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# GENERALIZED INVERSE SCATTERING TRANSFORM FOR THE NONLINEAR SCHRÖDINGER EQUATION FOR BOUND STATES WITH HIGHER MULTIPLICITIES

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Abstract. We consider a generalization of the inverse scattering transform for the nonlinear Schrödinger (NLS) equation when bound states have multiplicities greater than one. This generalization is accomplished by deriving an explicit compact formula for the time evolution of the norming constants in the presence of nonsimple bound states. Such a formula helps to find explicit solutions to the NLS equation and obtain a generalization of soliton solutions.

## 1. Introduction

The initial-value problem for the focusing nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1.1}$$

where the subscripts denote partial derivatives, was solved by the inverse scattering transform method, by Zakharov and Shabat [11] in 1972, although certain details were filled in later papers [1, 2]. Equation (1.1) has important applications to signal propagation along optical fibers under conditions of anomalous dispersion ([9], [3, Ch. 10]) and to 1-D wave propagation on the surface of deep waters [3, Ch. 6].

The inverse scattering transform method consists of converting the initial-value problem for (1.1) into the elementary initial-value problem for the scattering data of the focusing Zakharov-Shabat system

$$\frac{d}{dx}\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}-i\lambda & u(x,t)\\ -u(x,t)^* & i\lambda\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}, \quad -\infty < x < \infty. \tag{1.2}$$

To do so, it is important to solve, for each $t$, the direct scattering problem (converting $u(x,t)$ into scattering data) and the inverse scattering problem (converting scattering data into $u(x,t)$), where the scattering data consist of one of the reflection coefficients, the poles of the transmission coefficient, and the so-called norming constants. In the case where the poles are all simple, Zakharov and Shabat [11] have formulated the inverse scattering transform method for (1.1), with one norming constant for each simple pole. The inverse scattering problem can then be solved by solving the Marchenko integral equation whose kernel can be written in

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terms of reflection coefficient, poles, and norming constants. They also wrote, for each pole, the norming constant as the product of the residue of the transmission coefficient and the proportionality factor between the two Jost solutions of (1.1), the so-called dependency constant.

An alternative method to solve the inverse scattering problem is called the Riemann-Hilbert method [1, 2]. It consists of converting algebraic relations between Jost solutions involving reflection and transmission coefficients into a singular integral equation whose discrete terms contain poles and norming constants and whose continuous term contains one of the reflection coefficients. Thus the norming constants are indispensable to applying the Riemann-Hilbert method.

In the literature, the analysis of the inverse scattering transform with nonsimple bound states was mainly avoided due to technical complications. For example, in [11] a single nonsimple bound state was dealt with by coalescing two distinct simple bound states into one, and they illustrated this by a concrete example. As pointed out by Olmedilla [10], Zakharov and Shabat's "limiting process gives the appropriate value ... but their final result for the potential is mistaken." The error in the example by Zakharov and Shabat certainly does not diminish the importance of their paper, but rather illustrates the point that dealing with nonsimple bound state poles was not an easy procedure. It seems as if [10] was one of the few references in which a systematic method has been sought to determine the time evolution of the norming constants corresponding to nonsimple bound states. In [10] formulas were found for a bound state with multiplicities two and three, but Olmedilla added, "in an actual calculation it is very complex to exceed four or five." Using the computer algebra system REDUCE [10] he was able to reach a multiplicity of nine, but his formulas were too complicated to generalize to a bound state of any multiplicity.

The goal of this article is to present a complete generalization of the inverse scattering transform with bound states of any multiplicity by providing a compact formula for the time evolution of the associated norming constants. The time evolution of the bound-state norming constant associated with a simple pole $\lambda_j$ is known from [11] to be

$$c_j(t) = c_j(0)e^{-4i\lambda_j^2 t}. \tag{1.3}$$

Using matrix exponentials it is possible to obtain a straightforward generalization of (1.3). As seen in (4.9), such a generalization amounts to replacing the scalar $\lambda_j$ by the matrix $A_j$ defined in (4.8) and replacing the single bound-state norming constant $c_j$ in (1.3) by the row vector $\begin{bmatrix} c_{j(n_j-1)} & \dots & c_{j0} \end{bmatrix}$ consisting of the set of all norming constants associated with $\lambda_j$. At the same time we generalize the dependency constants. Although these generalizations appear straightforward, the proofs nevertheless are nontrivial. The work presented here was done in the author's thesis [7] and is related to the thesis of Francesco Demontis [8], where dependency constants were not studied.

This article is organized as follows. In Section 2 we set the notation and briefly review the transmission and reflection coefficients as well as the dependency constants and norming constants for the Zakharov-Shabat system from [11]. In Section 3 we first obtain (3.1) to define the dependency constants, which relate to the two types of Jost solutions and their derivatives. We then derive the time evolution of the dependency constants, which are given in (3.13). We do this using the properties of the Lax pair associated with the NLS equation from [1]. In Section 4 we relate the norming constants to the dependency constants. We then present the time evolution of the norming constants with the help of the governing evolution equation for the dependency constants.

