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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR ANISOTROPIC ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT AND NONLINEAR ROBIN BOUNDARY CONDITIONS

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Abstract. This article presents sufficient conditions for the existence of solutions of the anisotropic quasilinear elliptic equation with variable exponent and nonlinear Robin boundary conditions,

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u+\lambda|u|^{m(x)-2}u=\gamma g(x,u)$$
in Ω,

$$\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\upsilon_i=\mu|u|^{q(x)-2}u \quad \text{on } \partial\Omega.$$

Under appropriate assumptions on the data, we prove some existence and multiplicity results. The methods are based on Mountain Pass and Fountain theorems.

1. Introduction

Many problems in physics and mechanics can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, $L^p(\Omega)$ and $W^{1,p}(\Omega)$, where p is a fixed constant and Ω is an appropriate domain. But for the electrorheological fluids (Smart fluids), this is not adequate but rather, the exponent should be able to vary. This leads to study the problem in the frame-work of variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $p(\cdot)$ is a real-valued function; see, e.g. [12, 13].

On the other hand, it has been experimentally shown that the above-mentioned fluids may have their viscosity undergoing a significant change; see, e.g. [3]. Consequently, the mathematical modelling of such fluids requires the introduction of the so-called anisotropic variable spaces.Indeed, there is by now a large number of papers and increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces

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[20, 24] and some more recent regularity results for minimizers of anisotropic functionals [1, 6, 21].

Therefore, in the recent years, the study of various mathematical problems modeled by quasilinear elliptic and parabolic equations with both anisotropic and variable exponent has received considerable attention. Let us mention many works in that direction by Antontsev and Shmarev; see, e.g. [2] and the references therein.

Our paper is mainly devoted to the existence and multiplicity of solutions of quasilinear elliptic equations under nonlinear Robin boundary condition such as

$$\begin{gathered}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u+\lambda|u|^{m(x)-2}u \\ =\gamma g(x,u)\quad\text{in }\Omega, \\ \sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\upsilon_i=\mu|u|^{q(x)-2}u\quad\text{on }\partial\Omega,\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with $n\geq 2$, with smooth boundary $\partial\Omega$ and υ_i are the components of the outer normal unit vector and for $i\in\{1,\dots,N\}$, $p_i,m\in\mathcal{C}(\bar\Omega)$, $q\in\mathcal{C}(\partial\Omega)$. The functions p_i and g are supposed to satisfy some conditions to be specified below, while λ,γ, and μ are real parameters, with $\gamma,\mu>0$.

We shall give conditions under which problem (1.1) has infinitely many solutions. According to the behaviour of g and to the kind of results we want to prove, variational methods turn out to be more appropriate.
When $\lim_{s\to 0}g(x,s)/|s|^{\sigma}=0$, σ to be made precise later, Mountain Pass theorem provides the existence of at least a solution of (1.1) and, on the other hand, when g is an odd function, Fountain's theorem yields the existence of infinitely many solutions.

A host of publications exist for this type of problems when the boundary condition is replaced by $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$ and $\gamma=0$; see, e.g. [16] and the references therein, where the authors obtained existence results by means of standard variational tools. The associated problem with Dirichlet boundary conditions has also been treated by many authors; see, e.g. [3, 15]. Furthermore, existence of positive solutions for nonlinear Robin problem involving the $p(x)$-Laplacian have been studied by S. G. Deng; in [9], by using the sub-super solutions and variational methods. We consider here the case where μ is positive and g satisfies more hypotheses than in [17], to use the Mountain Passe and Fountain theorems. It turns out that the condition $q^->P_+^+$ plays an important role in the proofs of our main results.

This article is divided into four sections. In the second section, we introduce some basic properties of the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ and anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(x)}(\Omega)$, and state the existence and multiplicity results concerning the problem (1.1). The third section is devoted to the proofs of the main results and finally, in the fourth section we deal with a generalized equation related to our problem (1.1).

2. Preliminaries and main results

To study problem (1.1), we need to introduce the notions of Sobolev space $W^{1,p(x)}(\Omega)$ and anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(x)}(\Omega)$, with variable exponent. For convenience, we only recall some basic facts which will be used later. Let

$\Omega \subset \mathbb{R}^N$ be a measurable subset with $\text{meas}(\Omega) > 0$. We write

$$\mathcal{C}(\bar{\Omega}) = \{u : u \text{ is a continuous function in } \bar{\Omega}\},$$
$$\mathcal{C}_+(\bar{\Omega}) = \{u \in \mathcal{C}(\bar{\Omega}) : \operatorname{ess\,inf}_\Omega u \geq 1\}.$$

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $p \in \mathcal{C}(\bar{\Omega}, \mathbb{R})$ with $p(x) > 1$, for any $x \in \Omega$.

Denote $p^- = \inf_{x\in\Omega} p(x)$ and $p^+ = \sup_{x\in\Omega} p(x)$; then we have, $p^- > 1$ and $p^+ < \infty$. Denote by $\mathcal{M}$ be either Ω or $\partial\Omega$. Define the variable exponent Lebesgue space

$$L^{p(x)}(\mathcal{M}) = \big\{u \mid u : \mathcal{M} \to \mathbb{R} \text{ is measurable and } \int_{\mathcal{M}} |u(x)|^{p(x)}\,\mathrm{d}x < \infty\big\},$$

endowed with the Luxembourg norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\mathcal{M})} = \inf\Big\{\tau > 0; \int_{\mathcal{M}} \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\,\mathrm{d}x \leq 1\Big\}.$$

Proposition 2.1 ([8]). *Let $\rho(u) = \int_{\mathcal{M}} \big|\frac{u(x)}{\tau}\big|^{p(x)}\,\mathrm{d}x$. For $u, u_k \in L^{p(x)}(\mathcal{M})(k = 1, 2, \dots)$, we have:*

1. $|u|_{L^{p(x)}(\mathcal{M})} \leq 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\mathcal{M})} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\mathcal{M})}$.
2. $|u|_{L^{p(x)}(\mathcal{M})} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\mathcal{M})} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\mathcal{M})}$.
3. $|u_k|_{L^{p(x)}(\mathcal{M})} \to 0 \Leftrightarrow \rho(u_k) \to 0$.
4. $|u_k|_{L^{p(x)}(\mathcal{M})} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

We define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\| = \inf\Big\{\tau > 0; \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\tau}\Big|^{p(x)} + \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\Big)\,\mathrm{d}x \leq 1\Big\}.$$

