*Electronic Journal of Differential Equations*, Vol. 2017 (2017), No. 203, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# LYAPUNOV-TYPE INEQUALITIES FOR $\alpha$-TH ORDER FRACTIONAL DIFFERENTIAL EQUATIONS WITH $2 < \alpha \leq 3$ AND FRACTIONAL BOUNDARY CONDITIONS

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*Communicated by Jerry Goldstein*

Abstract. We study linear fractional boundary value problems consisting of an $\alpha$-th order Riemann-Liouville fractional differential equation with $2 < \alpha \leq 3$ and certain fractional boundary conditions. We derive several Lyapunov-type inequalities and apply them to establish nonexistence, uniqueness, and existence-uniqueness of solutions for related homogeneous and nonhomogeneous linear fractional boundary value problems. As a special case, our work extends some existing results for third-order linear boundary value problems.

## 1. Introduction

We consider the $\alpha$-th order fractional linear differential equation

$$\Big(D^{\alpha}_{a^+}x\Big)(t) + q(t)x = 0, \quad 2 < \alpha \leq 3. \tag{1.1}$$

Recall that for any $\gamma \geq 0$ and $t > a$,

$$\Big(I^{\gamma}_{a^+}x\Big)(t) := \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}x(s)ds$$

denotes the $\gamma$-th order left-sided Riemann-Liouville fractional integral of $x(t)$ at $a$, and $\big(D^{\gamma}_{a^+}x\big)(t)$ denotes the $\gamma$-th order left-sided Riemann-Liouville fractional derivative of $x(t)$ at $a$ defined as

$$\Big(D^{\gamma}_{a^+}x\Big)(t) := \frac{d^n}{dt^n}\Big(I^{n-\gamma}_{a^+}x\Big)(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\gamma-1}x(s)ds, \tag{1.2}$$

where $n = \lfloor\gamma\rfloor + 1$ with $\lfloor\gamma\rfloor$ the integer part of $\gamma$ and $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt$ is the Gamma function. In the following, we denote

$$\Big(D^{\alpha-k}_{a^+}x\Big)(a^+) := \lim_{t\to a^+}\Big(D^{\alpha-k}_{a^+}x\Big)(t) \quad \text{for } k = 1, 2, 3$$

with

$$\Big(D^{\alpha-3}_{a^+}x\Big)(a^+) := \lim_{t\to a^+}\Big(I^{3-\alpha}_{a^+}x\Big)(t) \quad \text{for } 2 < \alpha < 3.$$

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2010 *Mathematics Subject Classification.* 34A08, 34A40, 26A33, 34B05.
*Key words and phrases.* Fractional differential equations; fractional boundary conditions; Lyapunov-type inequalities; boundary value problems; existence and uniqueness of solutions.
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Submitted June 2, 2017. Published September 6, 2017.

In this paper, we derive Lyapunov-type inequalities for the boundary value problems (BVPs) consisting of (1.1) and one of the following boundary conditions (BCs):

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-3}x\right)(c) = 0, \quad a \le c \le b; \tag{1.3}$$

$$\left(D_{a^+}^{\alpha-3}x\right)(a^+) = \left(D_{a^+}^{\alpha-3}x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-2}x\right)(a^+) = 0, \quad a < b; \tag{1.4}$$

$$\left(D_{a^+}^{\alpha-3}x\right)(a^+) = \left(D_{a^+}^{\alpha-3}x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-1}x\right)(a) = 0, \quad a < b. \tag{1.5}$$

Lyapunov-type inequalities have been used as an important tool in oscillation, disconjugacy, control theory, eigenvalue problems, and many other areas of differential equations. Because of their importance, these inequalities have been extended and generalized in many directions by several authors. Now we briefly review some existing results on Lyapunov-type inequalities for both integer-order and fractional-order differential equations.

For the second-order linear differential equation

$$x'' + q(t)x = 0 \quad \text{on } (a,b) \tag{1.6}$$

with $q \in C([a,b],\mathbb{R})$, the following result is known as the Lyapunov inequality, see [17, 2].

**Theorem 1.1.** *Assume* (1.6) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a,b)$*. Then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}. \tag{1.7}$$

It was first noted by Wintner [24] and later by several other authors that inequality (1.7) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{1.8}$$

Inequality (1.8) was generalized to a more general form of second-order linear differential equations by Hartman [11, Chapter XI], and improved by Harris and Kong [12] and Brown and Hinton [1] later on. We note that the number 4 in the above inequalities is the best in the sense that if it is replaced by any larger number, then the inequalities fail to hold, see [11, p. 345] and [16]) for examples.

Lyapunov-type inequalities have been further extended to higher order linear differential equations and half-linear differential equations by many authors. See [5, 7, 19, 20, 26, 27, 25, 28] for the higher order linear case, [3, 4] for the half-linear case, and Pinasco [21] for an excellent survey on various Lyapunov-type inequalities. Among the above, Dhar and Kong [7] established Lyapunov-type inequalities for odd order linear differential equations. Restricting their results to the third-order equation

$$x''' + q(t)x = 0, \tag{1.9}$$

we have the following result.