## 2. Preliminaries

The inverse scattering transform associates the NLS equation with the Zakharov-Shabat system (1.1), where $\lambda$ is the complex-valued spectral parameter, $u(x,t)$ is a complex-valued potential integrable in $x \in \mathbb{R}$, and the asterisk denotes complex conjugation. Then for each $\lambda$ in the closed upper half complex plane there exist the so-called Jost solutions $\varphi(\lambda,x,t)$ and $\psi(\lambda,x,t)$ which satisfy the asymptotic conditions

$$\begin{gathered}\varphi(\lambda,x,t) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty, \\ \psi(\lambda,x,t) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to +\infty.\end{gathered}$$

Under the assumption that there do not exist discontinuities of $\varphi(\lambda,x,t)$ and $\psi(\lambda,x,t)$ at real values of $\lambda$, we obtain the transmission coefficient $T(\lambda,t)$, the right reflection coefficient $R(\lambda,t)$, and the left reflection coefficient $L(\lambda,t)$, all three continuous in $(\lambda,t) \in \mathbb{R}^2$, such that the Jost solutions satisfy the asymptotic conditions

$$\begin{gathered}\varphi(\lambda,x,t) = \begin{bmatrix} \frac{1}{T(\lambda,t)} e^{-i\lambda x} \\ \frac{R(\lambda,t)}{T(\lambda,t)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to +\infty, \\ \psi(\lambda,x,t) = \begin{bmatrix} \frac{L(\lambda,t)}{T(\lambda,t)} e^{-i\lambda x} \\ \frac{1}{T(\lambda,t)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to -\infty,\end{gathered}$$

at the other end of the real line. It appears that the transmission coefficient $T(\lambda,t)$ is time independent but, generally, the reflection coefficients are not. Besides scattering solutions to (1.1) there are also square-integrable solutions pertaining to $\lambda$ in the upper half complex plane which are known as bound-state solutions. Bound-state solutions occur at the poles of the transmission coefficient $T(\lambda)$ in the upper half complex plane; in these poles (and only there) the Jost solutions are proportional. We denote these finitely many bound-state poles as $\lambda_j$, where $j = 1,2,\dots,N$. We use $n_j$ to denote the algebraic multiplicity of $\lambda_j$. Since the two Jost solutions $\varphi(\lambda,x,t)$ and $\psi(\lambda,x,t)$ are linearly dependent at the bound state corresponding to $\lambda_j$ with algebraic multiplicity $n_j$, for $s = 0,1,\dots,n_j-1$ we have the constants $\gamma_{js}(t)$ in (3.1) which are called the dependency constants associated with the bound state $\lambda_j$.

It is known that when $u(x,t)$ is integrable in $x$ for each fixed $t$, the Jost solutions $\varphi(\lambda,x,t)$ and $\psi(\lambda,x,t)$ are analytic in $\lambda$ in the upper half complex plane. Hence, for each fixed $x$ and $t$, those two Jost solutions have Taylor series expansions at any point in the upper half of the complex-$\lambda$ plane [1].

The transmission coefficient $T(\lambda,t)$ can be expanded around $\lambda_j$ as in (4.3), where $t_{js}$ are complex-valued parameters and the $o(1)$ indicates the regular part of the expansion. We will refer to the parameters $t_{js}$ as generalized residues. It is known from [11] that $T(\lambda,t)$ is independent of $t$ and therefore the coefficients $t_{js}$ are also independent of $t$. From this point on we will refer to the transmission coefficient simply as $T(\lambda)$.

Associated with each bound state $\lambda_j$ having multiplicity $n_j$ is a corresponding set of bound-state norming constants $\{c_{js}\}_{s=0}^{n_j-1}$. It is appropriate, as in [6], to define $c_{js}$ in terms of the generalized residues $t_{js}$ and the dependency constants $\gamma_{js}(t)$ as in (4.1) so that the kernel of the corresponding Marchenko equation can be written in a simple form as

$$\Omega(y,t) = \int_{-\infty}^{\infty} R(\lambda,t)e^{i\lambda y}\frac{d\lambda}{2\pi} + \sum_{j=1}^{N}\sum_{m=0}^{n_j-1} c_{jm}(t)\frac{y^m}{m!}e^{i\lambda_j y}. \tag{2.1}$$

It will be shown that when the Marchenko integral equation is solved the result will give the exact solution to the NLS equation for bound states with multiplicity.

## 3. Dependency constants

In this article we derive the evolution of $\gamma_{js}(t)$ and $c_{js}(t)$ from their initial values $\gamma_{js}(0)$ and $c_{js}(0)$, respectively. We begin with the formulation of the dependency constants $\gamma_{js}(t)$ associated with a bound state, $\lambda_j$, with higher multiplicity, consider the case of $T(\lambda)$ with a set of poles $\{\lambda_j\}_{j=1}^{N}$ each with multiplicity $n_j$.