Proposition 2.2 (See [12]). *Both $(L^{p(x)}(\mathcal{M})), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive and uniformly convex Banach spaces.*

Proposition 2.3 (See [12]). *The Hölder inequality holds, namely*

$$\int_{\mathcal{M}} |uv|\,\mathrm{d}x \leq 2|u|_{p(x)}|v|_{q(x)}; \quad \forall u \in L^{p(x)}(\mathcal{M}),\ \forall v \in L^{q(x)}(\mathcal{M}),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Proposition 2.4 (See [8]). *Let $\rho(u) = \int_\Omega (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})\,\mathrm{d}x$. For $u, u_k \in W^{1,p(x)}(\Omega)(k = 1, 2, \dots)$, we have*

1. $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$.
2. $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$.
3. $\|u_k\| \to 0 \Leftrightarrow \rho(u_k) \to 0$.
4. $\|u_k\| \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Let $\overrightarrow{p}(\cdot) : \overline{\Omega} \to \mathbb{R}^N$ be a vectorial function, $\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot))$ such that $2 \leq p_i \leq N$ and put

$$p_M(x) = \max\{p_1(x), \dots, p_N(x)\}.$$

The anisotropic Sobolev space with variable exponent is defined by

$$W^{1,\overrightarrow{p}(x)}(\Omega)=\{u\in L^{p_M(x)}(\Omega):\frac{\partial u}{\partial x_i}\in L^{p_i(\cdot)}(\Omega),\forall i\in\{1,\dots,N\}\},$$

endowed with the norm

$$\|u\|_{\overrightarrow{p}(\cdot)}=\sum_{i=1}^N\big|\frac{\partial u}{\partial x_i}\big|_{p_i(\cdot)}+\sum_{i=1}^N|u|_{p_i(\cdot)}\,.$$

For convenience, we denote:

$$P_-^-=\inf\{p_1^-,p_2^-,\dots,p_N^-\},\quad P_+^+=\sup\{p_1^+,p_2^+,\dots,p_N^+\}$$

and write $X=W^{1,\overrightarrow{p}(x)}(\Omega)$. We know that X is reflexive if $P_-^->1$, (see e.g [22]). We define

$$J(u)=\int_\Omega\Big(\sum_{i=1}^N\frac{1}{p_i(x)}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)}+\sum_{i=1}^N\frac{1}{p_i(x)}|u|^{p_i(x)}\Big)\,\mathrm{d}x,$$

$$G(x,u)=\int_0^u g(x,s)\mathrm{d}s.$$

We have

$$(J'u,v)=\int_\Omega\Big(\sum_{i=1}^N\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+\sum_{i=1}^N|u|^{p_i(x)-2}uv\Big)\,\mathrm{d}x,$$

for all $v\in X$. In all this paper C, $C_i(i=0,1,2,\dots)$ represents different positive real constants.

We make the following assumptions on the functions q and g.

- (H0) $q\in\mathcal{C}(\partial\Omega)$ satisfies : $1\le q(x)\le\frac{(N-1)P_-^-}{N-P_-^-}$ for all $x\in\partial\Omega$ and $q^-<P_+^+$.
- (H1) $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory type function and there exist a constant $C>0$ and a function $\alpha\in\mathcal{C}(\bar\Omega)$ such that: $1<\alpha(x)<\frac{NP_-^-}{N-P_-^-}$, for all $x\in\bar\Omega$ and
$$|g(x,s)|\le C(1+|s|^{\alpha(x)-1})\quad\text{for all }(x,s)\in\Omega\times\mathbb{R}.$$
- (H2) There exists $M>0$ and $\theta_\lambda\ge m^+$ (resp $\theta_\lambda\le m^-$) if $\lambda\ge0$ (resp $\lambda<0$). such that for all s with $|s|\ge M$ and $x\in\Omega$, we have
$$0<\theta_\lambda G(x,s)\le sg(x,s).$$
- (H3) $g(x,s)=\circ(|s|^{P_+^+})$ as $s\to0$ and uniformly for $x\in\Omega$,
- (H4) $g(x,-s)=-g(x,s)$, $x\in\Omega,s\in\mathbb{R}$.

We say that $u\in X$ is a weak solution of (1.1) if

$$\begin{aligned}&\int_\Omega\Big(\sum_{i=1}^N\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+\sum_{i=1}^N|u|^{p_i(x)-2}uv\Big)\,\mathrm{d}x+\lambda\int_\Omega|u|^{m(x)-2}uv\,\mathrm{d}x\\&=\int_\Omega\gamma g(x,u)v\,\mathrm{d}x+\mu\int_{\partial\Omega}|u|^{q(x)-2}uv\,\mathrm{d}x,\end{aligned}$$

for all $v\in X$.

The energy functional associated with problem (1.1) is

$$\begin{aligned}\Phi(u) = &\int_\Omega \Big(\sum_{i=1}^N \frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big| + \sum_{i=1}^N \frac{1}{p_i(x)}|u|^{p_i(x)}\Big)\,\mathrm{d}x \\ &+ \lambda \int_\Omega \frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x - \gamma \int_\Omega G(x,u)\,\mathrm{d}x - \mu \int_{\partial\Omega} \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x.\end{aligned} \tag{2.1}$$

Proposition 2.5 (See [14, 11])**.**
1. *$L \equiv J' : X \to X^*$ is a continuous, bounded and strictly monotone operator;*
2. *L is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in X, and $\overline{\lim}_{n\to+\infty}(L(u_n) - L(u), u_n - u) \le 0$, then $u_n \to u$ in X;*
3. *$L : X \to X^*$ is a homeomorphism.*

The following are embedding results on anisotropic generalized Sobolev spaces and will be used later.

Proposition 2.6 (See [21])**.** *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. For any $q \in \mathcal{C}_+(\overline{\Omega})$ satisfying $q(x) < \frac{NP_-^-}{N-P_-^-}$ for all $x \in \overline{\Omega}$, the embedding*

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

Proposition 2.7 (See [21])**.** *Assume that the boundary of Ω possesses the cone property and $p_i \in \mathcal{C}(\overline{\Omega}), 2 \le p_i < N$ for all $i \in \{1, 2, \dots, N\}$. If $q \in \mathcal{C}(\partial\Omega)$ satisfies the hypothesis $1 < q(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$, then the embedding*

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$$

is continuous and compact.

