EVEN-ORDER SELF-ADJOINT BOUNDARY VALUE PROBLEMS FOR PROPORTIONAL DERIVATIVES

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ABSTRACT. In this study, even order self-adjoint differential equations incorporating recently introduced proportional derivatives, and their associated self-adjoint boundary conditions, are discussed. Using quasi derivatives, a Lagrange bracket and bilinear functional are used to obtain a Lagrange identity and Green's formula; this also leads to the classification of self-adjoint boundary conditions. Next we connect the self-adjoint differential equations with the theory of Hamiltonian systems and (n, n)-disconjugacy. Specific formulas of Green's functions for two and four iterated proportional derivatives are also derived.

1. INTRODUCTION

We study the $2n$th order differential expression

$$
\begin{aligned}
Ly(t) &= \sum_{j=0}^{n} (-D^\alpha)^j \left[p_j \left(D^\alpha\right)^j y \right](t) \\
&= (-D^\alpha)^n \left[p_n \left(D^\alpha\right)^n y \right](t) + \cdots - \left(D^\alpha\right)^3 \left[p_3 \left(D^\alpha\right)^3 y \right](t) \\
&\quad + \left(D^\alpha\right)^2 \left[p_2 \left(D^\alpha\right)^2 y \right](t) - D^\alpha \left[p_1 D^\alpha y \right](t) + p_0(t)y(t),
\end{aligned} \tag{1.1}
$$

for continuous functions p_i with $p_n \neq 0$, and show that it is formally self adjoint with respect to the inner product

$$
\langle y, z \rangle = \int_a^b y(t)z(t)e_0^2(b, t) d_\alpha t, \quad d_\alpha t := \frac{dt}{\kappa_0(t)};
$$

that is, the identity

$$
\langle Ly, z \rangle = \langle y, Lz \rangle
$$

holds provided that y and z satisfy some appropriate self-adjoint boundary conditions at a and b. Here D^α is a proportional derivative operator [2, 3, 5] modeled after a proportional-derivative controller (PD controller) [9]. This proportional derivative D^α of order $\alpha \in [0, 1]$, where D^0 is the identity operator, and D^1 is the classical differential operator, will be used to explore corresponding higher-order

linear self-adjoint equations of the form (1.1). We will refer to an equation with $2n$ iterations of D^α as $2n$th-order equations.

Remark 1.1. [2, 3]In control theory, a PD controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and E the is input deviation, or the error between the state variable and the process variable; see [9], for example. This is the impetus for the next definition.

Definition 1.2 (A Class of Proportional Derivatives [2, 3]). Let $\alpha \in [0,1]$, $\mathcal{I} \subseteq \mathbb{R}$, and let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathcal{I} \to [0,\infty)$ be continuous such that

$$\lim_{\alpha \to 0^+} \kappa_1(\alpha, t) = 1, \quad \lim_{\alpha \to 0^+} \kappa_0(\alpha, t) = 0, \quad \forall\, t \in \mathcal{I},$$
$$\lim_{\alpha \to 1^-} \kappa_1(\alpha, t) = 0, \quad \lim_{\alpha \to 1^-} \kappa_0(\alpha, t) = 1, \quad \forall\, t \in \mathcal{I}, \tag{1.2}$$
$$\kappa_1(\alpha, t) \neq 0, \alpha \in [0,1), \quad \kappa_0(\alpha, t) \neq 0, \alpha \in (0,1], \quad \forall\, t \in \mathcal{I}.$$

Define the proportional differential operator D^α via

$$D^\alpha f(t) = \kappa_1(\alpha, t) f(t) + \kappa_0(\alpha, t) f'(t), \quad t \in \mathcal{I} \tag{1.3}$$

provided the right-hand side exists at t, where $f' := \frac{d}{dt} f$.

Remark 1.3 ([2, 3]). For the operator given in (1.3), κ_1 is a type of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, f is the error, and $u = D^\alpha f$ is the controller output. To illustrate, one could take $\kappa_1 \equiv \cos(\alpha\pi/2)$ and $\kappa_0 \equiv \sin(\alpha\pi/2)$, or $\kappa_1 \equiv (1-\alpha)\omega^\alpha$ and $\kappa_0 \equiv \alpha\omega^{1-\alpha}$ for any $\omega \in (0,\infty)$; or, $\kappa_1 = (1-\alpha)|t|^\alpha$ and $\kappa_0 = \alpha|t|^{1-\alpha}$ on $\mathcal{I} = \mathbb{R}\backslash\{0\}$, so that

$$D^\alpha f(t) = (1-\alpha)|t|^\alpha f(t) + \alpha|t|^{1-\alpha} f'(t).$$

If κ_1 and κ_0 are constant with respect to the independent variable, then $D^\beta D^\alpha = D^\alpha D^\beta$, but $D^\beta D^\alpha \neq D^\alpha D^\beta$ for $\alpha, \beta \in [0,1]$ in general; see also [15]. By (1.2) and (1.3),

$$\lim_{\alpha \to 0^+} D^\alpha f = D^0 f = f \quad \text{and} \quad \lim_{\alpha \to 1^-} D^\alpha f = D^1 f = f'.$$

Throughout the discussion to follow we will need a vital definition [3, Definition 1.6], which establishes a type of exponential function for derivative (1.3).

Definition 1.4 (Proportional Exponential Function [2, 3]). Let $\alpha \in (0,1]$, the points $s, t \in \mathbb{R}$ with $s \leq t$, and let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (1.2), with p/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. Then the conformable exponential function with respect to D^α in (1.3) is defined to be

$$e_p(t,s) := e^{\int_s^t \frac{p(\tau) - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \quad e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \tag{1.4}$$

and satisfies

$$D^\alpha e_p(t,s) = p(t) e_p(t,s), \quad D^\alpha e_0(t,s) = 0. \tag{1.5}$$

The following fundamental theorem, given in [2, Theorem 2.4] and [3, Lemma 1.9 (ii)], relates the proportional derivative and the proportional integral using the above proportional exponential function.

Theorem 1.5 (Fundamental Theorem of Integral Calculus)**.** *Let $\alpha \in (0, 1]$. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' is integrable on $[a, b]$. Then*

$$\int_a^b D^\alpha[f(t)]e_0(b, t)d_\alpha t = f(b) - f(a)e_0(b, a),$$

where $d_\alpha t := dt/\kappa_0(t)$.