**Theorem 3.1.** *The dependency constants $\gamma_{jk}(t)$ associated with a bound state $\lambda_j$ with multiplicity $n_j$ can be expressed for $l = 0,1,\dots,n_j-1$ as*

$$\varphi^{(l)}(\lambda_j,x,t) = \sum_{k=0}^{l}\binom{l}{k}\gamma_{j(l-k)}(t)\psi^{(k)}(\lambda_j,x,t). \tag{3.1}$$

*where the superscript denotes the $\lambda$-derivative and the constants $\gamma_{jk}(t)$ are the bound-state dependency constants.*

*Proof.* For the case of simple poles it is known from [11] that

$$\varphi(\lambda,x,t) = \gamma_{j0}(t)\psi(\lambda,x,t). \tag{3.2}$$

The goal is to have a similar representation for $\varphi(\lambda_j,x,t)$ and its derivatives. To find this we consider the expansion of $1/T(\lambda)$ about $\lambda_j$. By letting

$$a(\lambda) = 1/T(\lambda)$$

we obtain

$$a(\lambda) = a(\lambda_j) + \dot{a}(\lambda_j)(\lambda-\lambda_j) + \frac{\ddot{a}(\lambda_j)}{2!}(\lambda-\lambda_j)^2 + \dots + \frac{a^{(n_j)}(\lambda_j)}{(n_j)!}(\lambda-\lambda_j)^{n_j} + \dots,$$

where the overdot denotes the $\lambda$-derivative. Since $\lambda_j$ is a pole of $T(\lambda)$ with multiplicity $n_j$, it is a zero of $a(\lambda)$ with multiplicity $n_j$. Therefore

$$a(\lambda_j) = \dot{a}(\lambda_j) = \dots = a^{(n_j-1)}(\lambda_j) = 0. \tag{3.3}$$

It is known [11] that $a(\lambda)$ can be written as the Wronskian of the Jost solutions, Let us define the Wronskian of the solutions $\zeta = \begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}$ and $\eta = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$ as

$$[\zeta;\eta] := \zeta_1\eta_2 - \eta_1\zeta_2$$

That is,

$$a(\lambda) = [\varphi(\lambda,x,t);\psi(\lambda,x,t)] \tag{3.4}$$

where $\varphi(\lambda, x, t)$ and $\psi(\lambda, x, t)$ are the Jost solutions. Consider the expansions of $\varphi(\lambda, x, t)$ and $\psi(\lambda, x, t)$ about the value $\lambda = \lambda_j$ and we have

$$\varphi(\lambda, x, t) = \varphi(\lambda_j, x, t) + \dot{\varphi}(\lambda_j, x, t)(\lambda - \lambda_j) + \frac{\ddot{\varphi}(\lambda_j, x, t)}{2!}(\lambda - \lambda_j)^2 + \dots,$$
$$\psi(\lambda, x, t) = \psi(\lambda_j, x, t) + \dot{\psi}(\lambda_j, x, t)(\lambda - \lambda_j) + \frac{\ddot{\psi}(\lambda_j, x, t)}{2!}(\lambda - \lambda_j)^2 + \dots.$$

For notational simplicity we use $\varphi(\lambda, x, t)$ as $\varphi$ and $\psi(\lambda, x, t)$ as $\psi$. Similarly $\varphi(\lambda_j, x, t)$ will be denoted as $\varphi(\lambda_j)$ and $\psi(\lambda_j, x, t)$ as $\psi(\lambda_j)$. We will let

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

and hence the subscript relates to the appropriate component of the Jost solution. We then have

$$\begin{aligned} a(\lambda) = \Big[&\varphi_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\psi_1(\lambda_j)] + (\lambda - \lambda_j)[\varphi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) \\ &- \varphi_2(\lambda_j)\dot{\psi}_1(\lambda_j) - \psi_1(\lambda_j)\dot{\varphi}(\lambda_j)] + (\lambda - \lambda_j)^2[\varphi_1(\lambda_j)\frac{\ddot{\psi}(\lambda_j)}{2!} + \psi_2(\lambda_j)\frac{\ddot{\varphi}_1(\lambda_j)}{2!} \\ &+ \dot{\varphi}_1(\lambda_j)\dot{\psi}_2(\lambda_j) - \psi_1(\lambda_j)\frac{\ddot{\varphi}(\lambda_j)}{2!} - \varphi_2(\lambda_j)\frac{\ddot{\psi}_1(\lambda_j)}{2!} - \dot{\psi}_1(\lambda_j)\dot{\varphi}(\lambda_j)] + \dots. \end{aligned}$$

Using this equality and (3.3), we obtain

$$0 = a(\lambda_j) = \varphi_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\psi_1(\lambda_j).$$

This can be expressed as

$$\begin{vmatrix} \varphi_1(\lambda_j) & \psi_2(\lambda_j) \\ \varphi_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = 0.$$

Hence we find that $\varphi(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent, therefore there exists a value $\gamma_{j0}(t)$ such that

$$\varphi(\lambda_j) = \gamma_{j0}(t)\psi(\lambda_j). \tag{3.5}$$

Similarly from (3.3),

$$0 = \dot{a}(\lambda_j) = \varphi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\dot{\psi}_1(\lambda_j) - \psi_1(\lambda_j)\dot{\varphi}(\lambda_j). \tag{3.6}$$