**Theorem 1.2.** *Assume* (1.9) *has a nontrivial solution* $x(t)$ *satisfying* $x'(a) = x'(b) = 0$ *and* $x(c) = 0$ *for* $c \in [a,b]$*. Then one of the following two statements holds:*

(i) $\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}$,

(ii) $\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}$,

(iii) $\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{8}{(b-a)^2}$.

*As a result,*

$$\int_a^b |q(t)|dt > \frac{8}{(b-a)^2}.$$

Dhar and Kong [3] also established Lyapunov-type inequalities for third-order half-linear differential equations. The following is one of their results restricted to the linear case.

**Theorem 1.3.** *Assume* (1.9) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = x''(a) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a, b)$*. Then*

$$\int_a^b q_+(t)dt > \frac{4}{(b-a)^2}.$$

Although Lyapunov-type inequalities have been developed in many directions for the integer-order differential equations, there are only a few known results for the fractional differential equations. In [9], Ferreira obtained such inequalities for a Riemann-Liouville fractional BVP for the equation

$$\Big(D^{\alpha}_{a^+}x\Big)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \tag{1.10}$$

where $q \in C([a,b],\mathbb{R})$.

**Theorem 1.4.** *Assume* (1.10) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a,b)$*. Then*

$$\int_a^b |q(t)|dt > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{1.11}$$

It was indicated in Dhar and Kong [6] that (1.11) can be improved by replacing $|q(t)|$ by $q_+(t)$ to become

$$\int_a^b q_+(t)dt > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

Moreover, Dhar and Kong [6] obtained Lyapunov-type inequalities for the fractional BVP consisting of (1.10) and the fractional integral BC

$$\left(D^{\alpha-2}_{a^+}x\right)(a^+) = \left(D^{\alpha-2}_{a^+}x\right)(b) = 0. \tag{1.12}$$

**Theorem 1.5.** *(a) Assume* (1.10) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.12)*. Then*

$$\max_{t\in[a,b]}\Big\{\int_a^b \big|D^{2-\alpha}_{b^-}[G(t,s)q(s)]\big|\,ds\Big\} > 1.$$

*(b) Assume* (1.10) *has a solution* $x(t)$ *satisfying BC* (1.12) *and* $\left(D^{\alpha-2}_{a^+}x\right)(t) \neq 0$ *on* $(a,b)$*. Then*

$$\max_{t\in[a,b]}\Big\{\int_a^b \Big[D^{2-\alpha}_{b^-}[G(t,s)q(s)]\Big]_+ ds\Big\} > 1.$$

Here,

$$G(t,s) := \frac{1}{b-a}\begin{cases}(s-a)(b-t), & a \leq s \leq t \leq b\\ (t-a)(b-s), & a \leq t \leq s \leq b\end{cases} \tag{1.13}$$

is the Green's function for the BVP

$$-u'' = h(t), \quad u(a) = u(b) = 0 \tag{1.14}$$

with $h \in L([a,b)],\mathbb{R})$; $D_{b^-}^{2-\alpha}[G(t,s)q(s)]$ denotes the $(2-\alpha)$-th order right-sided fractional derivative of $G(t,s)q(s)$ with respect to $s$ at $b$ for fixed $t \in [a,b]$, i.e.,

$$D_{b^-}^{2-\alpha}[G(t,s)q(s)] := \frac{-1}{\Gamma(-1+\alpha)} \frac{d}{ds} \int_s^b (\tau - s)^{-2+\alpha} G(t,\tau) q(\tau) d\tau; \tag{1.15}$$

and $\big[D_{b^-}^{2-\alpha}[G(t,s)q(s)]\big]_+$ represents the positive part of $D_{b^-}^{2-\alpha}[G(t,s)q(s)]$. Recently, Lyapunov-type inequalities for Riemann-Liouville fractional differential equations with $3 < \alpha \le 4$ and pointwise BCs were established by O'Regan and Samet [18]. For Caputo fractional differential equations, Lyapunov-type inequalities were derived so far only for $1 < \alpha \le 2$, see [8, 13, 14, 15, 22]. To the best of our knowledge, no Lyapunov-type inequalities have been found for $2 < \alpha \le 3$.

In this article, we derive Lyapunov-type inequalities for each of the BVPs (1.1), (1.3); (1.1), (1.4) and (1.1), (1.5) and utilize them to establish the existence and uniqueness for solutions of related homogeneous and nonhomogeneous linear BVPs. Our work covers the result of Theorem 1.2 and improves that of Theorem 1.3 for the third-order linear differential equation (1.9).

This article is organized as follows: After this introduction, we present the Lyapunov-type inequalities for each of the BVPs (1.1), (1.3); (1.1), (1.4) and (1.1), (1.5) in Section 2; and then apply them to establish the existence and uniqueness for solutions of certain related homogeneous and nonhomogeneous linear fractional BVPs in Section 3.