The main results of this article are as follows:

Theorem 2.8. *Suppose that* (H0)–(H2), (H4) *hold with $m^+ < \frac{NP_-^-}{N-P_-^-}$ and $P_+^+ < \min(\alpha^-, m^-)$. Then, for any $\lambda \in \mathbb{R}$ and $\mu, \gamma > 0$, problem* (1.1) *has at least a nontrivial weak solution.*

Theorem 2.9. *Suppose that* (H0)–(H2) , (H5) *hold with $m^+ < \frac{NP_-^-}{N-P_-^-}$ and $P_+^+ < \min(\alpha^-, m^-)$. Then, for any $\lambda \in \mathbb{R}$ and $\mu, \gamma > 0$, problem* (1.1) *has infinite many pairs of weak solutions.*

3. Proofs of main results

To prove Theorem 2.8, we shall use the Mountain Pass theorem [25]. We first start with the following lemmas.

Lemma 3.1. *If* (H0)–(H2) *hold, then for any $\lambda \in \mathbb{R}$, the functional Φ satisfies the Palais Smale condition (PS).*

Proof. Suppose that $(u_n) \subset X$ is a Palais Smale sequence , ie,

$$\sup|\Phi(u_n)| \le C, \Phi'(u_n) \to 0, \text{ as } n \to \infty.$$

We shall prove that (u_n) has a convergent subsequence.

Let us show that (u_n) is bounded in X. Denote by $\widetilde{m} :\equiv m^+$ if $\lambda > 0$ and $\widetilde{m} :\equiv m^-$ if $\lambda \le 0$. Since $\Phi(u_n)$ is bounded, then by using (H1), we have for large n,

$$
\begin{aligned}
C + C\|u_n\| &\ge \Phi(u_n) - \theta_\lambda \Phi'(u_n) \\
&\ge \Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big) \sum_{i=1}^N \int_\Omega \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u_n|^{p_i(x)}\Big)\,\mathrm{d}x \\
&\quad + \lambda\Big(\frac{1}{\widetilde{m}} - \frac{1}{\theta_\lambda}\Big) \int_\Omega |u_n|^{m(x)}\,\mathrm{d}x - \gamma \int_\Omega (G(x,u_n) - \theta_\lambda g(x,u_n)u_n)\,\mathrm{d}x \\
&\quad - \frac{1}{\theta_\lambda}\langle \Phi'(u_n), u_n\rangle + \mu \int_{\partial\Omega} \Big(\frac{1}{\theta_\lambda} - \frac{1}{q(x)}\Big)|u_n|^{q(x)}\,\mathrm{d}x \\
&\ge \Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big) \sum_{i=1}^N \big|\frac{\partial u}{\partial x_i}\big|_{p_i(x)}^{P_-^-} - \frac{1}{\theta_\lambda}(\Phi'(u_n), u_n) \\
&\quad + \mu\Big(\frac{1}{\theta_\lambda} - \frac{1}{q^-}\Big) \int_{\partial\Omega} |u_n|^{q(x)}\,\mathrm{d}x.
\end{aligned}
$$

Now, according to [4, page 6], we have

$$
\begin{aligned}
\frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-}}{2^{P_-^- -1} N^{P_-^- -1}} &\le \sum_{i=1}^N \Big(\big|\frac{\partial u}{\partial x_i}\big|_{p_i(\cdot)}^{P_-^-} + |u|_{p_i(\cdot)}^{P_-^-}\Big) \\
&\le \sum_{i=1}^N \int_\Omega \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x.
\end{aligned}
$$

Then,

$$
C + C\|u_n\| \ge \frac{1}{2^{P_-^- -1} N^{P_-^- -1}} \Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big)\|u_n\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - \frac{C_1}{\theta_\lambda}\|u_n\|_{\overrightarrow{p}(\cdot)} - C.
$$

Since $\mu > 0$, then by using condition (H2) and the inequality above, we deduce that u_n is bounded in X. The proof is complete. $\square$

Lemma 3.2. *There exist $r_1, C' > 0$ such that $\Phi(u) \ge C'$, for all $u \in X$ such that $\|u\| = r_1$.*

Proof. Conditions (H0), (H1) and (H2) ensure that, for any $\epsilon > 0$, we have

$$
|G(x,s)| \le \epsilon |s|^{P_+^+} + C(\epsilon)|s|^{\alpha(x)}, \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}.
$$

For $\|u\|$ small enough, we thus obtain

$$
\begin{aligned}
\Phi(u) &\ge \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x + \lambda \int_\Omega \frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x \\
&\quad - \int_\Omega (\epsilon|u|^{P_+^+} + C(\epsilon)|u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega} |u|^{q(x)}\,\mathrm{d}x \\
&\ge \frac{1}{P_+^+ 2^{P_+^+ -1} N^{P_+^+ -1}} \|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+} - \frac{|\lambda|}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x - \int_\Omega \epsilon |u|^{P_+^+}\,\mathrm{d}x \\
&\quad - \int_\Omega C(\epsilon)|u|^{\alpha(x)}\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega} |u|^{q(x)}\,\mathrm{d}x.
\end{aligned} \tag{3.1}
$$

Since $P_+^+ < \alpha^- \le \alpha(x) < \frac{NP_-^-}{N-P_-^-}$, for all $x \in \Omega$ and $q(x) < \frac{(N-1)P_-^-}{N-P_-^-}$, for all $x \in \partial\Omega$; then, we have

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega) \quad \text{and} \quad W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega),$$

with continuous and compact embeddings. Consequently, there exist two constants $C_1' > 0$ and $C_2' > 0$ such that

$$|u|_{L^{P_+^+}(\Omega)} \le C_2'\|u\|, \quad |u|_{L^{q(x)}(\Omega)} \le C_1'\|u\|, \text{ for all } u \in X. \tag{3.2}$$

By using (3.2) for $\|u\|$ small enough, we obtain from (3.1) that

$$\begin{aligned}\Phi(u) \ge& \frac{1}{P_+^+ 2^{P_+^+-1} N^{P_+^+-1}}\|u\|^{P_+^+} - \frac{|\lambda|}{m^-}\max\{|u|^{m^+}_{L^{m(x)}(\Omega)}, |u|^{m^-}_{L^{m(x)}(\Omega)}\}\\ &- \epsilon C_2'\|u\|^{P_+^+} - C(\epsilon)C_3'\|u\|^{\alpha^-} - \frac{\mu}{q^-}C_1'\|u\|^{q^-}.\end{aligned}$$