By substituting (3.5) into (3.6) we find

$$\gamma_{j0}(t)\psi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) - \gamma_{j0}(t)\psi_2(\lambda_j)\dot{\psi}_1(\lambda_j)\psi_1(\lambda_j)\dot{\varphi}_1(\lambda_j) = 0.$$

This can be written as

$$\begin{vmatrix} \dot{\varphi}_1(\lambda_j) - \gamma_{j0}(t)\dot{\psi}_1(\lambda_j) & \psi_1(\lambda_j) \\ \dot{\varphi}_2(\lambda_j) - \gamma_{j0}(t)\dot{\psi}_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = 0. \tag{3.7}$$

From (3.7) we see that $\dot{\varphi}(\lambda_j) - \gamma_{j0}(t)\dot{\psi}(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent. Therefore there exists a value $\gamma_{j1}(t)$ such that

$$\dot{\varphi}(\lambda_j) - \gamma_{j0}(t)\psi(\lambda_j) = \gamma_{j1}(t)\psi(\lambda_j). \tag{3.8}$$

We rewrite (3.8) as

$$\dot{\varphi}(\lambda_j) = \gamma_{j0}(t)\psi(\lambda_j) + \gamma_{j1}(t)\psi(\lambda_j). \tag{3.9}$$

Again from (3.3),

$$\begin{aligned} 0=\ddot a(\lambda_j)&=\varphi_1(\lambda_j)\frac{\ddot\psi(\lambda_j)}{2!}+\psi_2(\lambda_j)\frac{\ddot\varphi_1(\lambda_j)}{2!}+\dot\varphi_1(\lambda_j)\dot\psi_2(\lambda_j)\\ &\quad-\psi_1(\lambda_j)\frac{\ddot\varphi(\lambda_j)}{2!}-\varphi_2(\lambda_j)\frac{\ddot\psi_1(\lambda_j)}{2!}-\dot\psi_1(\lambda_j)\dot\varphi(\lambda_j). \end{aligned}\tag{3.10}$$

By substituting (3.5) and (3.9) into (3.10) we have

$$\begin{aligned} 0&=\gamma_{j0}(t)\psi_1(\lambda_j)\frac{\ddot\psi_2(\lambda_j)}{2!}+\psi_2(\lambda_j)\frac{\ddot\varphi_1(\lambda_j)}{2!}+\gamma_{j0}(t)\psi_1(\lambda_j)\dot\psi_2(\lambda_j)\\ &\quad+\gamma_{j2}(t)\psi_1(\lambda_j)\dot\psi_2(\lambda_j)-\psi_2\frac{\ddot\varphi_2(\lambda_j)}{2!}-\gamma_{j0}(t)\psi_2(\lambda_j)\frac{\ddot\psi_1(\lambda_j)}{2!}\\ &\quad-\dot\psi(\lambda_j)\gamma_{j0}(t)\dot\psi_2(\lambda_j)-\dot\psi_1(\lambda_j)\gamma_{j1}(t)\psi_2(\lambda_j). \end{aligned}$$

This can be written in terms of a determinant as

$$\begin{vmatrix}\ddot\varphi_1(\lambda_j)-2\gamma_{j1}(t)\dot\psi_1(\lambda_j)-\gamma_{j0}(t)\ddot\psi_1(\lambda_j) & \psi_1(\lambda_j)\\ \ddot\varphi_2(\lambda_j)-2\gamma_{j1}(t)\dot\psi_2(\lambda_j)-\gamma_{j0}(t)\ddot\psi_2(\lambda_j) & \psi_2(\lambda_j)\end{vmatrix}=0. \tag{3.11}$$

From (3.11) we can see that $\ddot\varphi(\lambda_j)-2\gamma_{j1}(t)\dot\psi(\lambda_j)-\gamma_{j0}(t)\ddot\psi(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent. Therefore, there exists a value, $\gamma_{j2}(t)$, such that

$$\ddot\varphi(\lambda_j)-2\gamma_{j1}(t)\dot\psi(\lambda_j)-\gamma_{j0}(t)\ddot\psi(\lambda_j)=\gamma_{j2}(t)\psi(\lambda_j),$$

or equivalently

$$\ddot\varphi(\lambda_j)=\gamma_{j0}(t)\ddot\psi(\lambda_j)+2\gamma_{j1}(t)\dot\psi(\lambda_j)+\gamma_{j2}(t)\psi(\lambda_j).$$

Now we must prove that there is a similar representation for $\varphi^{(n_j-1)}(\lambda_j)$. For $n=0,1,\dots,n_j-1$ from (3.4) we will use the identities

$$0=a^{(n)}(\lambda_j)=\sum_{l=0}^{n}\binom{n}{l}[\varphi^{(n-l)}(\lambda_j),\psi^{(l)}(\lambda_j)]. \tag{3.12}$$