## 2. Main results

In this Section, we let $-\infty < a < b < \infty$ and assume $q \in L([a,b],\mathbb{R})$. To prove our results, we will need the following fractional integration by parts formula, see [23, (2.64)]:

$$\int_a^b \phi(s) \big(D_{a^+}^{\gamma} \psi\big)(s) ds = \int_a^b \psi(s) \big(D_{b^-}^{\gamma} \phi\big)(s) ds, \quad 0 < \gamma < 1, \tag{2.1}$$

for any $\phi \in L_p(a,b)$, $\psi \in L_r(a,b)$ such that $p^{-1} + r^{-1} \le 1 + \gamma$, where

$$\big(D_{b^-}^{\gamma} \phi\big)(s) := \frac{-1}{\Gamma(1-\gamma)} \frac{d}{ds} \int_s^b (\tau - s)^{-\gamma} \phi(\tau) d\tau.$$

Similar to the notation in Section 1, we denote by $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ the $(3-\alpha)$-th order right-sided fractional derivative of $G(t,s)q(s)$ with respect to $s$ at $b$ for fixed $t \in [a,b]$ and by $\big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\big]_\pm$ the positive and negative parts of $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$, respectively. First we present the Lyapunov-type Inequalities for BVP (1.1), (1.3).

**Theorem 2.1.** *Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.3). *Then*

$$\int_a^b \int_a^b \big| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \big| \, ds \, dt > 1. \tag{2.2}$$

*Proof.* Let

$$y(t) := \big(D_{a^+}^{\alpha-3} x\big)(t) \quad \text{for } a < t \le b \text{ and } y(a) = \big(D_{a^+}^{\alpha-3} x\big)(a^+). \tag{2.3}$$

Then $y(t)$ is continuous on $[a,b]$. Note that $x(t) = (D_{a^+}^{3-\alpha} y)(t)$. We claim that

$$\Big(D_{a^+}^{\alpha} x\Big)(t) = y'''(t). \tag{2.4}$$

In fact, for $2 < \alpha < 3$, from (1.2) we have

$$\left(D^{\alpha}_{a^+}x\right)(t) = \frac{d^3}{dt^3}\left(I^{3-\alpha}_{a^+}x\right)(t) = \frac{d^3}{dt^3}\left(D^{\alpha-3}_{a^+}x\right)(t) = y'''(t);$$

and (2.4) holds clearly when $\alpha = 3$ since $y(t) = x(t)$. Also,

$$\left(D^{\alpha-2}_{a^+}x\right)(t) = \frac{d}{dt}\left(D^{\alpha-3}_{a^+}x\right)(t) = y'(t).$$

Then BVP (1.1), (1.3) becomes the third-order linear BVP

$$-y''' = q(t)x, \quad y'(a) = y'(b) = 0, \quad y(c) = 0 \quad \text{for } c \in [a, b]. \tag{2.5}$$

We denote $z(t) = y'(t)$ and rewrite BVP (2.5) as

$$-z'' = q(t)x, \quad z(a) = z(b) = 0. \tag{2.6}$$

Using the Green's function $G(t, s)$ defined in (1.13) for BVP (1.14) we have

$$z(t) = \int_a^b G(t, s)q(s)x(s)ds = \int_a^b G(t, s)q(s)(D^{3-\alpha}_{a^+}y)(s)ds.$$

For a fixed $t \in [a, b]$ and $2 < \alpha \le 3$, applying (2.1) with $\phi(s) = G(t, s)q(s)$, $\psi(s) = y(s)$, and $\gamma = 3 - \alpha$, we obtain

$$z(t) = \int_a^b G(t, s)q(s)(D^{3-\alpha}_{a^+}y)(s)ds = \int_a^b y(s)D^{3-\alpha}_{b^-}[G(t, s)q(s)]ds. \tag{2.7}$$

Replacing $z(t)$ by $y'(t)$, and then integrating both sides from $c$ to $t$ and using the fact that $y(c) = 0$, we see that

$$y(t) = \int_c^t \int_a^b y(s)D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]\,ds\,d\tau. \tag{2.8}$$

Hence

$$\begin{aligned} |y(t)| &= \Big|\int_c^t \int_a^b y(s)D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]ds\,d\tau\Big| \\ &\le \int_a^b \int_a^b |y(s)|\,\big|D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]\big|\,ds\,d\tau. \end{aligned} \tag{2.9}$$

Let $m := \max_{t\in[a,b]} |y(t)|$. Then

$$m \le m\int_a^b \int_a^b \big|D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]\big|\,ds\,d\tau \tag{2.10}$$

from which it follows that

$$1 \le \int_a^b \int_a^b \big|D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]\big|\,ds\,d\tau.$$

Using the same argument as given in the proof of [6, Theorem 3.1], we come to the conclusion that

$$1 < \int_a^b \int_a^b \big|D^{3-\alpha}_{b^-}[G(\tau, s)q(s)]\big|\,ds\,d\tau.$$

We omit the details. □

In the following, we say that a function $u(t)$ does not change sign on an interval $J$ if $u(t) \ge 0$ on $J$ or $u(t) \le 0$ on $J$. Under the assumptions that $\left(D^{\alpha-3}_{a^+}x\right)(t)$ does not change sign on $(a, c)$ and on $(c, b)$, we derive sharper Lyapunov-type inequalities than (2.2).

