{t}{T})}((\tau+1)\delta)^{2\frac{t}{T}}. \tag{2.34}
$$

e238

From (2.31), (2.32) and (2.34), we complete the proof. □

3. Numerical aspects

In this section, we want to discuss some numerical aspects of the proposed method.

3.1. Numerical implementation.

3.1. Numerical implementation. Suppose that the vector $\{\Phi(x_i)\}_{i=1}^N$ represents samples from the function $\varphi_T(x)$, and N is even, then we add a random normal distribution to each data and obtain the perturbation data

$$
\Phi^\delta = \Phi + \epsilon\, \mathrm{randn}(\mathrm{size}(\Phi)), \tag{3.1}
$$

where the function "randn(·)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$.

The total noise δ can be measured in the sense of root mean square error according to

$$
\delta = \|\Phi^\delta - \Phi\|_{l^2} := \Big(\frac{1}{N}\sum_{i=1}^N (\Phi^\delta(x_i) - \Phi(x_i))^2\Big)^{1/2}. \tag{3.2}
$$

For a given measured function φ_T^δ, from Section 2, we have

$$
\widehat{A_{t,J}\varphi_T^\delta}(\xi) = \sum_{k\in Z} (\widehat{\varphi_T^\delta}, \widehat{\varphi}_{Jk}) e^{\xi^2(T-t)} \widehat{\varphi}_{Jk} = e^{\xi^2(T-t)} \sum_{k\in Z} (\widehat{\varphi_T^\delta}, \widehat{\varphi}_{Jk}) \widehat{\varphi}_{Jk}. \tag{3.3}
$$

e3.3

If no specific instructions are assumed, we will compute the regularization parameter J according to inequalities (2.23) with $\tau = 1.1$. By using the Discrete Meyer wavelet Transform (DMT) and the Fast Fourier Transform (FFT), we can easily compute the regularized solution according to formula (3.3). Algorithms for DMT are described in [13]. These algorithms are based on the FFT, and computing the DMT of a vector in $\mathbb{R}$ requires $\mathcal{O}(N\log_2^2 N)$ operations.

3.2. Numerical tests. In this subsection some numerical tests are presented to demonstrate the usefulness of our method. The tests are performed using Matlab 7.0 and the wavelet package WaveLab 850, which is downloaded from http://www-stat.stanford.edu/~wavelab/.

Examples 3.1 and 3.2 are from [9] and [16], respectively. In theoretical aspect, the error estimates of regularization solutions for the wavelet method (WM) and Fourier method (FM) both are sharper Hölder-logarithm type, but it is only weaker logarithm type for the Modified method (MM). Here some comparison of numerical result for these three methods are considered. The case with no explicit solutions is considered in Examples 3.3 and 3.4.

In the following tests, the initial time is chosen as $t = 0$, and the number of the discrete points N is 128. The selection of regularization parameters for Fourier method and Modified method are chosen according to [9, (2.8) with $s = 0$] and [16, (3.24)] in Examples 3.1 and 3.2, respectively.

Let u be the exact solution and v be the approximation of some regularization method. The absolute error $e_a(u)$ is defined as

$$e_a(u) := \|u - v\|_{l^2} = \Big(\frac{1}{N}\sum_{n=1}^{N} |u(n) - v(n)|^2\Big)^{1/2},$$

examp1 **Example 3.1** ([9])**.** It is easy to verify that the function

$$u(x,t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}} \tag{3.4}$$

is the unique solution of the problem

$$\begin{gathered} u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,T) = \varphi_T(x) := \frac{1}{\sqrt{1+4T}} e^{-\frac{x^2}{1+4T}}, \quad x \in \mathbb{R}. \end{gathered} \tag{3.5}$$ eq:1

tab:1 Table 1. Regularization parameters for different methods with $\epsilon = 10^{-2}$

T	$\xi_{\max}$(FM)	μ(MM)	J(WM)
0.01	21.4597	0.0046	4
0.04	10.7298	0.0185	4
0.09	7.1532	0.0416	4

Figure 1 illustrates the exact solution and the approximation corresponding to the three methods at different times $t = 0$ from $T = 0.01, T = 0.04, T = 0.09$ with $\epsilon = 10^{-2}$ for Example 3.1 in the interval $x \in [-10, 10]$. The regularization parameters are chosen as in Table 1. Here $\xi_{\max}$, μ and J are the regularization parameters of Fourier method, Modified method and Wavelet method, respectively. Figure 2 shows that the reconstruction error obtained by the different methods for different numbers of discrete points with $T = 0.01$ and $\epsilon = 10^{-3}$. It shows that the number N of discrete points (i.e., the step length) also plays the role of regularization parameter [5]. According to the general regularization theory, it should be neither too small nor too large. But usually, in some regularization methods, the influence of number N is less than the regularization parameter.