Remark 1.6. As in [5], consider (1.3) with $\kappa_1 = (1 - \alpha)$ and $\kappa_0 = \alpha$, so that

$$D^\alpha f(t) = (1 - \alpha)f(t) + \alpha f'(t).$$

Then using the FTC, Theorem 1.5, as motivation and simplifying $e_0(t, \tau)$ via (1.4), define this special case of the proportional integral of f as

$$_a\mathcal{I}_t^\alpha f(t) := \frac{1}{\alpha} \int_a^t f(\tau)e^{-\frac{1-\alpha}{\alpha}(t-\tau)}d\tau. \tag{1.6}$$

In two recent papers [6, 7], Caputo and Fabrizio introduce a new fractional time derivative of the form

$$\mathscr{D}_t^{(\alpha)} f(t) = \frac{1}{1 - \alpha} \int_a^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau,$$

with related fractional time integral

$$_a\mathscr{I}_t^\alpha f(t) = \frac{1}{\alpha} \int_a^t f(\tau)e^{-\frac{1-\alpha}{\alpha}(t-\tau)}d\tau.$$

Note that we then have the relationships

$$\mathscr{D}_t^{(\alpha)} f(t) = {}_a\mathcal{I}_t^{1-\alpha} f'(t) \quad \text{and} \quad {}_a\mathscr{I}_t^\alpha f(t) = {}_a\mathcal{I}_t^\alpha f(t)$$

using (1.6); further research needs to be done on connecting the results of [6, 7] with those to follow.

2. Self-adjoint proportional equations

For the theory of higher order differential equations refer to [8, 10, 12, 13, 14]. Consider the $2n$th-order proportional differential expression (1.1), in which the coefficient functions $p_j : \mathcal{I} \to \mathbb{R}$ are continuous for $0 \le j \le n$ and $p_n(t) \ne 0$ for all $t \in \mathcal{I}$.

Definition 2.1. Let $\mathbb{D}$ be the linear set of all functions $y : \mathcal{I} \to \mathbb{R}$ such that the function

$$(D^\alpha)^j \left[p_j (D^\alpha)^j y \right]$$

is defined on $\mathcal{I}$ and is continuous for $0 \le j \le n$.

For each $y \in \mathbb{D}$ the expression Ly is defined and presents a continuous function on $\mathcal{I}$.

Definition 2.2 (Quasi-Derivatives)**.** As in the traditional case when $\alpha = 1$ (see [13, pp. 49]), we introduce the functions $y^{[j]}$, $0 \le j \le 2n$, as the quasi-derivatives of y related to the expression Ly. Given $y \in \mathbb{D}$, set

$$y^{[j]} = (D^\alpha)^j y, \quad 0 \le j \le n - 1, \quad y^{[0]} = (D^\alpha)^0 y = y, \tag{2.1}$$

$$y^{[n]} = p_n (D^\alpha)^n y, \tag{2.2}$$

$$y^{[n+j]} = p_{n-j} (D^\alpha)^{n-j} y - D^\alpha [y^{[n+j-1]}], \quad 1 \le j \le n-1$$

$$= \sum_{i=0}^{j} (-D^\alpha)^{j-i} [p_{n-i} (D^\alpha)^{n-i} y], \quad 0 \le j \le n-1, \tag{2.3}$$

$$y^{[2n]} = p_0 y - D^\alpha [y^{[2n-1]}]$$

$$= \sum_{j=0}^{n} (-D^\alpha)^j [p_j (D^\alpha)^j y] = Ly. \tag{2.4}$$

Definition 2.3 (Lagrange Bracket). Assume $y, z \in \mathbb{D}$ and $t \in \mathcal{I}$. The Lagrange bracket of y and z is given by

$$\{y, z\}(t) = \sum_{j=1}^{n} \left\{ y^{[j-1]} z^{[2n-j]} - y^{[2n-j]} z^{[j-1]} \right\}(t). \tag{2.5}$$

Definition 2.4 (Bilinear Functional). Assume $y, z \in \mathbb{D}$ and $t \in \mathcal{I}$. The bilinear (in y and z) functional F is given by

$$F(y, z, t) = \sum_{j=1}^{n} \left(y^{[j-1]} z^{[2n-j]} \right)(t). \tag{2.6}$$

Note that by combining (2.5) and (2.6), we have the Lagrange bracket in terms of the bilinear functional, namely

$$\{y, z\}(t) = F(y, z, t) - F(z, y, t).$$

Using (2.1) and (2.3) we get that

$$F(y, z, t) = \sum_{j=0}^{n-1} (-1)^j (D^\alpha)^{n-j-1} y(t) \sum_{i=0}^{j} (-1)^i (D^\alpha)^{j-i} [p_{n-i} (D^\alpha)^{n-i} z](t). \tag{2.7}$$

Lemma 2.5. *The bilinear functional F in (2.6) satisfies*

$$e_0(t, a) D^\alpha \left[\frac{F(y, z, \cdot)}{e_0(\cdot, a)} \right](t) = \left(-yLz + \sum_{j=0}^{n} p_j (D^\alpha)^j y (D^\alpha)^j z \right)(t)$$

for $t, a \in \mathcal{I}$.

Proof. Differentiating both sides of (2.6), employing the quotient rule for α-derivatives, and taking into account the formulas (2.2) and (2.4), we get

$$e_0(t, a) D^\alpha \left[\frac{F(y, z, \cdot)}{e_0(\cdot, a)} \right](t) = D^\alpha F(y, z, t) + \kappa_1(t) F(y, z, t)$$

$$= \sum_{j=1}^{n} \left(y^{[j-1]} D^\alpha [z^{[2n-j]}] + z^{[2n-j]} D^\alpha [y^{[j-1]}] \right)(t)$$

$$= \left(y^{[0]} D^\alpha [z^{[2n-1]}] + \sum_{j=2}^{n} y^{[j-1]} D^\alpha [z^{[2n-j]}] \right.$$

$$\left. + z^{[n]} D^\alpha [y^{[n-1]}] + \sum_{j=1}^{n-1} z^{[2n-j]} D^\alpha [y^{[j-1]}] \right)(t)$$

$$= \Big(y(p_0 z - Lz) + \sum_{j=2}^{n} y^{[j-1]} D^\alpha \big[z^{[2n-j]} \big]$$

$$+ p_n \left(D^\alpha \right)^n y \left(D^\alpha \right)^n z + \sum_{j=2}^{n} z^{[2n-j+1]} D^\alpha \big[y^{[j-2]} \big] \Big)(t).$$

Further, by (2.1) we have

$$D^\alpha \big[y^{[j-2]} \big](t) = y^{[j-1]}(t) \quad \text{for } 2 \le j \le n, \ t \in \mathcal{I},$$

and from (2.3) for z, replacing the j by $n - j + 1$, we find

$$z^{[2n-j+1]} = p_{j-1} \left(D^\alpha \right)^{j-1} z - D^\alpha \big[z^{[2n-j]} \big] \text{ for } 2 \le j \le n.$$

Consequently we obtain the desired result.   □

Theorem 2.6 (Lagrange Identity). *If $y, z \in \mathbb{D}$, then for $t, a \in \mathcal{I}$ we have*

$$\left(zLy - yLz \right)(t) = e_0(t, a) D^\alpha \Big[\frac{\{y, z\}}{e_0(\cdot, a)} \Big](t), \tag{2.8}$$

where $\{y, z\}$ is the Lagrange bracket of y and z defined by (2.5).