**Theorem 2.2.** *Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.3) *with* $c \in (a,b)$. *Furthermore, assume* $\left(D_{a^+}^{\alpha-3}x\right)(t)$ *does not change sign on* $(a,c)$ *and on* $(c,b)$. *Then one of the following holds:*

(a) $\int_a^c \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- \,ds\,dt > 1$,
(b) $\int_c^b \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ \,ds\,dt > 1$,
(c) $\int_a^c \int_a^c \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- \,ds\,dt + \int_a^c \int_c^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ \,ds\,dt > 1$.
(d) $\int_c^b \int_a^c \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- \,ds\,dt + \int_c^b \int_c^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ \,ds\,dt > 1$.

*Proof.* Let $y(t)$ be defined by (2.3). As shown in the proof of Theorem 2.1, (2.8) holds. Since $y(t)$ is continuous on $[a,b]$ and $y(c)=0$, there exist $t_1 \in [a,c)$ and $t_2 \in (c,b]$ such that $|y(t_1)| = \max\{|y(t)| : t \in [a,c]\}$ and $|y(t_2)| = \max\{|y(t)| : t \in [c,b]\}$. Without loss of generality, we may assume $y(t)$ satisfies one of the following cases:

(I) $y(t) \ge 0$ on $(a,c)\cup(c,b)$ and $y(t_1) \ge y(t_2)$;
(II) $y(t) \ge 0$ on $(a,c)\cup(c,b)$ and $y(t_1) < y(t_2)$;
(III) $y(t) \ge 0$ on $(a,c)$ and $y(t) \le 0$ on $(c,b)$, and $y(t_1) \ge -y(t_2)$;
(IV) $y(t) \ge 0$ on $(a,c)$ and $y(t) \le 0$ on $(c,b)$, and $y(t_1) < -y(t_2)$.

In the sequel, we denote $m = \max\{|y(t_1)|, |y(t_2)|\}$.

**Case I:** In this case, $m = y(t_1)$. Then (2.8) with $t = t_1$ shows that

$$m = \int_{t_1}^c \int_a^b y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau \le m\int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_-\,ds\,d\tau.$$

Similar to the proof of Theorem 2.1, we have

$$1 < \int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_-\,ds\,d\tau$$

which shows that conclusion (a) holds.

**Case II:** In this case, $m = y(t_2)$. Then (2.8) with $t = t_2$ shows that

$$m = \int_c^{t_2} \int_a^b y(s)D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\,ds\,d\tau \le m\int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_+\,ds\,d\tau.$$

Again, this leads to

$$1 < \int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_+\,ds\,d\tau$$

which shows that conclusion (b) holds.

**Case III:** In this case, $m = y(t_1)$. Then (2.8) with $t = t_1$ shows that

$$\begin{aligned}
m &= \int_{t_1}^c \int_a^b y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau \\
&= \int_{t_1}^c \int_a^c y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau \\
&\quad + \int_{t_1}^c \int_c^b (-y(s))\Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau \\
&\le m\int_{t_1}^c \int_a^c \Big[-D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau + m\int_{t_1}^c \int_c^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]\,ds\,d\tau.
\end{aligned}$$

Once again, this shows that conclusion (c) holds.

**Case IV:** The same argument as in Case III shows that conclusion (d) holds. We omit the detail. ☐

As a consequence of Theorem 2.2, we have the following corollary.

**Corollary 2.3.** *Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.3). *Furthermore, assume* $\left(D_{a^+}^{\alpha-3}x\right)(t)$ *does not change sign on* $(a,c)$ *and on* $(c,b)$.

(a) *Suppose* $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \le 0$ *for* $(s,t) \in [a,b]\times[a,b]$. *Then*

$$\int_a^b (s-a)^{\alpha-2}(b-s)q_-(s)ds > 2\Gamma(\alpha-2). \tag{2.11}$$

(b) *Suppose* $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \ge 0$ *for* $(s,t) \in [a,b]\times[a,b]$. *Then*

$$\int_a^b (s-a)^{\alpha-2}(b-s)q_+(s)ds > 2\Gamma(\alpha-2). \tag{2.12}$$

(c) *Suppose that* $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \le 0$ *for* $(s,t) \in [a,c]\times[a,b]$ *and that* $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \ge 0$ *for* $(s,t) \in [c,b]\times[a,b]$. *Then*

$$\int_a^c (s-a)^{\alpha-2}(b-s)q_-(s)ds + \int_c^b (s-a)^{\alpha-2}(b-s)q_+(s)ds > 2\Gamma(\alpha-2). \tag{2.13}$$