examp2 **Example 3.2** ([16])**.** The function

$$u(x,t) = e^{-t}\sin(x) \tag{3.6}$$

is the unique solution of the problem

$$\begin{gathered} u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,T) = \varphi_T(x) := e^{-T}\sin(x), \quad x \in \mathbb{R}. \end{gathered} \tag{3.7}$$ eq:2

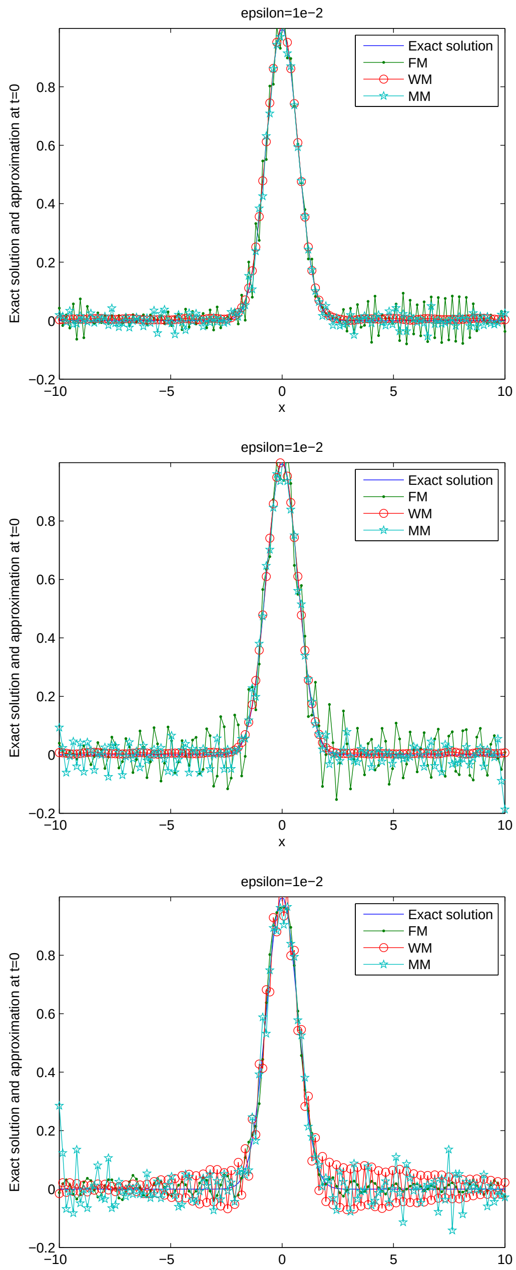

FIGURE 1. Example 3.1. Exact solution and approximation at $t = 0$ from $T = 0.01$ (top), $T = 0.04$ (middle), $T = 0.09$ (bottom) with $\epsilon = 10^{-2}$. fig:1

Figure 3 plots the exact solution and the approximation by the three methods at $t = 0$ from different times $T = 0.01, T = 0.04, T = 0.09$ with $\epsilon = 10^{-2}$ for Example

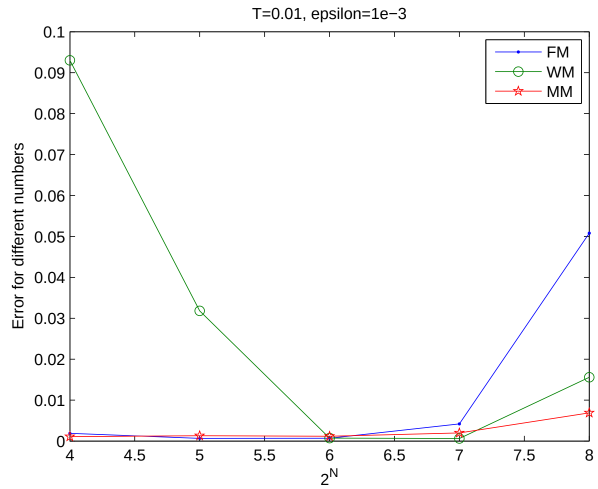

FIGURE 2. Example 3.1. Reconstruction error obtained by the different methods for different numbers of discrete points with $T = 0.01$, $\epsilon = 10^{-3}$. fig:2

3.2 in the interval $x \in [-3\pi, 3\pi]$. The regularization parameters are also chosen as in Table 1. Figure 4 illustrates that the reconstruction error obtained by the different methods for different noisy levels with $T = 0.01$, $N = 128$. It shows that, for the different methods, the error between the exact solution and the approximate solution gets smaller as the noise in the data decreases.