Using the following identities for $k=0,1,\dots n_j-1$

$$\begin{gathered} [\psi^{(k)};\psi^{(k)}]=0,\\ [\psi^{(k)};\psi^{(j)}]=-[\psi^{(j)};\psi^{(k)}],\\ \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(n-p)!(p-k)!}=\frac{n!}{(p-k)!(n-p+k)!}\frac{(n-p+k)!}{(n-p)!k!}, \end{gathered}$$

where we recall that $[*;*]$ is used to denote the Wronskian and $\psi^{(k)}$ denotes $\frac{\partial^k}{\partial\lambda^k}\psi(\lambda,x,t)$, we see that all but a select few of the terms in (3.12) are zero. Therefore

$$0=\Big[\varphi^{(n)}-\binom{n}{1}\gamma_{j,n-1}(t)\psi^{(1)}-\binom{n}{2}\gamma_{j,n-2}(t)\psi^{(2)}-\dots-\binom{n}{n}\gamma_{j0}(t)\psi^{(n)};\psi^{(0)}\Big].$$

Hence there exists a value $\gamma_{j,n+1}(t)$ such that

$$\varphi^{(n)}-\binom{n}{1}\gamma_{j,n-1}(t)\psi^{(1)}-\binom{n}{2}\gamma_{j,n-2}(t)\psi^{(2)}-\dots-\binom{n}{n}\gamma_{j0}(t)\psi^{(n)}=\gamma_{j,n+1}(t)\psi^{(0)},$$

or equivalently

$$\varphi^{(n)}=\binom{n}{1}\gamma_{j,n-1}(t)\psi^{(1)}+\binom{n}{2}\gamma_{j,n-2}(t)\psi^{(2)}+\dots+\binom{n}{n}\gamma_{j0}(t)\psi^{(n)}+\gamma_{j,n+1}(t)\psi^{(0)}.$$

Since this is true for any $n = 1, \dots, n_j - 1$, there exists a value $\gamma_{j,n_j}(t)$ such that

$$\varphi^{(n_j-1)} = \binom{n_j-1}{1}\gamma_{j,n_j-2}(t)\psi^{(1)} + \binom{n_j-1}{2}\gamma_{j,n_j-3}(t)\psi^{(2)} + \dots + \binom{n_j-1}{n_j-1}\gamma_{j0}(t)\psi^{(n_j-1)} + \gamma_{j,n_j}(t)\psi^{(0)}.$$

Thus (3.1) holds. $\square$

Recall that we are seeking to obtain the time dependence of the dependency constants, $\gamma_{js}(t)$, and norming constants, $c_{js}$ for $j = 1, \dots, N$ and $s = 0, \dots, n_j - 1$. Using the representation of the dependency constants in (3.1), the time evolution of the dependency constants $\gamma_{js}(t)$ is described in the next theorem.

**Theorem 3.2.** *The time evolution of the dependency constants $\gamma_{jk}(t)$ is governed by the recursive formula*

$$\frac{d}{dt}(\gamma_{jk}(t)) = 4i\lambda_j^2\gamma_{jk}(t) + 8ik\lambda_j\gamma_{j(k-1)}(t) + 4ik(k-1)\gamma_{j(k-2)}(t), \tag{3.13}$$

*for $j = 1, 2, \dots, N$ and $k = 0, 1, \dots, n_j - 1$.*

*Proof.* Using induction we will show (3.13) to be true. Let us use the subscript $t$ to denote the time-derivative. First consider (3.2) evaluated at $\lambda_j$, we find

$$\varphi(\lambda_j, x, t) = \gamma_{j0}(t)\psi(\lambda_j, x, t). \tag{3.14}$$

Taking the time derivative of (3.14) we have

$$\varphi_t(\lambda, x, t) = \gamma_{j0}(t)\psi_t(\lambda, x, t) + (\gamma_{j0}(t))_t\psi(\lambda, x, t). \tag{3.15}$$

The Lax pair $\mathcal{L}$ and $\mathcal{A}$ associated with (1.1) is given by [11],

$$\mathcal{L} := i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\partial_x - i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & q\\ -q^* & 0\end{bmatrix},$$

$$\mathcal{A} := 2i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\partial_x^2 + \begin{bmatrix}0 & -2iq\\ -2iq^* & 0\end{bmatrix}\partial_x + \begin{bmatrix}iqq^* & -q_x\\ -iq_x^* & -iqq^*\end{bmatrix}.$$

It is known from [1] that the time evolution of the two Jost solutions is given by

$$\begin{aligned}\psi_t - \mathcal{A}\psi &= -2i\lambda^2\psi,\\ \varphi_t - \mathcal{A}\varphi &= 2i\lambda^2\varphi.\end{aligned} \tag{3.16}$$

Now using both equations in (3.16) we can write (3.15) as

$$2i\lambda_j^2\gamma_{j0}(t)\psi + \mathcal{A}\gamma_{j0}(t)\psi = \gamma_{j0}(t)\mathcal{A}\psi - 2i\lambda_j^2\gamma_{j0}(t)\psi + (\gamma_{j0}(t))_t\psi,$$

which is equivalent to

$$(\gamma_{j0}(t))_t = 4i\lambda_j^2\gamma_{j0}(t). \tag{3.17}$$

The $\lambda$-derivative of $\varphi$ can be expressed as

$$\dot\varphi = \gamma_{j0}(t)\dot\psi + \gamma_{j1}(t)\psi. \tag{3.18}$$

Consider the time derivative of (3.18), we find

$$\dot\varphi_t = \gamma_{j0}(t)\dot\psi_t + (\gamma_{j0}(t))_t\dot\psi + \gamma_{j1}(t)\psi_t + (\gamma_{j0}(t))_t\psi. \tag{3.19}$$