*Proof.* (a) By assumption we have $\left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- = -D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ and $\left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ = 0$ for $(s,t) \in [a,b]\times[a,b]$. It is easy to see that in this case, all feasible inequalities in (a)-(d) of Theorem 2.2 lead to

$$-\int_a^b \int_a^b D_{b^-}^{3-\alpha}[G(t,s)q(s)]\,ds\,dt > 1. \tag{2.14}$$

By the definition of $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ given in (1.15) we have

$$\begin{aligned}
&-\int_a^b \int_a^b D_{b^-}^{3-\alpha}[G(t,s)q(s)]\,ds\,dt \\
&= \frac{1}{\Gamma(\alpha-2)} \int_a^b \int_a^b \Big(\int_s^b (\tau-s)^{\alpha-3}G(t,\tau)q(\tau)d\tau\Big)'\,ds\,dt \\
&= \frac{-1}{\Gamma(\alpha-2)} \int_a^b \int_a^b (\tau-a)^{\alpha-3}G(t,\tau)q(\tau)d\tau dt \\
&= \frac{-1}{\Gamma(\alpha-2)} \int_a^b \Big(\int_a^b G(t,\tau)dt\Big)(\tau-a)^{\alpha-3}q(\tau)d\tau.
\end{aligned}$$

Hence (2.14) becomes

$$\int_a^b \Big(\int_a^b G(t,\tau)dt\Big)(\tau-a)^{\alpha-3}(-q(\tau))d\tau > \Gamma(\alpha-2).$$

Using the facts that $-q(t) \le q_-(t)$, $G(t,\tau) \ge 0$ on $[a,b]\times[a,b]$ we have

$$\int_a^b \Big(\int_a^b G(t,\tau)dt\Big)(\tau-a)^{\alpha-3}q_-(\tau)d\tau > \Gamma(\alpha-2). \tag{2.15}$$

Note that for $\tau \in [a,b]$,

$$\int_a^b G(t,\tau)dt = \frac{1}{2}(\tau - a)(b-\tau).$$

Therefore, (2.15) leads to (2.11).

(b) The proof is similar to case (a) and hence is omitted.

(c) It is easy to see that Theorem 2.2, conclusions (a)-(d) leads to

$$\int_a^b \int_a^c \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_- \,ds\,dt + \int_a^b \int_c^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_+ \,ds\,dt > 1.$$

Then the proof is similar to case (a) and hence is omitted. $\square$

**Remark 2.4.** Let $g(s) := (s-a)^{\alpha-1}(b-s)$ for $2 < \alpha \le 3$. It is easy to see that the maximum of $g(s)$ occurs at $d = [(\alpha-2)b + a]/(\alpha-1)$. Hence for $s \in [a,b]$,

$$g(s) \le g(d) = \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{(\alpha-1)^{\alpha-1}}.$$

Therefore, (2.11)-(2.13) become respectively the following

(i) $\int_a^b q_-(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}$.
(ii) $\int_a^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}$.
(iii) $\int_a^c q_-(s)ds + \int_c^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}$.

The following provides supplements to Theorem 2.2.

**Theorem 2.5.** *(a) Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying*

$$\big(D_{a^+}^{\alpha-2}x\big)(a^+) = \big(D_{a^+}^{\alpha-2}x\big)(b) \quad \text{and} \quad \big(D_{a^+}^{\alpha-3}x\big)(a) = 0. \tag{2.16}$$

*Then*

$$\int_a^b \int_a^b \Big|D_{b^-}^{3-\alpha}[G(t,s)q(s)\Big| \,ds\,dt > 1. \tag{2.17}$$

*Furthermore, assume* $\big(D_{a^+}^{\alpha-3}x\big)(t)$ *does not change sign on* $(a,b)$*. Then*

$$\int_a^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_+ \,ds\,dt > 1. \tag{2.18}$$

*(b) Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying*

$$\big(D_{a^+}^{\alpha-2}x\big)(a^+) = \big(D_{a^+}^{\alpha-2}x\big)(b) \quad \text{and} \quad \big(D_{a^+}^{\alpha-3}x\big)(b) = 0.$$

*Then*

$$\int_a^b \int_a^b \big|D_{b^-}^{3-\alpha}[G(t,s)q(s)\big| \,ds\,dt > 1.$$

*Furthermore, assume* $\big(D_{a^+}^{\alpha-3}x\big)(t)$ *does not change sign on* $(a,b)$*. Then*

$$\int_a^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_- \,ds\,dt > 1.$$

*Proof.* (a) As in the proof of Theorem 2.1, we see that (2.8) holds with $c = a$, i.e.,

$$y(t) = \int_a^t \int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)] \,ds\,d\tau. \tag{2.19}$$

Hence

$$\begin{aligned}|y(t)| &= \Big|\int_a^t \int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\,ds\,d\tau\Big| \\ &\le \int_a^b \int_a^b |y(s)|\Big|D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big|\,ds\,d\tau.\end{aligned}$$

Let $m = \max_{t\in[a,b]} |y(t)|$. Then as shown in the proof of Theorem 2.1, this leads to (2.17).