From the Figures 1 and 3, we can also see that, for all methods, the smaller the T, the better the approximation. This phenomenon conforms to the theory that, the larger the T, the more ill-posed the problem.

Unfortunately, for general data φ_T, it is not easy to find an explicit analytical solution to problem (1.1), so we will construct new examples as follows: take a function $\varphi_0(x) \in L^2(\mathbb{R})$ and solve the well-posed problem

$$\begin{gathered} u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \varphi_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{3.8}$$ eq:1b

to get an approximation to $\varphi_T(x)$. To avoid the inverse crime, we use the finite difference to compute this well-posed problem. Here we discretize problem (3.8) only with respect to the spatial variable x and leave the time variable t continuous, and then we obtain a system of ordinary differential equations and we can solve it using an explicit Runge-Kutta method. See the details in [16]. Then we add a random noise to $\varphi_T(x)$ to get the noisy data $\varphi_T^\delta(x)$. At last, we use the proposed regularized technique to obtain the regularized solution at $t = 0$.

examp3 **Example 3.3.** We choose a non-smooth function

$$\varphi_0(x) = \begin{cases} 1 + \frac{x}{3}, & -3 \le x \le 0, \\ 1 - \frac{x}{3}, & 0 < x \le 3, \\ 0, & |x| > 3. \end{cases}$$

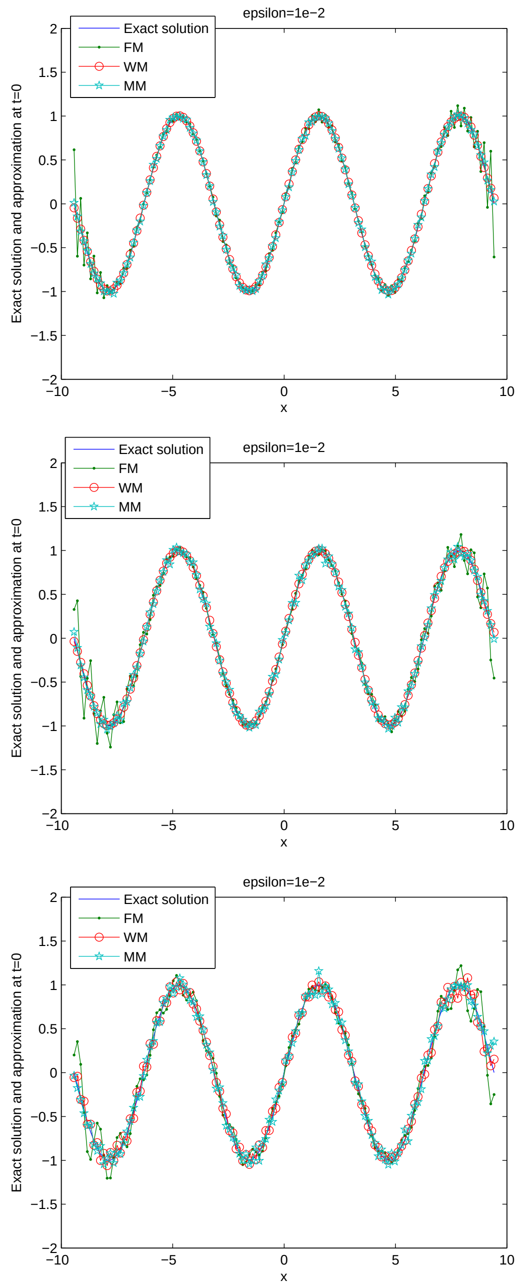

FIGURE 3. Example 3.2. Exact solution and approximation at $t = 0$ from $T = 0.01$ (top), $T = 0.04$ (middle), $T = 0.09$ (bottom) with $\epsilon = 10^{-2}$.