Since $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we have

$$\begin{aligned}\Phi(u) \ge& \frac{1}{P_+^+ 2^{P_+^+-1} N^{P_+^+-1}}\|u\|^{P_+^+} - \frac{|\lambda|C}{m^-}\max\{\|u\|^{m^+}, \|u\|^{m^-}\}\\ &- \epsilon C_2'^{P_+^+}\|u\|^{P_+^+} - C(\epsilon)C_3'\|u\|^{\alpha^-} - \frac{\mu}{q^-}C_1'\|u\|^{q^-}.\end{aligned}$$

Now, let $\epsilon > 0$ be small enough so that:

$$0 < \epsilon C_2'^{P_+^+} \le \frac{1}{2P_+^+ 2^{P_+^+-1} N^{P_+^+-1}} =: c_0.$$

We have

$$\begin{aligned}\Phi(u) &\ge c_0\|u\|^{P_+^+} - \frac{|\lambda|C}{m^-}\max\{\|u\|^{m^+}, \|u\|^{m^-}\} - C(\epsilon)\|u\|^{\alpha^-} - \frac{\mu C_1'}{q^-}\|u\|^{q^-}\\ &\ge \|u\|^{P_+^+}\Big(c_0 - \frac{|\lambda|C}{m^-}\max\{\|u\|^{m^+-P_+^+}, \|u\|^{m^--P_+^+}\}\Big)\\ &\quad - \|u\|^{P_+^+}\Big(C(\epsilon)\|u\|^{\alpha^--P_+^+} + \frac{\mu C_1'}{q^-}\|u\|^{q^--P_+^+}\Big).\end{aligned}$$

Since $P_+^+ < \min(\alpha^-, m^-, q^-)$, then there exist $r_1 > 0$ and $C' > 0$ such that

$$\Phi(u) \ge C' > 0, \quad \text{for any } u \in X.$$

Hence, the proof is complete. □

Proof of Theorem 2.8. To apply the Mountain Pass theorem ([25]), we have to prove that $\Phi(tu) \to -\infty$ as $t \to +\infty$, for some $u \in X$. From (H2), it follows that

$$G(x,s) \ge C|s|^{\theta_\lambda}, \quad \forall x \in \bar{\Omega}, \forall |s| \ge M.$$

For $u \in X$ and $t > 1$, we have

$$\begin{aligned}\Phi(tu) \le& \frac{1}{P_-^-}\sum_{i=1}^N \int_\Omega \Big(\Big|\frac{\partial(tu)}{\partial x_i}\Big|^{p_i(x)} + |tu|^{p_i(x)}\Big)\,\mathrm{d}x + \lambda\int_\Omega \frac{1}{m(x)}|tu|^{m(x)}\,\mathrm{d}x\\ &- \int_\Omega G(x,tu)\,\mathrm{d}x - \frac{\mu}{q^+}\int_{\partial\Omega}|tu|^{q(x)}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le \frac{t^{P_+^+}}{P_-^-}\sum_{i=1}^N\int_\Omega\Big(\big|\frac{\partial(tu)}{\partial x_i}\big|^{p_i(x)}+|u|^{p_i(x)}\Big)\,\mathrm{d}x+\lambda t^{\widetilde{m}}\int_\Omega\frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x\\
&\quad - Ct^{\theta_\lambda}\int_\Omega|u|^{\theta_\lambda}\,\mathrm{d}x-\frac{\mu t^{q^-}}{q^+}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x,
\end{aligned}
$$

where again $\widetilde{m}=m^+$ if $\lambda>0$ and $\widetilde{m}=m^-$ if $\lambda\le 0$.

By (H0) and (H2), it follows that, for any $\lambda\in\mathbb{R}$, $\Phi(tu)\to-\infty$ as $(t\to+\infty)$. Since $\Phi(0)=0$, it follows that Φ satisfies the condition of the Mountain Pass lemma, and so Φ admits at least one nontrivial critical point $u_0\in X$; which is characterized by

$$
\tau=\inf_{h\in\Gamma}\sup_{t\in[0,1]}\Phi(h(t)),
$$

where

$$
\Gamma=\{h\in\mathcal{C}([0,1],X);h(0)=0 \text{ and } h(1)=e\}.
$$

□

Proof of Theorem 2.9. Let X be a reflexive and separable Banach space. It is well know (see, e.g. [1]) that there are $\{e_j\}_{j=1}^\infty\subset X$ and $\{e_j^*\}_{j=1}^\infty\subset X^*$ (where X^* is the topological dual of X) such that

$$
X=\overline{\operatorname{span}}\{e_j:1,2,\dots\},\quad X^*=\overline{\operatorname{span}}\{e_j^*:1,2,\dots\},
$$

and

$$
\langle e_j^*,e_i\rangle=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases} \tag{3.3}
$$

For convenience, we write $X_j=\operatorname{span}\{e_j\}$, $Y_k=\oplus_{j=1}^k X_j$ and $Z_k=\oplus_{j=k}^\infty X_j$. Denote

$$
p^*(x)=\begin{cases}Np(x)/(N-p(x)) & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\ge N.\end{cases}
$$

Lemma 3.3 (See [8, 10])**.** *Let $\beta(x)\in\mathcal{C}_+(\bar\Omega)$, with $\beta(x)<p^*(x)$ for $x\in\bar\Omega$ and $\alpha_k:=\sup\{|u|_{L^{\beta(x)}(\Omega)};\|u\|=1,u\in Z_k\}$. Then, we have $\lim_{k\to\infty}\alpha_k=0$.*

To prove Theorem 2.9, we shall use the Fountain theorem (see [25, Theorem 3.6]). Obviously, $\Phi\in\mathcal{C}^1(X,\mathbb{R})$ is an even functional. By using (H0) and (H1) we first prove that if k is large enough, then there exist $\rho_k>\nu_k>0$ such that

$$
b_k:=\inf\{\Phi(u)/u\in Z_k,\|u\|=\nu_k\}\to+\infty\quad\text{as } k\to+\infty; \tag{3.4}
$$

$$
a_k:=\max\{\Phi(u)/u\in Y_k,\|u\|=\rho_k\}\to 0\quad\text{as } k\to+\infty. \tag{3.5}
$$