Furthermore, assume $y(t) = \big(D_{a^+}^{\alpha-3}x\big)(t)$ does not change sign on $(a,b)$. Without loss of generality, we may assume $y(t) \ge 0$ in $(a,b]$. Then there exists $t_2 \in (a,b]$ such that $m = y(t_2) = \max\{y(t) : t \in [a,b]\}$. Letting $t = t_2$ in (2.19) we obtain

$$m = \int_a^{t_2}\int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\,ds\,d\tau \le m\int_a^b\int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_+\,ds\,d\tau.$$

As shown in the proof of Theorem 2.1, this leads to (2.18).

(b) The proof is similar to Part (a) and hence is omitted. □

**Remark 2.6.** Now, we remark on the special case of Theorems 2.1-2.5 with $\alpha = 3$, where BVP (1.1), (1.3) becomes the third-order linear BVP

$$x''' + q(t)x = 0, \quad x'(a) = x'(b) = 0, \quad x(c) = 0 \quad \text{for } c \in [a,b]. \tag{2.20}$$

In this case, conclusion (2.2) in Theorem 2.1 becomes

$$\int_a^b\int_a^b |G(t,s)q(s)|\,ds\,dt > 1. \tag{2.21}$$

Note that $G(t,s) \ge 0$ for $(s,t) \in [a,b]\times[a,b]$. Hence

$$\int_a^b\int_a^b |G(t,s)q(s)|\,ds\,dt = \int_a^b\int_a^b G(t,s)|q(s)|\,ds\,dt = \int_a^b \Big(\int_a^b G(t,s)dt\Big)|q(s)|ds.$$

With a simple calculation we have

$$\int_a^b G(t,s)dt = \frac{1}{2}(b-s)(s-a) \le \frac{(b-a)^2}{8}$$

and hence (2.21) leads to

$$\int_a^b |q(s)|ds > \frac{8}{(b-a)^2}.$$

Similarly, conclusions (a)-(d) in Theorem 2.2 lead to

(i) $\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}$,

(ii) $\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}$,

(iii) $\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{8}{(b-a)^2}$.

The same remark applies to the case when $c = a$ or $b$ given in Theorem 2.5. It is easy to see that the condition $x(t)$ does not change sign on $(a,c)$ and on $(c,b)$ in Theorem 2.2 and its parallel conditions in Theorem 2.5 are not essential for the integer-order differential equations. Therefore, these results agree with Theorem 1.2.

Next we derive the Lyapunov-type inequalities for BVP (1.1), (1.4). Note from (1.13) that the Green's function of BVP

$$-u'' = h(t), \quad u(a) = u(\eta) = 0$$

is given by

$$G_\eta(t,s) := \frac{1}{\eta - a}\begin{cases}(s-a)(\eta - t), & a \le s \le t \le \eta,\\ (t-a)(\eta - s), & a \le t \le s \le \eta\end{cases}$$

for any $\eta \in (a,b)$. Then the following results for BVP (1.1), (1.4) are derived from Theorems 2.1 and 2.5 for BVP (1.1), (1.3). Here, we will use $D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]$ to denote the $(3-\alpha)$-th order right-sided fractional derivative of $G_\eta(t,s)q(s)$ with respect to $s$ at $\eta$ for fixed $t \in [a,b]$, i.e.,

$$D_{\eta^-}^{3-\alpha}[G(t,s)q(s)] := \frac{-1}{\Gamma(-2+\alpha)}\frac{d}{ds}\int_s^\eta (\tau - s)^{-3+\alpha} G_\eta(t,\tau)q(\tau)d\tau.$$

**Theorem 2.7.** *(a) Assume* (1.1) *has a nontrivial solution $x(t)$ satisfying BC* (1.4). *Then*

$$\sup_{\eta\in(a,b)}\int_a^\eta\int_a^\eta \left|D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right| ds\,dt > 1. \tag{2.22}$$

*(b) Assume* (1.1) *has a nontrivial solution $x(t)$ satisfying* (1.4) *and $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on $(a,b)$. Then*

$$\sup_{\eta\in(a,b)}\int_a^\eta\int_a^\eta \left[D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right]_+ ds\,dt > 1. \tag{2.23}$$

*Proof.* (a) Since $\left(D_{a^+}^{\alpha-3}x\right)(a^+) = \left(D_{a^+}^{\alpha-3}x\right)(b) = 0$, by Rolle's Theorem there exists a $\eta \in (a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(\eta) = \left(D_{a^+}^{\alpha-3}x\right)'(\eta) = 0$. Hence it satisfies

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(\eta) = 0 \text{ and } \left(D_{a^+}^{\alpha-3}x\right)(a^+) = 0. \tag{2.24}$$

Applying Theorem 2.1 to BVP (1.1), (2.24) we have

$$\int_a^\eta\int_a^\eta \left|D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right| ds\,dt > 1.$$

Then (2.22) follows.

(b) From the proof of Part (a) we see that there exists a $\eta \in (a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(\eta) = 0$. Hence $x(t)$ satisfies (2.24). By the assumption, $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on $(a,\eta)$. Applying the second part of Theorem 2.5, Part (a) to BVP (1.1), (2.24) we have

$$\int_a^\eta\int_a^\eta \left[D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right]_+ ds\,dt > 1.$$

Then (2.23) follows. □

With the same argument as in Corollary 2.3, we obtain the corollary below from Theorem 2.7.