fig:3

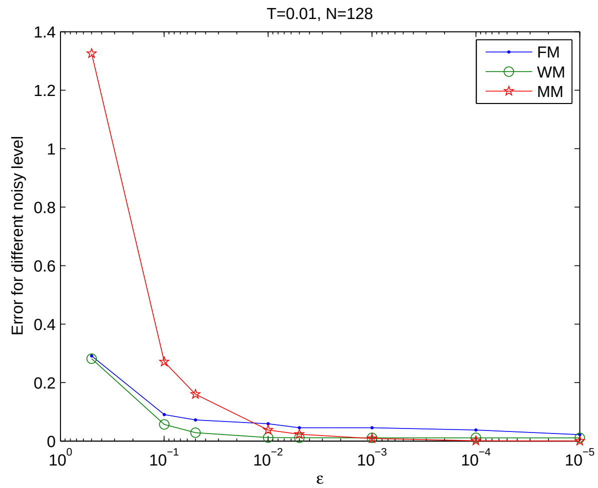

FIGURE 4. Example 3.3. Reconstruction error obtained by the different methods for different noisy levels with $T = 0.01$, $N = 128$. fig:4

examp4 **Example 3.4.** We consider a discontinuous function

$$\varphi_0(x) = \begin{cases} 1, & -3 \le x \le 0, \\ -1, & 0 < x \le 3, \\ 0, & |x| > 3. \end{cases}$$

Figures 5 and 6 illustrate the exact and regularized solutions corresponding to different ϵ for Examples 3.3 and 3.4, respectively. The results show that the approximation gets better as the noise level ϵ decreases. Although $\varphi_0(x)$ in Example 3.3 is not smooth and $\varphi_0(x)$ in Example 3.4 is even non-continuous, our method is also effective for them.

In summary, from the above different kinds of numerical examples, we can conclude that the numerical solution is stable, and the Meyer wavelet method is an applicable method. This accords with our theoretical results.

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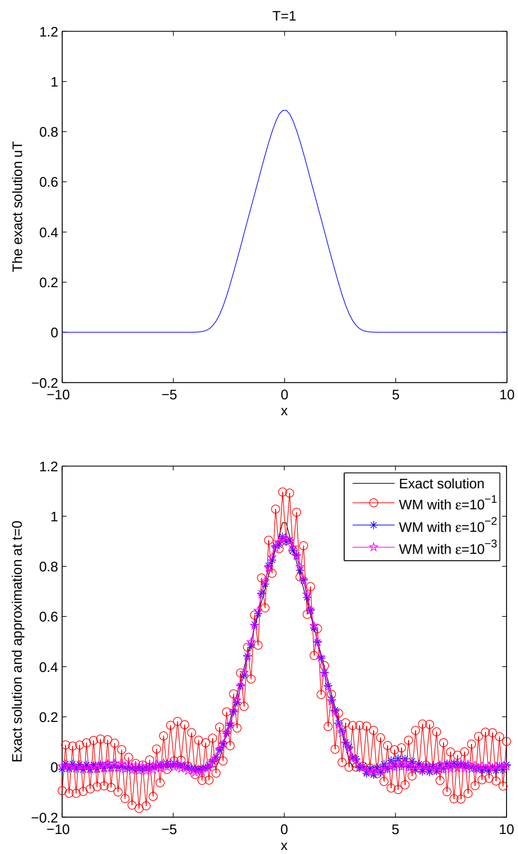

FIGURE 5. Example 3.3. Computed input data $\varphi_T(x)$ (top); exact and regularized solutions with different ϵ from $T = 0.09$ (bottom). fig:5

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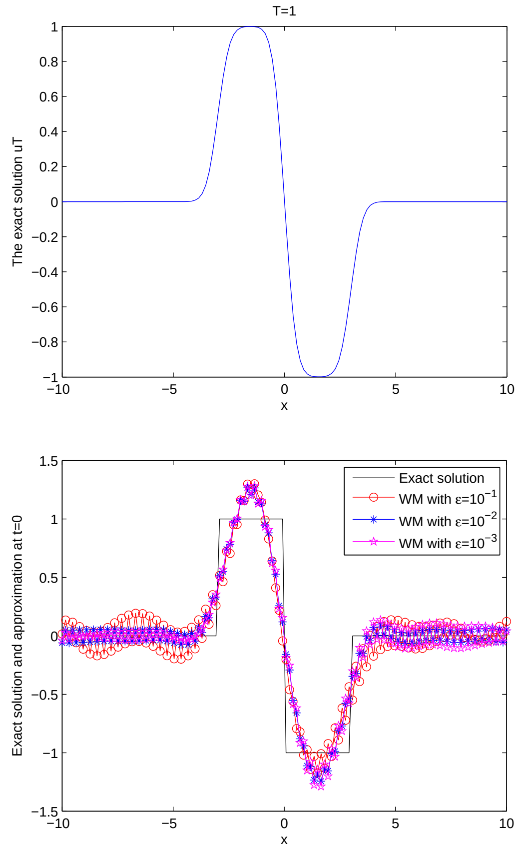

FIGURE 6. Example 3.4: Computed input data $\varphi_T(x)$ (top); Exact and regularized solutions with different ϵ from $T = 0.09$ (bottom). fig:6

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