The $\lambda$-derivative of the identities in (3.16) can now be written as

$$\begin{aligned}\dot\varphi_t - \mathcal{A}\dot\varphi &= 2i\lambda^2\dot\varphi + 4i\lambda\varphi,\\ \dot\psi_t - \mathcal{A}\dot\psi &= -2i\lambda^2\dot\psi - 4i\lambda\psi.\end{aligned} \tag{3.20}$$

By using (3.17), (3.19), and (3.20), we can write (3.18) as

$$(\gamma_{j1}(t))_t = 4i\lambda_j^2\gamma_{j1}(t) + 8i\lambda_j\gamma_{j0}(t). \tag{3.21}$$

Performing similar calculations for higher derivatives, from (3.1) we have

$$\ddot{\varphi} = \gamma_{j0}(t)\ddot{\psi} + 2\gamma_{j1}(t)\dot{\psi} + \gamma_{j2}(t)\psi. \tag{3.22}$$

Taking the $\lambda$-derivative of (3.20),

$$\begin{aligned} \ddot{\varphi}_t - \mathcal{A}\ddot{\varphi} &= 2i\lambda^2\ddot{\varphi} + 8i\lambda\dot{\varphi} + 4i\varphi, \\ \ddot{\psi}_t - \mathcal{A}\ddot{\psi} &= -2i\lambda^2\ddot{\psi} - 8i\lambda\dot{\psi} - 4i\psi. \end{aligned} \tag{3.23}$$

We can write (3.22) using (3.17), (3.19), (3.20), (3.21), and (3.23) as

$$(\gamma_{j2}(t))_t = 4i\lambda_j^2\gamma_{j2}(t) + 16i\lambda_j\gamma_{j1}(t) + 8i\gamma_{j0}(t). \tag{3.24}$$

Then (3.24) satisfies the recursive formula, which is the base case for induction. Assume the recursive formula in (3.13) is true for $k$ and considering higher order $\lambda$-derivatives of (3.16), for $m = 0, 1, \dots, n_j - 1$,

$$\begin{aligned} \varphi_t^{(m)} - \mathcal{A}\varphi^{(m)} &= 2i\lambda^2\varphi^{(m)} + \binom{m}{1}4i\lambda\varphi^{(m-1)} + \binom{m}{2}4i\varphi^{(m-2)}, \\ \psi_t^{(m)} - \mathcal{A}\psi^{(m)} &= -2i\lambda^2\psi^{(m)} - \binom{m}{1}4i\lambda\psi^{(m-1)} - \binom{m}{2}4i\psi^{(m-2)}. \end{aligned} \tag{3.25}$$

Take the time derivative of (3.1),

$$\varphi_t^{(l)}(\lambda_j, x, t) = \sum_{k=0}^{l}\binom{l}{k}\gamma_{j(l-k)}(t)\psi_t^{(k)}(\lambda_j, x, t) + \sum_{k=0}^{l}\binom{l}{k}(\gamma_{j(l-k)}(t))_t\psi^{(k)}(\lambda_j, x, t). \tag{3.26}$$

From (3.1) and (3.26),

$$\begin{aligned} &\varphi_t^{(l)}(\lambda_j) - \mathcal{A}\varphi^{(l)}(\lambda_j) \\ &= \sum_{k=0}^{l}\binom{l}{k}\gamma_{j(l-k)}(t)\psi_t^{(k)}(\lambda_j) + \sum_{k=0}^{l}\binom{l}{k}(\gamma_{j(l-k)}(t))_t\psi^{(k)}(\lambda_j) \\ &\quad - \mathcal{A}\sum_{k=0}^{l}\binom{l}{k}\gamma_{j(l-k)}(t)\psi^{(k)}(\lambda_j). \end{aligned} \tag{3.27}$$

By substituting (3.1) and (3.25) in (3.27) we have

$$\begin{aligned} &\sum_{k=0}^{l}\binom{l}{k}(\gamma_{j(l-k)}(t))_t\psi^{(k)}(\lambda_j) \\ &= 4i\lambda_j^2\sum_{k=0}^{l}\binom{l}{k}\gamma_{j(l-k)}(t)\psi^{(k)}(\lambda_j) \ + 4i\lambda_j\sum_{k=0}^{l-1}\frac{2l!}{k!(l-k-1)!}\gamma_{j(l-k)}(t)\psi^{(k-1)}(\lambda_j) \\ &\quad + 2i\sum_{k=0}^{l-2}\frac{2l!}{k!(l-k-2)!}\gamma_{j(l-k)}(t)\psi^{(k-2)}(\lambda_j). \end{aligned}$$