**Corollary 2.8.** *Assume* (1.1) *has a nontrivial solution $x(t)$ satisfying BC* (1.4). *Suppose $D_{\eta^-}^{3-\alpha}[G_\eta(\tau,s)q(s)] \ge 0$ on $[a,\eta]\times[a,\eta]$ for every $\eta \in (a,b)$. Then*

$$\int_a^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

In the last part of this section, we derive the Lyapunov-type inequalities for BVP (1.1), (1.5).

**Theorem 2.9.** *(a) Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.5). *Then*

$$\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)\Big|\Big(D_{s^-}^{3-\alpha}q\Big)(\tau)\Big|d\tau ds>1. \tag{2.25}$$

*(b) Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying* (1.5) *and* $\left(D_{a^+}^{\alpha-3}x\right)(t)$ *does not change sign on* $(a,b)$. *Then*

$$\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)\Big[\Big(D_{s^-}^{3-\alpha}q\Big)(\tau)\Big]_+d\tau ds>1. \tag{2.26}$$

*Proof.* Let $y(t)$ be defined by (2.3). As shown in the proof of Theorem 2.1, BVP (1.1), (1.5) becomes

$$-y'''=q(t)x,\quad y(a)=y(b)=0 \text{ and } y''(a)=0.$$

It follows that

$$-y''=\int_a^t q(\tau)x(\tau)d\tau=\int_a^t q(\tau)\left(D_{a^+}^{3-\alpha}y\right)(\tau)d\tau.$$

For any $t\in(a,b]$, applying (2.1) with $\phi(\tau)=q(\tau)$, $\psi(\tau)=\left(D_{a^+}^{3-\alpha}y\right)(\tau)$, $\gamma=3-\alpha$, and $b$ replaced by $t$, we obtain

$$-y''=\int_a^t y(\tau)\left(D_{t^-}^{3-\alpha}q\right)(\tau)d\tau.$$

Using the Green's function $G(t,s)$ given by (1.13) for BVP (1.14) we see

$$y(t)=\int_a^b\int_a^s G(t,s)y(\tau)\left(D_{s^-}^{3-\alpha}q\right)(\tau)d\tau ds. \tag{2.27}$$

Hence

$$\begin{aligned}|y(t)|&=\Big|\int_a^b\int_a^s G(t,s)y(\tau)\left(D_{s^-}^{3-\alpha}q\right)(\tau)d\tau ds\Big|\\&\le\int_a^b\int_a^s G(t,s)|y(\tau)|\left|\left(D_{s^-}^{3-\alpha}q\right)(\tau)\right|d\tau ds.\end{aligned}$$

Let $m=\max_{t\in[a,b]}|y(t)|$. Then

$$m\le m\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)\left|\left(D_{s^-}^{3-\alpha}q\right)(\tau)\right|d\tau ds$$

from which it follows that

$$1\le\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)\left|\left(D_{s^-}^{3-\alpha}q\right)(\tau)\right|d\tau ds.$$

This, together with the argument in the proof of Theorem 2.1, leads to (2.25).

(b) From the proof of Part (a) we see that (2.27) holds. Without loss of generality, assume that $y(t)\ge 0$ on $(a,b)$. Let $m=\max_{t\in[a,b]}y(t)$. Then from (2.27) we see that

$$m\le\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)y(\tau)\left[\left(D_{s^-}^{3-\alpha}q\right)(\tau)\right]_+d\tau ds.$$

After this the same argument as in the proof of Theorem 2.1 leads us to (2.26). $\square$

With the same argument as in Corollary 2.3, we obtain the corollary below from Theorem 2.9.

**Corollary 2.10.** *Assume* (1.1) *has a nontrivial solution* $x(t)$ *satisfying BC* (1.5). *Suppose* $\big(D_{s^-}^{3-\alpha}q\big)(\tau) \geq 0$ *for* $s \in [a,b]$. *Then*

$$\int_a^b (t-a)^{\alpha-3} q_+(t)dt > \frac{8\Gamma(\alpha-2)}{(b-a)^2}.$$

**Remark 2.11.** Here we remark on the special case of Theorem 2.9 with $\alpha = 3$ where BVP (1.1),(1.5) becomes the third-order linear BVP

$$x''' + q(t)x = 0, \quad x(a) = x(b) = 0 \text{ and } x''(a) = 0. \tag{2.28}$$

Note that

$$\begin{aligned} \max_{t\in[a,b]} \int_a^b \int_a^s G(t,s)|q(\tau)|d\tau ds &\leq \max_{t\in[a,b]} \int_a^b \int_a^b G(t,s)|q(\tau)|d\tau ds \\ &\leq \int_a^b \Big(\max_{t\in[a,b]} \int_a^b G(t,s)ds\Big)|q(\tau)|d\tau \end{aligned}$$

and by Remark 2.6

$$\max_{t\in[a,b]} \int_a^b G(t,s)ds = \frac{(b-a)^2}{8}.$$

Hence conclusion (2.25) in Theorem 2.9 leads to

$$\int_a^b |q(\tau)|d\tau > \frac{8}{(b-a)^2}.$$

Similarly, conclusion (2.26) in Theorem 2.9 leads to

$$\int_a^b q_+(\tau)d\tau > \frac{8}{(b-a)^2}.$$

These inequalities improve those in Theorem 1.3.