Proof. By (2.5) and (2.6) we have

$$\{y, z\}(t) = F(y, z, t) - F(z, y, t);$$

dividing both sides by $e_0(t, a)$, taking the α derivative, multiplying the result by $e_0(t, a)$ on both sides, and applying Lemma 2.5 we obtain (2.8).   □

Remark 2.7 (Green's Formula). Let the numbers $a, b, t \in \mathcal{I}$ with $a < b$. If we multiply both sides of (2.8) by $e_0^2(b, t) d_\alpha t$ and integrate from a to b, then we obtain Lagrange's identity in integral form, also called Green's formula,

$$\langle Ly, z \rangle - \langle y, Lz \rangle = \int_a^b \left(zLy \right)(t) e_0^2(b, t) d_\alpha t - \int_a^b \left(yLz \right)(t) e_0^2(b, t) d_\alpha t$$

$$= \{y, z\}(b) - e_0^2(b, a) \{y, z\}(a).$$

Let $g : \mathcal{I} \to \mathbb{R}$ be a continuous function, and consider the non-homogeneous equation

$$Ly(t) = g(t) \quad \text{for } t \in \mathcal{I}. \tag{2.9}$$

If $y \in \mathbb{D}$ and (2.9) holds for y, we say that y is a solution of (2.9). In order to obtain an existence and uniqueness theorem for initial value problems involving (2.9), it is necessary to rewrite (2.9) in the form of an equivalent system of first order equations. From (2.1), (2.3), and (2.4) we have the following system of equations

$$D^\alpha \big[y^{[j]} \big] = y^{[j+1]}, \quad 0 \le j \le n - 2,$$

$$D^\alpha \big[y^{[n-1]} \big] = \left(D^\alpha \right)^n y = \frac{y^{[n]}}{p_n},$$

$$D^\alpha \big[y^{[n+j-1]} \big] = p_{n-j} \left(D^\alpha \right)^{n-j} y - y^{[n+j]} = p_{n-j} y^{[n-j]} - y^{[n+j]}, \quad 1 \le j \le n - 1,$$

$$D^\alpha \big[y^{[2n-1]} \big] = p_0 y - Ly. \tag{2.10}$$

Define the following column vectors via

$$\vec{y} = \left(y^{[0]}, y^{[1]}, \dots, y^{[2n-1]} \right)^\top, \quad \vec{g} = (0, 0, \dots, 0, -g)^\top,$$

where $^\top$ indicates transpose. In addition, define the $n \times n$ matrix functions

$$A_1 = -A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{p_n} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & p_{n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_1 & 0 & 0 & \cdots & 0 & 0 \\ p_0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

so that

$$A(t) = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}$$

is a $(2n) \times (2n)$ variable matrix function on $\mathcal{I}$. From this we see that the equation (2.9) is equivalent to the first order system

$$D^\alpha \vec{y}(t) = A(t)\vec{y} + \vec{g}(t) \quad \text{for } t \in \mathcal{I}. \tag{2.11}$$

We are now able to prove the following theorem.

Theorem 2.8 (Existence and Uniqueness). *Fix $t_0 \in \mathcal{I}$ and let $c_j \in \mathbb{R}$, $0 \le j \le 2n-1$, be given. Then for $\alpha \in (0,1]$, equation (2.9) has a unique solution $y : \mathcal{I} \to \mathbb{R}$ such that*

$$y^{[j]}(t_0) = c_j, \quad 0 \le j \le 2n - 1.$$

Proof. Since equation (2.9) is equivalent to the system (2.11), and (2.11) is equivalent to

$$\frac{d}{dt}\vec{y} = \frac{1}{\kappa_0}\left(A - \kappa_1 I\right)\vec{y} + \frac{1}{\kappa_0}\vec{g},$$

the result follows from classical ODE theory. $\qquad\square$

Consider the homogeneous equation $Ly(t) = 0$.

Definition 2.9 (Wronskian). Let y_j, $1 \le j \le 2n$, be solutions of $Ly(t) = 0$. The Wronskian of these solutions is defined to be the determinant

$$W_t(y_1, \ldots, y_{2n}) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{2n} \\ y_1^{[1]} & y_2^{[1]} & \cdots & y_{2n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n-1]} & y_2^{[2n-1]} & \cdots & y_{2n}^{[2n-1]} \end{vmatrix}.$$

The proofs of the following two theorems follow in the same manner as the differential equations case; see [13, pp. 57–58].

Theorem 2.10. *If the solutions y_i, $1 \le i \le 2n$, of the homogeneous equation $Ly = 0$ are linearly dependent, then their Wronskian vanishes identically on $\mathcal{I}$. Conversely, if the Wronskian vanishes at at least one point in $\mathcal{I}$, then the solutions y_i, $1 \le i \le 2n$, are linearly dependent.*

We can easily construct a linearly independent system of solutions y_i, $1 \le i \le 2n$, of a homogeneous system. We need only choose a system of solutions which satisfy initial conditions of the form

$$y_i^{[j-1]}(t_0) = a_{ij}, \quad 1 \le i, j \le 2n,$$

where the determinant of the matrix $[a_{ij}]$ is different from zero. A linearly independent system of solutions y_i, $1 \le i \le 2n$, is a fundamental system.

Theorem 2.11. *Every solution of a homogeneous equation is a linear combination of a fixed, arbitrarily chosen, fundamental system.*

3. Self-adjoint boundary conditions and Green's functions

Let $a, b \in \mathcal{I}$ with $a < b$. If y and z are real valued continuous functions and bounded on $[a, b]$, define their inner product to be

$$\langle y, z \rangle = \int_a^b y(t)z(t)e_0^2(b,t)d_\alpha t, \quad d_\alpha t := \frac{dt}{\kappa_0(t)}.$$

Suppose for $0 \le j \le n-1$ that $p_j : [a,b] \to \mathbb{R}$ is continuous with $p_n(t) \ne 0$ on $[a,b]$.

Definition 3.1. Denote by $\mathbb{D}[a,b]$ the linear set of all continuous functions $y : [a,b] \to \mathbb{R}$ such that

$$(D^\alpha)^j \left[p_j \left(D^\alpha \right)^j y \right]$$

is defined on $\mathcal{I}$ and is continuous for $0 \le j \le n$.

For $y \in \mathbb{D}[a,b]$ let

$$Ly(t) = \sum_{j=0}^n (-D^\alpha)^j \left[p_j \left(D^\alpha \right)^j y \right](t), \quad t \in [a,b]. \tag{3.1}$$

Then Ly is continuous and bounded on $[a,b]$. Together with the equation (3.1), define the boundary conditions

$$U_i(y) := e_0(b,a) \sum_{j=1}^{2n} \eta_{ij} y^{[j-1]}(a) + e_0(a,b) \sum_{j=1}^{2n} \beta_{ij} y^{[j-1]}(b), \quad 1 \le i \le 2n, \tag{3.2}$$

where η_{ij}, β_{ij}, $1 \le i, j \le 2n$ are given real numbers.

Definition 3.2. The boundary conditions (3.2) are self adjoint with respect to the equation (3.1) if and only if

$$\langle Ly, z \rangle = \langle y, Lz \rangle \tag{3.3}$$

for all functions $y, z \in \mathbb{D}[a,b]$ satisfying the boundary conditions (3.2).