Proof of (3.4): For any $u \in Z_k$, $|u| = r_k > 1$, we have

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \int_\Omega \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)} \Big)\,\mathrm{d}x - \frac{|\lambda|}{m^-} \int_\Omega |u|^{m(x)}\,\mathrm{d}x \\
&\quad - C \int_\Omega (1 + |u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-} \int_{\partial\Omega} |u|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \int_\Omega \big|\frac{\partial u}{\partial x_i}\big|^{P_-^-}\,\mathrm{d}x - \frac{|\lambda|}{m^-} \int_\Omega |u|^{m(x)}\,\mathrm{d}x \\
&\quad - C \int_\Omega (1 + |u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-} \int_{\partial\Omega} |u|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{1}{P_+^+ 2^{P_-^- - 1} N^{P_-^- - 1}} \|u\|^{p_-^-} - \frac{C|\lambda|}{m^-} |u|_{L^{m(x)}}^{m(\xi)} \\
&\quad - C_1 |u|_{\alpha(x)}^{\alpha(\xi)} - \frac{\mu}{q^-} |u|_{L^{q(x)}(\partial\Omega)}^{q(\xi)} - C_2, \quad \text{for some } \xi \in \Omega.
\end{aligned} \tag{3.6}$$

So, for the study of the previous inequality, we only need to consider either the case where $m(x) \geq \alpha(x)$ or the case where $m(x) < \alpha(x)$ for all $x \in \Omega$.

Let us assume that $m(x) \leq \alpha(x)$ for all $x \in \Omega$. Then, we have $L^{\alpha(x)}(\Omega) \subset L^{m(x)}(\Omega)$. Thus, there is a positive constant $C_3 > 0$ such that

$$|u|_{L^{m(x)}(\Omega)} \leq C_3 |u|_{L^{\alpha(x)}(\Omega)} \quad \text{for all } u \in X.$$

So, for any $\xi \in \Omega$, we have

$$|u|_{L^{m(x)}(\Omega)}^{m(\xi)} \leq C^{m(\xi)} |u|_{L^{\alpha(x)}(\Omega)}^{m(\xi)}.$$

Let us denote $e := 1/(2^{P_-^- - 1} N^{P_-^- - 1})$. Then, for any $\xi \in \Omega$, we have

$$\begin{aligned}
&\Phi(u) \\
&\geq \frac{e}{P_+^+} \|u\|^{P_-^-} - C' |u|_{L^{\alpha(x)}}^{m(\xi)} - C_1 |u|_{L^{\alpha(x)}(\Omega)}^{\alpha(\xi)} - \frac{\mu}{q^-} |u|_{L^{q(x)}(\partial\Omega)}^{q(\xi)} - C_2 \\
&\geq \frac{e}{P_+^+} \|u\|^{P_-^-} - C \max\{ |u|_{L^{\alpha(x)}(\Omega)}^{\alpha(\xi)}, |u|_{L^{\alpha(x)}(\Omega)}^{m(\xi)} \} - \frac{\mu}{q^-} |u|_{L^{q(x)}(\partial\Omega)}^{q(\xi)} - C_2.
\end{aligned} \tag{3.7}$$

Denote

$$\begin{gathered}
E = L^{\alpha(x)}(\Omega) \cap L^{q(x)}(\partial\Omega), \\
A = \{ u \in E : |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1 \}, \\
B = \{ u \in E : |u|_{L^{\alpha(x)}(\Omega)} > 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1 \}, \\
C = \{ u \in E; |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1 \}, \\
D = \{ u \in E; |u|_{L^{\alpha(x)}(\Omega)} \geq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1 \}.
\end{gathered}$$

From (3.7), we have

$$\Phi(u) \geq \begin{cases}
\frac{e}{P_+^+} \|u\|^{P_-^-} - C_1 & \text{if } u \in A, \\
\frac{e}{P_+^+} \|u\|^{P_-^-} - C_1 (\alpha_k |u|)^{\alpha^+} - C_2 & \text{if } u \in B, \\
\frac{e}{P_+^+} \|u\|^{P_-^-} - \frac{\mu}{q^-} (\beta_k |u|)^{q^+} - -C_1 & \text{if } u \in C, \\
\frac{e}{P_+^+} \|u\|^{P_-^-} - C_1 (\alpha_k |u|)^{\alpha^+} - \frac{\mu}{q^-} (\beta_k |u|)^{q^+} - C_2, & \text{if } u \in D,
\end{cases}$$

where $\beta_k = \sup\{|u|_{L^{q(x)}(\partial\Omega)}; \|u\| = 1, u \in Z_k\}$. It is obvious that $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$ in A.

For $u \in B \cup C$, we have

$$\Phi(u) \geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C_2(\widetilde{\alpha_k}|u|)^{\widetilde{\alpha}^+} - C_3,$$

By taking $\widetilde{\alpha_k}$ to be either α_k or β_k and $\widetilde{\alpha} = \alpha^+$ or q^+, we obtain

$$\Phi(u) \geq e\Big(\frac{1}{P_+^+} - \frac{1}{\widetilde{\alpha}^+}\Big)\left(\frac{C_2}{e}\widetilde{\alpha}^+\widetilde{\alpha_k}^{\widetilde{\alpha}^+}\right)^{\frac{P_-^-}{P_-^- - \widetilde{\alpha}^+}} - C_3.$$

Since $\widetilde{\alpha_k} \to 0$ as $k \to \infty$ and $\widetilde{\alpha}^+ > P_+^+$, then $\left(\frac{C_2}{e}\widetilde{\alpha}^+\widetilde{\alpha_k}^{\widetilde{\alpha}^+}\right)^{\frac{1}{P_-^- - \widetilde{\alpha}^+}} \to \infty$ as $k \to \infty$. Consequently, we have

$$\Phi(u) \to +\infty \quad \text{as } \|u\| \to +\infty, u \in Z_k.$$

If $u \in D$, then

$$\Phi(u) \geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C(\alpha_k^{\alpha^+}|u|^{\alpha^+}) - \frac{\mu}{q^-}(\beta_k^{q^+}|u|^{q^+}) - C_1.$$

By assuming $\alpha^+ \leq q^+$, we obtain

$$\begin{aligned}
\Phi(u) &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C(\alpha_k^{\alpha^+}|u|^{q^+}) - \frac{\mu}{q^-}(\beta_k^{q^+}|u|^{q^+}) - C_1 \\
&\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - (C\alpha_k^{\alpha^+} + \frac{\mu}{q^-}\beta_k^{q^+})|u|^{q^+} - C_1 \\
&\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C_3(\alpha_k^{\alpha^+} + \beta_k^{q^+})|u|^{q^+} - C_1 \\
&\geq e\Big(\frac{1}{P_+^+} - \frac{1}{q^+}\Big)\Big[C_3q^+(\alpha_k^{\alpha^+} + \beta_k^{q^+})\Big]^{\frac{P_-^-}{P_-^- - q^+}} - C_1.
\end{aligned}$$