Multiplying both sides of (3.13) by the term $\binom{k+1}{s}\psi^{(k+1-s)}(\lambda_j)$ and applying the summation over $s = 0, \dots, k$ we obtain

$$\begin{aligned}
&\sum_{s=0}^{k}\binom{k+1}{s}(\gamma_{jk}(t))_t\psi^{(k+1-s)}(\lambda_j) \\
&= \sum_{s=0}^{k}\binom{k+1}{s}4i\lambda_j^2\gamma_{jk}(t)\psi^{(k+1-s)}(\lambda_j) \\
&\quad + \sum_{s=0}^{k}\binom{k+1}{s}8ik\lambda_j\gamma_{j(k-1)}(t)\psi^{(k+1-s)}(\lambda_j) \\
&\quad + \sum_{s=0}^{k}\binom{k+1}{s}4ik(k-1)\gamma_{j(k-2)}(t)\psi^{(k+1-s)}(\lambda_j).
\end{aligned} \tag{3.28}$$

Thus, (3.13) can then be rewritten as

$$\begin{aligned}
&(\gamma_{j(k+1)}(t))_t\psi^{(0)}(\lambda_j) + \sum_{s=0}^{k}\binom{k+1}{s}(\gamma_{js}(t))_t\psi^{(k+1-s)}(\lambda_j) \\
&= 4i\lambda_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}(t)\psi^{(k-s+1)}(\lambda_j) \\
&\quad + 4i\lambda_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}(t)\psi^{(k-s)}(\lambda_j) \\
&\quad + 2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\gamma_{js}(t)\psi^{(k-s-1)}(\lambda_j).
\end{aligned} \tag{3.29}$$

By substituting (3.28) into the second term of (3.29) and rearranging terms, we obtain

$$\begin{aligned}
(\gamma_{j(k+1)}(t))_t\psi^{(0)}(\lambda_j) &= 4i\lambda_j^2\Big[\gamma_{j(k+1)}(t)\psi^{(0)}(\lambda_j)\Big] + 8i\lambda_j\Big[(k+1)\gamma_{jk}(t)\psi^{(0)}(\lambda_j)\Big] \\
&\quad + 4i\Big[(k+1)k\gamma_{j(k-1)}(t)\psi^{(0)}(\lambda_j)\Big].
\end{aligned}$$

Therefore,

$$(\gamma_{j(k+1)}(t))_t = 4i\lambda_j^2\gamma_{j(k+1)}(t) + 8i\lambda_j(k+1)\gamma_{jk}(t) + 4i(k+1)k\gamma_{j(k-1)}(t).$$

Hence the theorem is proved. $\square$

Using the recursive formula for the time derivatives for $\gamma_{jk}(t)$ we can now consider the norming constants $c_{jk}$ and the time evolution of these constants in the next section.

## 4. Bound-state norming constants

This section begins by considering the representation of the bound-state norming constants, $\{c_{js}\}_{s=0}^{n_j-1}$, in the presence of poles $\lambda_j$ with multiplicity $n_j$. Then as in the previous section, the time evolution of these constants is considered.

**Theorem 4.1.** *The bound state norming constants associated with the bound state at $\lambda_j$ of multiplicity $n_j$ for $m = 0, 1, \dots, n_j - 1$ and $j = 1, 2, \dots, N$ can be written as*

$$c_{jm} := \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} i^m t_{j(m+k+1)}. \tag{4.1}$$

*Proof.* We look at the formulation of the norming constants associated with bound state poles of higher multiplicity with respect to the Marchenko integral equation since they arise in the formulation of the kernel of this integral equation. Considering the Marchenko integral equation for simple poles in [1, section 1.3], the only term that will be affected by the presence of bound state poles of higher multiplicity is

$$\int_{-\infty}^{\infty} [T(\lambda) - 1]\, \varphi(\lambda, x) e^{i\lambda y} \frac{d\lambda}{2\pi}. \tag{4.2}$$

In the case of simple bound state poles this term can be found by evaluating the residue of $T(\lambda)$ at $\lambda_j$. However when a bound state pole $\lambda_j$ has multiplicity greater than one consider the expansions of the three functions that make up this integral term about $\lambda_j$.

$$\begin{aligned} T(\lambda) - 1 &= \frac{t_{jn_j}}{(\lambda - \lambda_j)^{n_j}} + \frac{t_{j(n_j-1)}}{(\lambda - \lambda_j)^{n_j-1}} + \dots + \frac{t_{j1}}{(\lambda - \lambda_j)} + O(1), \\ e^{i\lambda y} &= e^{i\lambda_j y}\Big(1 + iy(\lambda - \lambda_j) + \dots + \frac{(iy)^{n_j-1}}{(n_j-1)!}(\lambda - \lambda_j)^{n_j-1} + O(1)\Big), \\ \varphi(\lambda, x, t) &= \varphi(\lambda_j, x, t) + \dot\varphi(\lambda_j, x, t)(\lambda - \lambda_j) + \dots \\ &\quad + \frac{\varphi^{(n_j-1)}(\lambda_j, x, t)}{(n_j-1)!}(\lambda - \lambda_j)^{n_j-1} + O(1). \end{aligned} \tag{4.3}$$