By Green's formula given in Remark 2.7 we have, for all $y, z \in \mathbb{D}[a, b]$,

$$\langle Ly, z \rangle - \langle y, Lz \rangle = \{y, z\}(b) - e_0^2(b, a)\{y, z\}(a),$$

where the Lagrange bracket $\{y, z\}$ is as defined previously in (2.5). Therefore boundary conditions (3.2) are self adjoint if and only if

$$\{y, z\}(b) = e_0^2(b, a)\{y, z\}(a)$$

for all functions $y, z \in \mathbb{D}[a, b]$ satisfying (3.2). For example the boundary conditions

$$y^{[j]}(a) = 0 = y^{[j]}(b), \quad 0 \le j \le n - 1,$$

and also the boundary conditions

$$e_0(b, a)y^{[j]}(a) = e_0(a, b)y^{[j]}(b), \quad 0 \le j \le 2n - 1,$$

are self adjoint. The boundary value problem $Ly(t) = 0$, $U_i(y) = 0$, $1 \le i \le 2n$ has Green's function $G(t, s)$ if for any continuous and bounded function $g : [a, b] \to \mathbb{R}$ the nonhomogeneous boundary value problem $Ly(t) = g(t)$, $U_i(y) = 0$, $1 \le i \le 2n$, has a unique solution $y : [a, b] \to \mathbb{R}$ which is given by

$$y(t) = \int_a^b G(t, s)g(s)d_\alpha s.$$

4. Self-adjoint equations as Hamiltonian systems

One important type of differential system is a Hamiltonian system [1, 11]. Let us show that the $2n$th order self-adjoint equation $Ly = 0$, in which Ly is of the form (1.1), can be written as an equivalent complex linear Hamiltonian system given by

$$D^\alpha \vec{x}(t) = \mathcal{A}(t)\vec{x}(t) + \mathcal{B}(t)\vec{u}(t), \quad D^\alpha \vec{u}(t) = \mathcal{C}(t)\vec{x}(t) - \mathcal{A}^*(t)\vec{u}(t), \quad t \in \mathcal{I}, \quad (4.1)$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are $n \times n$ complex matrices with $\mathcal{B}$ and $\mathcal{C}$ Hermitian; $\mathcal{A}^*$ denotes the complex conjugate of $\mathcal{A}$; $\mathcal{I} \subseteq [a, \infty)$. In particular, we will show (1.1) can be written in the form of (4.1), where

$$\mathcal{A} = (a_{ij})_{1 \le i, j \le n} \quad \text{with} \quad a_{ij} = \begin{cases} 1 : & \text{if } j = i + 1, \ 1 \le i \le n - 1, \\ 0 : & \text{otherwise,} \end{cases}$$

$$\mathcal{B} = \text{diag}\left\{0, \ldots, 0, \frac{1}{p_n}\right\}, \quad \mathcal{C} = \text{diag}\{p_0, p_1, p_2, \ldots, p_{n-1}\}.$$

Recall for any function $y \in \mathbb{D}$ the system of equations in (2.10). Then using the substitution

$$\vec{x} = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \ldots \\ y^{[n-1]} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} y^{[2n-1]} \\ y^{[2n-2]} \\ \ldots \\ y^{[n]} \end{pmatrix}, \quad (4.2)$$

and the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ above, we have that $Ly(t) = 0$, $t \in \mathcal{I}$ is equivalent to the linear Hamiltonian system (4.1).

Now let us present some properties of solutions to the homogeneous equation $Ly(t) = 0$, $t \in \mathcal{I}$. From the Lagrange identity (2.8) we immediately get the following theorem.

Theorem 4.1. *If y and z are solutions of $Ly(t) = 0$ for $t \in \mathcal{I}$, then the Lagrange bracket of y and z satisfies*

$$\{y, z\}(t) = ce_0^2(t, a), \quad t \in \mathcal{I},$$

where $a \in \mathcal{I}$ and $c \in \mathbb{R}$.

Lemma 2.5 yields the following result.

Theorem 4.2. *Let $F(y, z, t)$ be defined as in (2.6) (see also (2.7)), and let $a \in \mathcal{I}$. If y is a solution of $Ly(t) = 0$, $t \in \mathcal{I}$, then*

$$e_0(t, a) D^\alpha \Big[\frac{F(y, y, \cdot)}{e_0(\cdot, a)} \Big](t) = \sum_{j=0}^n p_j(t) \big[(D^\alpha)^j y \big]^2(t), \quad t \in \mathcal{I}.$$

In particular, if $p_j(t) \geq 0$ for $0 \leq j \leq n$ and $t \in \mathcal{I}$, then $F(y, y, t)$ satisfies

$$e_0(a, t) F(y, y, t) \geq e_0(t, a) F(y, y, a)$$

along solutions of $Ly(t) = 0$ for all $t \in \mathcal{I}$ with $t \geq a$.

Proof. If y is a solution of $Ly(t) = 0$, then by Lemma 2.5 we know that $F(y, y, t)$ satisfies

$$e_0(t, a) D^\alpha \Big[\frac{F(y, y, \cdot)}{e_0(\cdot, a)} \Big](t) = \sum_{j=0}^n p_j \big[(D^\alpha)^j y \big]^2(t)$$

for $t, a \in \mathcal{I}$. Furthermore, if $p_j(t) \geq 0$ for $0 \leq j \leq n$ and $t \in \mathcal{I}$, then

$$D^\alpha \Big[\frac{F(y, y, \cdot)}{e_0(\cdot, a)} \Big](t) \geq 0, \quad t \in \mathcal{I},$$

and the function $F(y, y, \cdot)/e_0(\cdot, a)$ is α-increasing on $\mathcal{I}$. Thus,

$$e_0(t_1, t_2) F(y, y, t_2)/e_0(t_2, a) \geq F(y, y, t_1)/e_0(t_1, a),$$

whenever $t_2 > t_1$, $t_1, t_2 \in \mathcal{I}$. The result follows if we take $t_1 = a$ and $t_2 = t$. $\square$

Lemma 4.3. *Assume $\eta \in \mathbb{D}[a, b]$. Then*

$$F(\eta, \eta, b) - F(\eta, \eta, a) e_0^2(b, a) = -\langle \eta, L\eta \rangle + \sum_{j=0}^n \langle p_j, [(D^\alpha)^j \eta]^2 \rangle. \qquad (4.3)$$

Proof. Setting $y = z = \eta$ in Lemma 2.5 we have

$$e_0(t, a) D^\alpha \Big[\frac{F(\eta, \eta, \cdot)}{e_0(\cdot, a)} \Big](t) = \Big(-\eta L\eta + \sum_{j=0}^n p_j \big[(D^\alpha)^j \eta \big]^2 \Big)(t)$$

for $t, a \in \mathcal{I}$. If we multiply both sides by $e_0^2(b, t) d_\alpha t$ and then integrate from a to b we get the desired result. $\square$

Definition 4.4. The set of admissible variations is given by

$$\mathcal{S} = \big\{ \eta \in \mathbb{D}[a, b] : (D^\alpha)^j \eta(a) = (D^\alpha)^j \eta(b) = 0, \quad 0 \leq j \leq n - 1 \big\},$$

with corresponding functional

$$\mathcal{F}(\eta) = \sum_{j=0}^n \langle p_j, [(D^\alpha)^j \eta]^2 \rangle. \qquad (4.4)$$

For an admissible variation $\eta \in \mathcal{S}$, Lemma 4.3 implies that

$$\mathcal{F}(\eta) = \langle \eta, L\eta \rangle.$$

The functional $\mathcal{F}$ is positive definite on the set of admissible variations $\mathcal{S}$ if $\mathcal{F}(\eta) \geq 0$ for all $\eta \in \mathcal{S}$, and $\mathcal{F}(\eta) = 0$ if and only if $\eta = 0$.