Since $q^+ > P_+^+$, we then have $[C_3q^+(\alpha_k^{\alpha^+} + \beta_k^{q^+})]^{\frac{1}{P_-^- - q^+}} \to \infty$, as $k \to \infty$. Consequently, we obtain $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in Z_k$. Now, from condition (H2), we have

$$G(x,s) \geq C_1|s|^{\theta_\lambda} - C_2, \quad \text{for any } (x,s) \in \Omega \times \mathbb{R}.$$

Then there exist constants $C_1', C_3' > 0$ such that

$$\Phi(u) \leq \frac{C_1'}{P_-^-}\sum_{i=1}^N \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,\mathrm{d}x + \frac{\lambda}{\widehat{m}}\int_\Omega |u|^{m(x)}\,\mathrm{d}x - C_2'\|u\|^{\theta_\lambda} - C_3',$$

where $\widehat{m} = m^-$ if $\lambda > 0$ and $\widehat{m} = m^+$ if $\lambda \leq 0$. Hence, we obtain the inequality

$$\sum_{i=1}^N \Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i(x)}(\Omega)}^{P_+^+} \leq C\Big(\sum_{i=1}^N \Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i(x)}(\Omega)}\Big)^{P_+^+},$$

Where C is a positive constant.

In the case $\lambda > 0$, we obtain

$$\Phi(u) \leq \frac{C'}{P_-^-}\|u\|^{P_+^+} + \frac{C_4\lambda}{\widehat{m}}\|u\|^{m^+} - C_2'\|u\|^{\theta_\lambda} - C_3'.$$

But we have $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$, and $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. Then, as $\theta_\lambda > \max(P_+^+, m^+)$ and $\dim Y_k = k$, it is easy to see that

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \text{ for } u \in Y_k.$$

For the case $\lambda \leq 0$, we have

$$\Phi(u) \leq \frac{C'}{P_-^-}\|u\|^{P^+} - C_2'\|u\|^{\theta_\lambda} - C_3'.$$

Now, as we have $\theta_\lambda > P_+^+$ and $\dim Y_k = k$, it is also easy to see that

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \text{ for } u \in Y_k.$$

4. A generalized equation

We shall now consider the generalized equation

$$\begin{aligned} &-\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{n}|u|^{p_i(x)-2}u \\ &= \lambda g_1(x,u) + \nu g_2(x,u) \quad \text{in } \Omega, \\ &\sum_{i=1}^{n}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\upsilon_i = \mu f(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\lambda, \nu, \mu > 0$ are real numbers, $p_i(x) \in \mathcal{C}(\bar{\Omega})$ with $2 \leq p_i(x) \leq N$ for all $i \in \{1, 2, \dots, N\}$, and $g_1, g_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are two functions of class $\mathcal{C}^1$ with respect to the Ω-variable, and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ with respect to the $\partial\Omega$-variable. We make the following assumptions on the functions q, g_1, g_2 and f.

(H5) For $i = 1, 2$, $q_i \in \mathcal{C}(\bar{\Omega})$ satisfies $1 < q_i(x) < \frac{NP_-^-}{N-P_-^-}$ for all $x \in \Omega$.

(H6) (i) For $i = 1, 2$, $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, and there exist some positive constant C_i such that

$$|g_i(x,s)| \leq C_1 + C_2|s|^{q_i(x)-1} \quad \text{for } (x,s) \in \Omega \times \mathbb{R}.$$

(ii) There exists $M > 0$, $\sigma > P_+^+$ such that for all $|s| \geq M$ and $x \in \Omega$,

$$0 < \sigma G_2(x,s) \leq g_2(x,s)s.$$

(H7) There exist $\delta_1 > 0, C_3 > 0$ and $q_3 \in \mathcal{C}(\bar{\Omega})$ such that

$$G_1(x,s) \geq C_3|s|^{q_3(x)}, \forall (x,s) \in \Omega \times (0, \delta_1],$$

where $\max(q_3^-, q_1^+) < P_-^- < P_+^+ < q_2^-$.

(H8) $g_2(x,s) = \circ(|s|^{P_+^+-1})$ as $s \to 0$ uniformly for $x \in \Omega$.

(H9) (i) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exists a constant $C > 0$ such that

$$|f(x,s)| \leq C(1 + |s|^{\beta(x)-1}), \quad \forall (x,s) \in \partial\Omega \times \mathbb{R};$$

where $\beta(x) \in \mathcal{C}(\partial\Omega)$ with $1 < \beta^- \leq \beta^+ < P_-^-$ and $\beta(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$.

(ii) There exist $R > 0$, such that for all $|s| \geq R$ and $x \in \partial\Omega$

$$0 < \sigma F(x,s) \leq f(x,s)s.$$

(H10) There exist $\delta_2 > 0$, $C_4 > 0$ and $q_4(x) \in \mathcal{C}(\partial\Omega)$ such that
$$F(x,s) \geq C_4|s|^{q_4(x)}, \quad \forall x \in \partial\Omega,\ \forall |s| \leq \delta_2,$$
where $1 < q_4 < \frac{(N-1)P_-^-}{N-P_-^-}$ and $q_4^+ < P_-^-$ for all $x \in \partial\Omega$.

(H11) For $i = 1,2$, $g_i(x,-s) = -g_i(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$, and $f(x,-s) = -f(x,s)$ for all $(x,s) \in \partial\Omega \times \mathbb{R}$.

We denote
$$g(x,s) = \lambda g_1(x,s) + \gamma g_2(x,s), \quad G_i(x,s) = \int_0^s g_i(x,t)\,\mathrm{d}t \quad (i = 1,2),$$
$$G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t, \quad F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t;$$
and the associated functional
$$\Phi(u) = \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x - \mu\int_{\partial\Omega} F(x,u)\,\mathrm{d}x.$$

Proposition 4.1. *If* (H5), (H6) *and* (H9) *hold, then for every* $\lambda,\gamma,\mu \geq 0$ *the functional* Φ *satisfies the Palais Small condition (PS).*