## 3. Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type Inequalities obtained in Section 2 to study the nonexistence, uniqueness, and existence-uniqueness of solutions of certain fractional order linear BVPs.

**Definition 3.1.** A nontrivial solution $x(t)$ of (1.1) is said to be $I$-positive if $\big(I_{a^+}^{n-\alpha}x\big)(t) \geq 0$ on $[a,b]$, where $n = \lfloor \alpha \rfloor + 1$.

The following result is on the nonexistence of certain solutions of BVP (1.1), (1.3).

**Theorem 3.2.** *(a) Assume*

$$\int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right| ds\, dt \leq 1. \tag{3.1}$$

*Then BVP* (1.1), (1.3) *has no nontrivial solution.*

*(b) Assume*

$$\int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_- ds\, dt \leq 1, \tag{3.2}$$

$$\int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_+ \,ds\,dt \le 1 \tag{3.3}$$

*Then BVP* (1.1)*,* (1.3) *has no I-positive solution.*

*Proof.* (a) Assume the contrary, i.e., BVP (1.1), (1.3) has a nontrivial solution $x(t)$. Then by Theorem 2.1, (2.2) holds. This contradicts assumption (3.1).

(b) Assume the contrary, i.e., BVP (1.1), (1.3) has an $I$-positive solution $x(t)$. Then from the proof of Theorem 2.2, we see that only Cases I and II in the proof are feasible. Hence either

$$\int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_- \,ds\,dt > 1$$

or

$$\int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_+ \,ds\,dt > 1.$$

This contradicts the assumptions. □

Next we apply the results of Theorem 3.2 to study the nonhomogeneous linear BVP consisting of the equation

$$\Big(D_{a^+}^{\alpha} x\Big)(t) + q(t)x = w(t), \quad \text{on } (a,b) \tag{3.4}$$

and the BC

$$\big(D_{a^+}^{\alpha-2} x\big)(a^+) = k_1, \quad \big(D_{a^+}^{\alpha-2} x\big)(b) = k_2, \quad \big(D_{a^+}^{\alpha-3} x\big)(c) = k_3, \tag{3.5}$$

where $q, w \in L((a,b),\mathbb{R})$, $2 < \alpha \le 3$, and $k_1, k_2, k_3 \in \mathbb{R}$. Based on Theorem 3.2, we obtain a criterion for BVP (3.4), (3.5) to have a unique solution and reveal a relation among the solutions if the problem has more than one solution.

**Theorem 3.3.** *(a) Assume*

$$\int_a^b \int_a^b \big|D_{b^-}^{3-\alpha}[G(t,s)q(s)]\big| \,ds\,dt \le 1. \tag{3.6}$$

*Then BVP* (3.4)*,* (3.5) *has a unique solution on* $(a,b)$ *for any* $k_1, k_2, k_3 \in \mathbb{R}$*.*

*(b) Assume*

$$\int_a^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_\pm \,ds\,dt \le 1 < \int_a^b \int_a^b \big|D_{b^-}^{3-\alpha}[G(t,s)q(s)]\big| \,ds\,dt.$$

*If BVP* (3.4)*,* (3.5) *has two solutions* $x_1(t)$ *and* $x_2(t)$*, then there exists a* $d \in (a,b)$ *such that* $\Big(I_{a^+}^{3-\alpha} x_1\Big)(d) = \Big(I_{a^+}^{3-\alpha} x_2\Big)(d)$*.*

*Proof.* (a) By Theorem 3.2, Part (a), BVP (1.1), (1.3) has only the zero solution. Then by the Fredholm alternative theorem [10], we conclude that BVP (3.4), (3.5) has a unique solution.

(b) The conclusion is clearly true when $x_1(t) \equiv x_2(t)$ on $[a,b]$. Assume $x_1(t) \not\equiv x_2(t)$ on $[a,b]$ and let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a nontrivial solution of BVP (1.1), (1.3). By Theorem 3.2, Part (b), $x(t)$ is not $I$-positive on $[a,b]$. With the same reason, $-x(t)$ is not an $I$-positive solution on [a,b] either. Then there exists a $d \in (a,b)$ such that $\big(I_{a^+}^{3-\alpha} x\big)(d) = 0$, i.e., $\big(I_{a^+}^{3-\alpha} x_1\big)(d) = \big(I_{a^+}^{3-\alpha} x_2\big)(d)$. □

The results in this section can be easily extended to the homogeneous linear BVPs (1.1), (1.4) and (1.1), (1.5), and their corresponding nonhomogeneous linear BVPs. We left the details to the interested reader.

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