Note that the bilinear functional F in (2.6) and the vector-valued functions $\vec{x}$ and $\vec{u}$ given above in (4.2) satisfy the dot product equation

$$(\vec{x} \cdot \vec{u})(t) = F(y, y, t).$$

We will use this in the proof of the next theorem.

Theorem 4.5. *Assume $p_j(t) \geq 0$ for $0 \leq j \leq n$ and $t \in \mathcal{I}$, and $p_n(t) > 0$ for $t \in \mathcal{I}$. Then the functional $\mathcal{F}$ is positive definite on $\mathcal{S}$ and the linear Hamiltonian system (4.1) being considered for $t \in [a, b]$ is disconjugate on $[a, b]$. In particular the self-adjoint BVP*

$$Ly(t) = 0, \quad t \in [a, b],$$
$$(D^\alpha)^j y(a) = 0 = (D^\alpha)^j y(b), \quad j = 0, 1, \ldots, n - 1,$$

has only the trivial solution.

Proof. Let $t \in \mathcal{I}$. From $p_j(t) \geq 0$ for $0 \leq j \leq n$ and (4.4), it is clear that $\mathcal{F}(\eta) \geq 0$ for all $\eta \in \mathcal{S}$, and that $\mathcal{F}(0) = 0$. Now suppose $\eta \in \mathcal{S}$ and $F(\eta) = 0$. Then

$$0 = \sum_{j=0}^{n} \langle p_j, [(D^\alpha)^j \eta]^2 \rangle \geq \langle p_n, [(D^\alpha)^n \eta]^2 \rangle,$$

and since $p_n(t) > 0$, we have that $(D^\alpha)^n \eta(t) = 0$ for $t \in [a, b]$. Because η is admissible, it solves the initial value problem

$$(D^\alpha)^n \eta(t) = 0, \quad t \in [a, b]$$
$$(D^\alpha)^j \eta(a) = 0, \quad 0 \leq j \leq n - 1.$$

By uniqueness of solutions to initial value problems, η is the trivial solution in the set of admissible functions, whence $\mathcal{F}$ is positive definite on that set. By (4.3), if y is a solution of $Ly(t) = 0$, $t \in [a, b]$, then

$$(\vec{x} \cdot \vec{u})(b) - (\vec{x} \cdot \vec{u})(a)e_0^2(b, a) = F[y, y, b] - F[y, y, a]e_0^2(b, a)$$
$$= \sum_{j=0}^{n} \langle p_j, [(D^\alpha)^j y]^2 \rangle$$
$$= \mathcal{F}(y).$$

Note that the Hamiltonian system (4.1) is disconjugate on $[a, b]$ if and only if for a vector solution $\vec{x}$, $\vec{u}$ of (4.1), the following is positive definite:

$$\int_a^b \left(\vec{x}^\top \mathcal{C} \vec{x} + \vec{u}^\top \mathcal{B} \vec{u} \right)(t) e_0^2(b, t) d_\alpha t = \sum_{j=0}^{n-1} \langle p_j, (y^{[j]})^2 \rangle + \langle 1/p_n, (y^{[n]})^2 \rangle = \mathcal{F}(y).$$

This completes the proof. $\square$

The point $t = t_0$ is a zero of order (at least) n of y if

$$(D^\alpha)^j y(t_0) = 0, \quad j = 0, 1, \ldots, n - 1.$$

The equation $Ly = 0$ is (n, n) disconjugate on $[a, b]$ provided there is no nontrivial solution of $Ly = 0$ with a zero of order (at least) n in $(a, b]$ preceded by a zero of order (at least) n in $[a, b)$. These ideas lead to the next conclusion.

Theorem 4.6. *If $p_n(t) > 0$ for $t \in [a, b]$, then $Ly(t) = 0$ is (n, n) disconjugate on $[a, b]$.*

Proof. Suppose y is a solution of $Ly = 0$, and without loss of generality assume y has a zero of order n at a, namely $(D^\alpha)^j y(a) = 0$, $j = 0, 1, \ldots, n - 1$. Then from (2.7) we have $F(y, y, a) = 0$, and $F(y, y, t) \geq 0$ for all $t \in [a, b]$ by Theorem 4.2. If y has a zero at $t_0 \in (a, b]$ of order n, then

$$(D^\alpha)^j y(t_0) = 0, \quad j = 0, 1, \ldots, n - 1.$$

But then y is a trivial solution of $Ly = 0$ by the previous theorem. $\square$

5. Second-order proportional equations

Analogous to the classic and time scales cases [4], in this section we find Green's function associated to second-order proportional equations. With this in mind, again consider (1.1). Taking $n = 1$, we find that

$$Ly(t) = -D^\alpha \left[p_1 D^\alpha y \right](t) + p_0(t) y(t), \quad t \in \mathcal{I},$$

and for each function $y \in \mathbb{D}$,

$$y^{[0]} = y, \quad y^{[1]} = p_1 D^\alpha y, \quad y^{[2]} = p_0 y - D^\alpha \left[y^{[1]} \right].$$

Then

$$Ly = y^{[2]}$$

as expected. In addition, the equation $Ly(t) = g(t)$ for $t \in \mathcal{I}$ is equivalent to the first order system

$$D^\alpha \vec{y}(t) = A(t) \vec{y}(t) + \vec{g}(t), \quad t \in \mathcal{I},$$

where

$$\vec{y} = \begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & \frac{1}{p_1(t)} \\ p_0(t) & 0 \end{pmatrix}.$$

The Wronskian of two solutions y, z, is

$$W_t(y, z) = \begin{vmatrix} y^{[0]}(t) & z^{[0]}(t) \\ y^{[1]}(t) & z^{[1]}(t) \end{vmatrix} = p_1(t) \left(y D^\alpha z - z D^\alpha y \right)(t) = \{y, z\}(t),$$

the Lagrange bracket (2.5) of y and z, giving rise to the following theorem.

Theorem 5.1. *The Wronskian of any two solutions y, z of $Ly(t) = 0$ satisfies*

$$W_t(y, z) = e_0^2(t, a) W_a(y, z).$$

The following theorem presents a variation of constants formula for the nonhomogeneous equation $Ly(t) = g(t)$.