Proof. We use the following inequalities: For $x \in \Omega, s \in \mathbb{R}$
$$\sigma G_2(x,s) \leq g_2(x,s)s + C_3,$$
$$\begin{aligned}\sigma G(x,s) - g(x,s)s &\leq [\sigma G_1(x,s) - sg_1(x,s)] + [\sigma G_2(x,s) - sg_2(x,s)]\\ &\leq (C_1 + C_2|s|^{q_1(x)}) + C_3.\end{aligned}$$
Suppose that $(u_n) \subset X$ is a (PS) sequence ; i.e,
$$\sup|\Phi(u_n)| \leq C, \Phi'(u_n) \to 0, \text{ as } n \to \infty.$$
Let us show that (u_n) is bounded in X. Since $\Phi(u_n)$ is bounded, then by using hypothesis (H6) and (H9), we have for n large enough
$$\begin{aligned}C + C\|u_n\| &\geq \sigma\Phi(u_n) - \Phi'(u_n)\\ &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\sigma}\Big)\sum_{i=1}^N \int_\Omega \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u_n|^{p_i(x)}\Big)\,\mathrm{d}x\\ &\quad - \int_\Omega (\sigma G(x,u_n) - g(x,u_n)u_n)\ \mathrm{d}x\\ &\quad - \mu\int_{\partial\Omega} (\sigma f(x,u_n) - f(x,u_n)u_n)\ \mathrm{d}x\\ &\geq \frac{1}{2^{P_-^- -1}N^{P_-^- -1}}\Big(\frac{1}{p_+^+} - \frac{1}{\sigma}\Big)\|u_n\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - C'\int_\Omega |u_n|^{q_1}\,\mathrm{d}x\\ &\quad - C' - \int_{\partial\Omega} (\sigma f(x,u_n) - f(x,u_n)u_n)\ \mathrm{d}x.\end{aligned} \tag{4.2}$$
Applying (H9) for $\|u_n\|$ large enough, we then get
$$C + C\|u_n\| \geq \frac{1}{2^{P_-^- -1}N^{P_-^- -1}}\Big(\frac{1}{p_+^+} - \frac{1}{\sigma}\Big)\|u_n\|^{P_-^-} - C'\|u_n\|^{q_1^+} - C_3'.$$
Now, as $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q_1^+}(\Omega)$ is a continuous and compact embedding, from the inequality above, we deduce that u_n is bounded in X. The proof is complete. □

Remark 4.2. It follows from (H6) that

$$G_2(x,s) \geq C_5|s|^{1/\sigma} - C_6, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}.$$

The main results of this section are as follows:

Proposition 4.3 ([14]). *Assume that $\psi : X \to \mathbb{R}$ is weakly-strongly continuous and that $\psi(0) = 0$. Let $\nu > 0$ be given. Set*

$$\beta_k = \beta_k(\nu) = \sup_{u \in Z_k, \|u\| \leq \nu} |\psi(u)|.$$

Then $\beta_k \to 0$ as $k \to \infty$.

Theorem 4.4. *Assume that* (H5), (H6) *and* (H9) *hold.*

1. *If in addition,* (H10) *holds, then for every $\gamma, \mu > 0$, there exists $r_0(\gamma) > 0$ such that when $0 \leq \lambda, \mu \leq r_0(\gamma)$, problem* (4.1) *has a nontrivial solution u_1 such that $\Phi(u_1) > 0$.*
2. *If in addition,* (H7) *and* (H10) *hold, then for every $\gamma, \mu > 0$, there exists $r_0(\gamma) > 0$ such that when $0 \leq \lambda, \mu \leq r_0(\gamma)$, problem* (4.1) *has two nontrivial solutions u_1, v_1 such that $\Phi(u_1) > 0$ and $\Phi(v_1) < 0$.*
3. *If in addition,* (H7), (H10) *and* (H11) *hold, then for every $\lambda, \gamma, \mu > 0$, problem* (4.1) *has a sequence of solutions $\{\pm u_k\}$ such that $\Phi(\pm u_k) \to +\infty$ as $k \to +\infty$.*

Proof. (1) We denote

$$\psi_1(u) = \lambda \int_\Omega G_1(x, u(x))\,\mathrm{d}x, \quad \psi_2(u) = \gamma \int_\Omega G_2(x, u(x))\,\mathrm{d}x.$$

When the assumptions in (1) hold, then for sufficiently small $\|u\|$, we get

$$G_2(x,u) \leq \epsilon |u|^{P_+^+} + C(\epsilon)|u|^{q_2}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

Then

$$\psi_2(u) \leq \gamma\epsilon \int_\Omega |u|^{P_+^+}\,\mathrm{d}x + \gamma C(\epsilon)\int_\Omega |u|^{q_2(x)}\,\mathrm{d}x.$$

Since $1 < q_2 < \frac{NP_-^-}{N-P_-^-}$, for all $x \in \Omega$, then we have

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega), \quad \text{and} \quad W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q_2(x)}(\Omega),$$

with continuous and compact embeddings. This implies the existence of $C_1, C_2 > 0$ such that

$$\psi_2(u) \leq \gamma\epsilon C_1 \|u\|^{P_+^+} + \gamma C(\epsilon) C_2 \|u\|^{q_2^-}.$$

Choose $\epsilon > 0$ small enough so that $0 < \gamma\epsilon C_2 < \frac{1}{2^{P_+^+-1} N^{P_+^+-1}}$. Then, we have

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x - \psi_2(u)\\ &\geq \frac{1}{2^{P_+^+} N^{P_+^+-1}}\|u\|^{P_+^+} - \gamma C(\epsilon) C_2 \|u\|^{q_2^-}.\end{aligned}$$

Since $q_2^- > P_+^+$, there exist $r_1 > 0$ and $\alpha > 0$ such that

$$\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x - \psi_2(u) \geq \alpha > 0, \quad \text{for } \|u\| = r_1.$$

We can find $r_0(\gamma) > 0$ such that when $\mu, \lambda \le r_0(\gamma)$, we obtain

$$\psi_1(u) \le \frac{\alpha}{2}, \quad \forall u \in S_{r_1} = \{u \in X; \|u\| = r_1\}.$$

Therefore, $\lambda, \mu \le r_0(\gamma)$. So, we obtain

$$\Phi(u) \ge \frac{\alpha}{2} > 0, \quad \forall u \in S_{r_1}.$$

Let $u \in X$ and $t > 1$, we have

$$\begin{aligned} \Phi(tu) = \sum_{i=1}^{N} \int_\Omega \frac{t^{p_i(x)}}{p_i(x)} \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)} \Big) \, \mathrm{d}x - \lambda \int_\Omega G_1(x, tu) \, \mathrm{d}x \\ - \gamma \int_\Omega G_2(x, tu) \, \mathrm{d}x - \mu \int_{\partial\Omega} F(x, tu) \, \mathrm{d}x. \end{aligned} \tag{4.3}$$