Theorem 5.2 (Variation of Constants)**.** *Suppose that y_1, y_2 form a fundamental system of solutions of the homogeneous equation $Ly(t) = 0$. Then the general solution of the nonhomogeneous equation $Ly(t) = g(t)$ is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \int_{t_0}^{t} \frac{y_1(t) y_2(s) - y_1(s) y_2(t)}{W_s(y_1, y_2)} g(s) d_\alpha s,$$

where $t_0 \in \mathcal{I}$ and c_1, c_2 are real constants.

Proof. It suffices to show that the function

$$z(t) = \int_{t_0}^t \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{W_s(y_1, y_2)} g(s) d_\alpha s$$

is a particular solution of the nonhomogeneous equation $Ly(t) = g(t)$. Differentiating both sides yields

$$D^\alpha z(t) = \int_{t_0}^t \frac{y_2(s)D^\alpha y_1(t) - y_1(s)D^\alpha y_2(t)}{W_s(y_1, y_2)} g(s) d_\alpha s.$$

Hence

$$D^\alpha \left[p_1 D^\alpha z \right](t) = \frac{y_2(t)p_1(t)D^\alpha y_1(t) - y_1(t)p_1(t)D^\alpha y_2(t)}{W_t(y_1, y_2)} g(t)$$
$$+ \int_{t_0}^t \frac{y_2(s)D^\alpha [p_1 D^\alpha y_1](t) - y_1(s)D^\alpha [p_1 D^\alpha y_2](t)}{W_s(y_1, y_2)} g(s) d_\alpha s$$
$$= -g(t) + p_0(t)z(t),$$

that is z satisfies $Ly(t) = g(t)$.   □

For $y \in \mathbb{D}[a, b]$ let

$$Ly(t) = -D^\alpha \left[p_1 D^\alpha y \right](t) + p_0(t)y(t), \quad t \in [a, b],$$

together with the boundary conditions

$$\eta_{11}e_0(b, a)y(a) + \eta_{12}e_0(b, a)y^{[1]}(a) + \beta_{11}e_0(a, b)y(b) + \beta_{12}e_0(a, b)y^{[1]}(b) = 0,$$
$$\eta_{21}e_0(b, a)y(a) + \eta_{22}e_0(b, a)y^{[1]}(a) + \beta_{21}e_0(a, b)y(b) + \beta_{22}e_0(a, b)y^{[1]}(b) = 0, \tag{5.1}$$

where η_{ij}, β_{ij} are given real numbers, $i, j = 1, 2$. Set

$$N = \begin{pmatrix} \eta_{11} & \eta_{12} & \beta_{11} & \beta_{12} \\ \eta_{21} & \eta_{22} & \beta_{21} & \beta_{22} \end{pmatrix}.$$

We will assume that the matrix N has rank 2. This means that the two boundary conditions (5.1) are linearly independent. As before, we call the boundary conditions (5.1) self adjoint with respect to the expression Ly if

$$\langle Ly, z \rangle - \langle y, Lz \rangle = \{y, z\}(b) - e_0^2(b, a)\{y, z\}(a)$$

for all functions $y, z \in \mathbb{D}[a, b]$ satisfying the boundary conditions (5.1). Recall that by Green's formula, the boundary conditions (5.1) are self adjoint if and only if

$$e_0(a, b)\{y, z\}(b) = e_0(b, a)\{y, z\}(a).$$

Set

$$N_1 = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}, \quad N_2 = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

Theorem 5.3. *If $\det N_1 = \det N_2$, then the boundary conditions (5.1) are self adjoint.*

Proof. Let $y, z \in \mathbb{D}[a, b]$, be functions which satisfy boundary conditions (5.1). Then we have

$$e_0(b, a)N_1 \begin{pmatrix} y(a) & z(a) \\ y^{[1]}(a) & z^{[1]}(a) \end{pmatrix} = e_0(a, b)N_2 \begin{pmatrix} -y(b) & -z(b) \\ -y^{[1]}(b) & -z^{[1]}(b) \end{pmatrix}.$$

Passing to determinants we have

$$(\det N_1)e_0(b, a)\{y, z\}(a) = (\det N_2)e_0(a, b)\{y, z\}(b).$$

If $\det N_1 = \det N_2 \neq 0$, then

$$e_0(b,a)\{y,z\}(a) = e_0(a,b)\{y,z\}(b).$$

Suppose $\det N_1 = \det N_2 = 0$. Since N has rank 2, it is clear that the boundary conditions (5.1) are equivalent to separated boundary conditions of the form

$$\begin{aligned}
\eta_1 y(a) + \eta_2 y^{[1]}(a) &= 0, \quad |\eta_1| + |\eta_2| \neq 0, \\
\beta_1 y(b) + \beta_2 y^{[1]}(b) &= 0, \quad |\beta_1| + |\beta_2| \neq 0,
\end{aligned} \tag{5.2}$$

where η_i, β_i, $i = 1, 2$ are real numbers. It can easily be verified that for any functions $y, z \in \mathbb{D}[a,b]$ satisfying boundary conditions (5.1) we have

$$\{y,z\}(a) = 0 = \{y,z\}(b),$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.4. As was noted above, the separated boundary conditions (5.2), in particular the boundary conditions $y(a) = y(b) = 0$ are self adjoint. The "periodic" boundary conditions

$$e_0(b,a)y(a) = e_0(a,b)y(b), \quad e_0(b,a)y^{[1]}(a) = e_0(a,b)y^{[1]}(b)$$

which are non-separated, are also self adjoint.

We will now construct Green's function for the self-adjoint (separated) BVP

$$-D^\alpha\left[p_1 D^\alpha y\right](t) + p_0(t)y(t) = g(t) \tag{5.3}$$

$$\eta y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0, \tag{5.4}$$

where $\eta, \beta, \gamma, \delta$ are real numbers such that $|\eta| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$.

Remark 5.5. The minus sign on the left hand side of (5.3), as well as in the first boundary condition of (5.4), is taken so that the positivity of Green's function can be formulated in terms of $p_1(t) > 0$, $p_0(t) \geq 0$, for $\eta, \beta, \gamma, \delta \geq 0$.

Denote by ϕ and ψ the solutions of the corresponding homogeneous equation

$$-D^\alpha\left[p_1 D^\alpha y\right](t) + p_0(t)y(t) = 0, \quad t \in [a,b], \tag{5.5}$$

under the initial conditions

$$\phi(a) = \beta, \quad \phi^{[1]}(a) = \eta, \tag{5.6}$$

$$\psi(b) = \delta, \quad \psi^{[1]}(b) = -\gamma, \tag{5.7}$$

so that ϕ and ψ satisfy the first and second boundary conditions in (5.4), respectively. From Theorem 5.1 we have that the Wronskian of ϕ and ψ satisfies

$$W_t(\phi,\psi) = \phi(t)\psi^{[1]}(t) - \phi^{[1]}(t)\psi(t) = e_0^2(t,a)W_a(\phi,\psi);$$

evaluating this expression at $t = a$, $t = b$, and using the boundary conditions (5.6), (5.7) yields

$$W_a(\phi,\psi) = \beta\psi^{[1]}(a) - \eta\psi(a) = \frac{-\gamma\phi(b) - \delta\phi^{[1]}(b)}{e_0^2(b,a)}.$$

Additionally, $W_a(\phi,\psi) \neq 0$ if and only if the homogeneous equation (5.5) has only the trivial solution satisfying the boundary conditions (5.4).

Theorem 5.6. *If $W_a(\phi, \psi) \neq 0$, then the nonhomogeneous BVP (5.3), (5.4), has a unique solution y for which the formula*

$$y(t) = \int_a^b G(t, s)g(s)d_\alpha s, \quad t \in [a, b]$$

holds, where the function $G(t, s)$ is given by

$$G(t, s) = \frac{-1}{W_s(\phi, \psi)} \begin{cases} \phi(t)\psi(s): & a \leq t \leq s \leq b, \\ \phi(s)\psi(t): & a \leq s \leq t \leq b, \end{cases}$$

and this $G(t, s)$ is Green's function of the BVP (5.3), (5.4). Furthermore the Green function satisfies the property $e_0(s, t)G(t, s) = e_0(t, s)G(s, t)$ for all $t, s \in [a, b]$.

Proof. Since $W_a(\phi, \psi) \neq 0$, the solutions ϕ and ψ of the homogeneous equation (5.5) are linearly independent. Thus the general solution of the nonhomogeneous equation (5.3) has the variation of constants form

$$y(t) = c_1\phi(t) + c_2\psi(t) + \int_a^t \frac{\phi(t)\psi(s) - \phi(s)\psi(t)}{W_s(\phi, \psi)}g(s)d_\alpha s, \tag{5.8}$$

where c_1 and c_2 are real constants. We now construct c_1 and c_2 so that the function y satisfies the boundary conditions (5.1). Using (5.8) we have

$$y^{[1]}(t) = c_1\phi^{[1]}(t) + c_2\psi^{[1]}(t) + \int_a^t \frac{\phi^{[1]}(t)\psi(s) - \phi(s)\psi^{[1]}(t)}{W_s(\phi, \psi)}g(s)d_\alpha s. \tag{5.9}$$

Consequently,

$$y(a) = c_1\phi(a) + c_2\psi(a) = c_1\beta + c_2\psi(a),$$

$$y^{[1]}(a) = c_1\phi^{[1]}(a) + c_2\psi^{[1]}(a) = c_1\eta + c_2\psi^{[1]}(a).$$

Substituting these values of $y(a)$ and $y^{[1]}(a)$ into the first condition of (5.4) we have

$$c_2\left(\eta\psi(a) - \beta\psi^{[1]}(a)\right) = 0.$$

On the other hand, using the definition of $W_a(\phi, \psi)$,

$$\eta\psi(a) - \beta\psi^{[1]}(a) = -W_a(\phi, \psi) \neq 0.$$

Consequently $c_2 = 0$, and (5.8), (5.9), take the form

$$y(t) = c_1\phi(t) + \int_a^t \frac{\phi(t)\psi(s) - \phi(s)\psi(t)}{W_s(\phi, \psi)}g(s)d_\alpha s,$$

$$y^{[1]}(t) = c_1\phi^{[1]}(t) + \int_a^t \frac{\phi^{[1]}(t)\psi(s) - \phi(s)\psi^{[1]}(t)}{W_s(\phi, \psi)}g(s)d_\alpha s,$$

respectively. Hence

$$y(b) = c_1\phi(b) + \int_a^b \frac{\phi(b)\psi(s) - \phi(s)\psi(b)}{W_s(\phi, \psi)}g(s)d_\alpha s,$$

$$y^{[1]}(b) = c_1\phi^{[1]}(b) + \int_a^b \frac{\phi^{[1]}(b)\psi(s) - \phi(s)\psi^{[1]}(b)}{W_s(\phi, \psi)}g(s)d_\alpha s.$$

Substituting these values into the second condition of (5.4) yields

$$c_1\left(\gamma\phi(b) + \delta\phi^{[1]}(b)\right) + \int_a^b \frac{\left(\gamma\phi(b) + \delta\phi^{[1]}(b)\right)}{W_s(\phi, \psi)}\psi(s)g(s)d_\alpha s = 0.$$

Again using the definition of $W_a(\phi, \psi)$,

$$\gamma\phi(b) + \delta\phi^{[1]}(b) = -e_0^2(b, a)W_a(\phi, \psi) \neq 0.$$

Hence

$$c_1 = -\int_a^b \frac{\psi(s)}{W_s(\phi, \psi)} g(s) d_\alpha s.$$

Thus y has the desired form, and $G(t, s)$ satisfies $e_0^2(s, a)G(t, s) = e_0^2(t, a)G(s, t)$; this is equivalent to $e_0(s, t)G(t, s) = e_0(t, s)G(s, t)$, completing the proof. $\square$

Corollary 5.7 (Green's Function for the Two-Point Problem). *If*

$$d := \beta\gamma + \eta\delta + \eta\gamma \int_a^b \frac{d_\alpha \tau}{p_1(\tau)} \neq 0,$$

then the nonhomogeneous BVP (5.3), (5.4) with $p_0 \equiv 0$ has a unique solution y for which the formula

$$y(t) = \int_a^b G(t, s)g(s)d_\alpha s, \quad t \in [a, b]$$

holds, where the function $G(t, s)$ is given by

$$G(t, s) = \frac{e_0(t, s)}{d} \begin{cases} \left[\beta + \eta \int_a^t \frac{d_\alpha \tau}{p_1(\tau)}\right]\left[\delta + \gamma \int_s^b \frac{d_\alpha \tau}{p_1(\tau)}\right] : & a \leq t \leq s \leq b, \\ \left[\beta + \eta \int_a^s \frac{d_\alpha \tau}{p_1(\tau)}\right]\left[\delta + \gamma \int_t^b \frac{d_\alpha \tau}{p_1(\tau)}\right] : & a \leq s \leq t \leq b. \end{cases}$$

This $G(t, s)$ is Green's function of the BVP (5.3), (5.4) with $p_0 \equiv 0$.