From Remark 4.2, we obtain

$$\begin{aligned} \Phi(tu) \le t^{P_+^+} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)} + |u|^{p_i(x)} \Big) \, \mathrm{d}x - \lambda \int_\Omega G_1(x, tu) \, \mathrm{d}x \\ - \gamma C_5 t^{1/\sigma} \int_\Omega |u|^{1/\sigma} \, \mathrm{d}x - \mu \int_{\partial\Omega} F(x, tu) \, \mathrm{d}x. \end{aligned} \tag{4.4}$$

Now, since

$$G_1(x, tu) = o((t|u|)^{q_1^+}), \quad F(x, tu) = o((t|u|)^{\beta^+}) \quad \text{when } t \to +\infty,$$

(because $P_+^+ \le q_1^+$ and $\beta^+ < P_+^+ < \frac{1}{\sigma}$), we obtain

$$\Phi(tu) \to -\infty, \quad \text{when } t \to +\infty.$$

Hence, It follows that there exist $u_0 \in X$ such that $\|u_0\| > r_1$ and $\Phi(u_0) < 0$. Therefore, By the Mountain Pass theorem, problem (4.1) has a nontrivial solution u_1 such that $\Phi(u_1) > 0$.

(2) Under the assumpti9ns in (2) hold, (1), we know that there exist $r_0(\gamma) > 0$ such that when $0 \le \lambda, \mu \le r_0(\gamma)$, problem has a nontrivial solution u_1 such that $\Phi(u_1) > 0$. For $t \in (0, 1)$ small enough, and $v_0 \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \le v_0(x) \le \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} &\Phi(tv_0) \\ &= \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big(\big|\frac{\partial (tv_0)}{\partial x_i}\big|^{p_i(x)} + |tv_0|^{p_i(x)} \Big) \, \mathrm{d}x - \lambda \int_\Omega G_1(x, tv_0) \, \mathrm{d}x \\ &\quad - \gamma \int_\Omega G_2(x, tv_0) \, \mathrm{d}x - \mu \int_{\partial\Omega} F(x, tv_0) \, \mathrm{d}x \\ &\le t^{P_-^-} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big(\big|\frac{\partial v_0}{\partial x_i}\big|^{p_i(x)} + |v_0|^{p_i(x)} \Big) \, \mathrm{d}x - \lambda C_3 \int_\Omega |tv_0|^{q_3(x)} \, \mathrm{d}x \\ &\quad - \gamma \int_\Omega G_2(x, tv_0) \, \mathrm{d}x - \mu C_4 \int_{\partial\Omega} |tv_0|^{q_4(x)} \, \mathrm{d}x. \end{aligned} \tag{4.5}$$

For $t \in (0, 1)$ small enough, we obtain

$$G_2(x, tv_0) = o(|tv_0|^{P_+^+}), \quad \text{as } t \to \infty$$

So, we have

$$\begin{aligned} &\Phi(tv_0)\\ &\leq t^{P_-^-}\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}\Big(\big|\frac{\partial tv_0}{\partial x_i}\big|^{p_i(x)}+|tv_0|^{p_i(x)}\Big)\,\mathrm{d}x-\lambda C_3 t^{q_3^+}\int_\Omega |v_0|^{q_3(x)}\,\mathrm{d}x \\ &\quad -\gamma M t^{P_+^+}\int_\Omega v_0\,\mathrm{d}x-\mu C_4 t^{q_4^+}\int_{\partial\Omega}|v_0|^{q_4(x)}\,\mathrm{d}x. \end{aligned} \tag{4.6}$$

Since $\min(q_3^+,q_4^+)<P_-^-$, by factoring the right side of (4.6) by $t^{q_3^+}$ if $q_4^+>q_3^+$, and by $t^{q_4^+}$ if $q_3^+>q_4^+$, we obtain

$$\lim_{t\to 0}\Phi(tv_0)<0.$$

Then there exist $w\in X$ such that $\|w\|\leq r_1$, and $\Phi(w)<0$.

(3) Φ is an even functional. We denote

$$\psi(u)=\lambda\int_\Omega G_1(x,u)\,\mathrm{d}x+\gamma\int_\Omega G_2(x,u)\,\mathrm{d}x+\mu\int_{\partial\Omega}F(x,u)\,\mathrm{d}x$$

As $\beta_k(\nu)$ is defined in Proposition 2.6, for each positive integer, there exist a positive integer k_0 such that $\beta_k(n)\leq 1$ for all $k\geq k_0(n)$. We can assume $k_0(n)<k_0(n+1)$ for each n. We define $\{\nu_k : k=1,2,\dots\}$ by

$$\nu_k=\begin{cases} n & \text{if } k_0\leq k<k_0(n+1),\\ 1 & \text{if } 1\leq k<k_0. \end{cases} \tag{4.7}$$

We see that $\nu_k\to\infty$ when $k\to\infty$, then for $u\in Z_k$ *with* $\|u\|=\nu_k$, we obtain

$$\begin{aligned} \Phi(u)&=\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)}+|u|^{p_i(x)}\Big)\,\mathrm{d}x-\psi(u)\\ &\geq \frac{1}{P_+^+2^{P_-^--1}N^{P_-^--1}}(\nu_k)^{P_-^-}-1. \end{aligned}$$

Consequently,

$$\inf_{u\in Z_k,\|u\|=\nu_k}\Phi(u)\to\infty \quad \text{as } k\to\infty.$$

So the hypotheses (3.5) of Fountain theorem are satisfied. Indeed, by (H6), (H9) and Remark 4.2, for $\|u\|\geq 1$ we obtain

$$\Phi(u)\leq \frac{C}{P_-^-}\|u\|^{P_+^+}+C_1\lambda|u|_{q_1(x)}^{q_1^+}-C_5\gamma|u|_{\frac{1}{\sigma}}^{1/\sigma}+C_6\mu|u|_{\beta(x)}^{\beta^+}+C_7.$$

As the space Y_k has finite dimension i.e all norms are equivalents, we then have

$$\Phi(u)\leq \frac{C}{P_-^-}\|u\|^{P_+^+}+C_1'\lambda\|u\|^{q_1^+}-C_5'\gamma\|u\|^{1/\sigma}+C'\mu\|u\|^{\beta^+}+C_7.$$

Since $\min(q_1^+,q_2^+)<P_+^+<\frac{1}{\sigma}$, we obtain $\Phi(u)\to-\infty$ as $\|u\|\to+\infty$, $u\in Y_k$. Finally, the proof of (3) is complete. □

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