Proof. Assume

$$d := \beta\gamma + \eta\delta + \eta\gamma \int_a^b \frac{d_\alpha \tau}{p_1(\tau)} \neq 0.$$

Note that

$$\phi(t) = \eta e_0(t, a) \int_a^t \frac{d_\alpha \tau}{p_1(\tau)} + \beta e_0(t, a), \quad \psi(t) = \gamma e_0(t, b) \int_t^b \frac{d_\alpha \tau}{p_1(\tau)} + \delta e_0(t, b)$$

satisfy (5.5) with $p_0 \equiv 0$, along with conditions (5.6) and (5.7). The result then follows from Theorem 5.6. $\square$

Corollary 5.8 (Green's Function for the Conjugate Problem). *Green's function for the conjugate boundary value problem*

$$-D^\alpha \left[pD^\alpha y\right](t) = 0, \quad y(a) = y(b) = 0 \tag{5.10}$$

is given by

$$G(t, s) = \frac{e_0(t, s)}{\int_a^b \frac{1}{p(\tau)} d_\alpha \tau} \begin{cases} \int_a^t \frac{1}{p(\tau)} d_\alpha \tau \int_s^b \frac{1}{p(\tau)} d_\alpha \tau & : a \leq t \leq s \leq b, \\ \int_a^s \frac{1}{p(\tau)} d_\alpha \tau \int_t^b \frac{1}{p(\tau)} d_\alpha \tau & : a \leq s \leq t \leq b. \end{cases}$$

Proof. By Theorem 4.5, the BVP (5.10) has only the trivial solution. Due to the boundary conditions $y(a) = y(b) = 0$, we see that $\eta = \gamma = 1$ and $\beta = \delta = 0$ in (5.6) and (5.7). The result then follows from Corollary 5.7. $\square$

Corollary 5.9 (Green's Function for the Focal Problem)**.** *Green's function for the focal boundary value problem*

$$-D^\alpha\left[pD^\alpha y\right](t) = 0, \quad y(a) = D^\alpha y(b) = 0 \tag{5.11}$$

is given by

$$G(t,s) = e_0(t,s)\begin{cases} \int_a^t \frac{1}{p(\tau)}d_\alpha\tau & : a \le t \le s \le b, \\ \int_a^s \frac{1}{p(\tau)}d_\alpha\tau & : a \le s \le t \le b. \end{cases}$$

Proof. The boundary conditions imply $\eta = \delta = 1$ and $\beta = \gamma = 0$ in (5.6) and (5.7). The result again follows from Corollary 5.7. □

6. Fourth-order proportional equations

In equation (1.1) let $n = 2$, and consider the fourth order expression

$$Ly(t) = (D^\alpha)^2\left[p_2(D^\alpha)^2 y\right](t) - D^\alpha\left[p_1 D^\alpha y\right](t) + p_0(t)y(t). \tag{6.1}$$

For $y \in \mathbb{D}$ we have by definition

$$y^{[0]} = y, \quad y^{[1]} = D^\alpha y, \quad y^{[2]} = p_2(D^\alpha)^2 y,$$
$$y^{[3]} = p_1 D^\alpha y - D^\alpha\left[y^{[2]}\right], \quad y^{[4]} = p_0 y - D^\alpha\left[y^{[3]}\right].$$

It follows that

$$Ly = y^{[4]}.$$

In this case, for $y, z \in \mathbb{D}$ the Lagrange bracket of y and z is

$$\{y,z\}(t) = y(t)z^{[3]}(t) - y^{[3]}(t)z(t) + y^{[1]}(t)z^{[2]}(t) - y^{[2]}(t)z^{[1]}(t),$$

and the Lagrange identity

$$(zLy - yLz)(t) = e_0(t,a)D^\alpha\left[\frac{\{y,z\}}{e_0(\cdot,a)}\right](t)$$

holds. Using the same techniques as in previous sections, for each function $y \in \mathbb{D}$ we have the following system of relations at $t \in \mathcal{I}$,

$$D^\alpha\left[y^{[0]}\right] = y^{[1]}, \quad D^\alpha\left[y^{[1]}\right] = \frac{y^{[2]}}{p_2},$$
$$D^\alpha\left[y^{[2]}\right] = p_1 y^{[1]} - y^{[3]}, \quad D^\alpha\left[y^{[3]}\right] = p_0 y - Ly.$$

Thus the equation $Ly(t) = g(t)$ for $t \in \mathcal{I}$ where $g : \mathcal{I} \to \mathbb{R}$ is a continuous function is equivalent to the first order system

$$D^\alpha \vec{y}(t) = A(t)\vec{y}(t) + \vec{g}(t), \quad t \in \mathcal{I},$$

where

$$\vec{y} = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{p_2} & 0 \\ 0 & p_1 & 0 & -1 \\ p_0 & 0 & 0 & 0 \end{pmatrix}.$$

Together with the expression (6.1), take boundary conditions of the form

$$e_0(b,a)\sum_{j=1}^{4}\eta_{ij}y^{[j-1]}(a) + e_0(a,b)\sum_{j=1}^{4}\beta_{ij}y^{[j-1]}(b) = 0, \quad 1 \le i \le 4. \tag{6.2}$$

These boundary conditions are self adjoint if and only if

$$0 = e_0(a,b)\Big\{ y(b)z^{[3]}(b) - y^{[3]}(b)z(b) + y^{[1]}(b)z^{[2]}(b) - y^{[2]}(b)z^{[1]}(b) \Big\}$$

$$- e_0(b,a)\Big\{ y(a)z^{[3]}(a) + y^{[3]}(a)z(a) - y^{[1]}(a)z^{[2]}(a) + y^{[2]}(a)z^{[1]}(a) \Big\}$$

for all $y, z \in \mathbb{D}_{[a,b]}$. As is the case when $\alpha = 1$, it follows that by joining any one of the four types of conditions

(i) $y(a) = y^{[1]}(a) = 0$,
(ii) $y^{[1]}(a) = y^{[3]}(a) = 0$,
(iii) $y(a) = y^{[2]}(a) = 0$,
(iv) $y^{[2]}(a) = y^{[3]}(a) = 0$,

with any one of the four types of conditions

(i) $y(b) = y^{[1]}(b) = 0$,
(ii) $y^{[1]}(b) = y^{[3]}(b) = 0$,
(iii) $y(b) = y^{[2]}(b) = 0$,
(iv) $y^{[2]}(b) = y^{[3]}(b) = 0$,

yields the sixteen types of self-adjoint boundary conditions. The "periodic" boundary conditions

$$e_0(b,a)y(a) = e_0(a,b)y(b), \quad e_0(b,a)y^{[1]}(a) = e_0(a,b)y^{[1]}(b),$$

$$e_0(b,a)y^{[2]}(a) = e_0(a,b)y^{[2]}(b), \quad e_0(b,a)y^{[3]}(a) = e_0(a,b)y^{[3]}(b),$$

are also self adjoint.

Example 6.1. The Green function $G(t,s)$ for

$$(D^\alpha)^2[p(D^\alpha)^2 y](t), \quad t \in [a,b],$$

with the boundary conditions

$$y(a) = y^{[1]}(a) = y^{[2]}(b) = y^{[3]}(b) = 0$$

is given by

$$G(t,s) = \begin{cases} e_0(t,s) \int_a^t \big(\int_a^\tau \frac{h_1(s,\xi)}{p(\xi)} d_\alpha \xi \big) d_\alpha \tau & : a \le t \le s \le b, \\ e_0(t,s) \int_a^s \big(\int_a^\tau \frac{h_1(t,\xi)}{p(\xi)} d_\alpha \xi \big) d_\alpha \tau & : a \le s \le t \le b, \end{cases}$$

where $h_1(v,\xi) := \int_\xi^v 1 \, d_\alpha w$.

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