When the terms of (4.3) are multiplied and integrated, only the coefficient of $\frac{1}{(\lambda-\lambda_j)}$ will be nonzero. Then (4.2) can be written as

$$\begin{aligned} &\int_{-\infty}^{\infty} [T(\lambda) - 1]\varphi(\lambda, x, t) e^{i\lambda y} \frac{d\lambda}{2\pi} \\ &= \sum_{j=1}^{N} ie^{i\lambda_j y}\Big[\varphi(\lambda_j, x, t)\Big(t_{j1} + \dots + t_{jn_j}\frac{(iy)^{n_j-1}}{(n_j-1)!}\Big) \\ &\quad + \frac{\dot\varphi(\lambda_j, x, t)}{1!}\Big(t_{j2} + \dots + t_{jn_j}\frac{(iy)^{n_j-2}}{(n_j-2)!}\Big) + \dots + \frac{\varphi^{(n_j-1)}(\lambda_j, x, t)}{(n_j-1)!} t_{jn_j}\Big]. \end{aligned}$$

Rewriting this in matrix form we have

$$\int_{-\infty}^{\infty} [T(\lambda) - 1]\varphi(\lambda, x, t) e^{i\lambda y} \frac{d\lambda}{2\pi} = \sum_{j=1}^{N} ie^{i\lambda_j y} \Phi_j F_j T_j Y_j,$$

where the matrices $\Phi_j$, $F_j$, $T_j$, and $Y_j$ are defined as

$$\Phi_j := \begin{bmatrix} \varphi(\lambda_j) & \dot\varphi(\lambda_j) & \dots & \varphi^{(n_j-1)}(\lambda_j) \end{bmatrix},$$

$$F_j := \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{(n_j-1)!} \end{bmatrix},$$

$$T_j := \begin{bmatrix} \frac{t_{j1}}{0!} & \frac{t_{j2}}{1!} & \dots & \frac{t_{j(n_j-1)}}{(n_j-2)!} & \frac{t_{jn_j}}{(n_j-1)!} \\ \frac{t_{j2}}{0!} & \frac{t_{j3}}{1!} & \dots & \frac{t_{jn_j}}{(n_j-2)!} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{t_{jn_j}}{0!} & 0 & \dots & 0 & 0 \end{bmatrix}, \quad Y_j := \begin{bmatrix} 1 \\ (iy)^1 \\ \vdots \\ (iy)^{n_j-1} \end{bmatrix}.$$

We can now express (3.1) in matrix form as

$$\Phi_j = \Psi_j \Gamma_j,$$

where $\Psi_j$ and $\Gamma_j$ are defined as

$$\Psi_j := \begin{bmatrix} \psi(\lambda_j) & \dot{\psi}(\lambda_j) & \dots & \psi^{(n_j-1)}(\lambda_j) \end{bmatrix},$$

$$\Gamma_j := \begin{bmatrix} \gamma_{j0}(t) & \gamma_{ji}(t) & \dots & \gamma_{j,n_j-1}(t) \\ 0 & \binom{1}{1}\gamma_{j0}(t) & \dots & \binom{n_j-1}{1}\gamma_{j,n_j-2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & & \binom{n_j-1}{n_j-1}\gamma_{j0}(t) \end{bmatrix}. \tag{4.4}$$

Using (4.4), we can write (4.2) as

$$\int_{-\infty}^{\infty} [T(\lambda) - 1]\varphi(\lambda, x, t) e^{i\lambda y} \frac{d\lambda}{2\pi} = \sum_{j=1}^{N} i e^{i\lambda_j y} \Psi_j \Gamma_j F_j T_j Y_j. \tag{4.5}$$

Recall the representation of the Jost solution $\psi(\lambda_j, x, t)$ (2.1),

$$\psi(\lambda_j, x, t) = \begin{bmatrix} 0 \\ e^{i\lambda_j x} \end{bmatrix} + \int_{-\infty}^{\infty} K(x, z, t) e^{i\lambda_j z}\, dz, \tag{4.6}$$

where

$$K(x, z, t) = \int_{-\infty}^{\infty} \Big(\psi(\lambda, x, t) - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\Big) e^{-i\lambda y} \frac{d\lambda}{2\pi}.$$

Taking the $\lambda$-derivative of (4.5) for any $n$, we get

$$\psi^{(n)}(\lambda_j, x, t) = \begin{bmatrix} 0 \\ (ix)^n e^{i\lambda_j x} \end{bmatrix} + \int_{-\infty}^{\infty} K(x, z, t) e^{i\lambda_j z} (iz)^n\, dz.$$

Now $\Psi_j$ can be expressed as

$$\Psi_j = \begin{bmatrix} 0 \\ e^{i\lambda_j x} \end{bmatrix} X_j + \int_{-\infty}^{\infty} K(x, z, t) e^{i\lambda_j z} Z_j\, dz, \tag{4.7}$$

where the matrices $X_j$ and $Z_j$ are defined as

$$X_j := \begin{bmatrix} 1 & ix & \cdots & (ix)^{n_j-1} \end{bmatrix}, \quad Z_j := \begin{bmatrix} 1 & iz & \cdots & (iz)^{n_j-1} \end{bmatrix}.$$

Then (4.2) can be written as

$$\int_{-\infty}^{\infty} [T(\lambda) - 1]\varphi(\lambda, x) e^{i\lambda y} \frac{d\lambda}{2\pi}$$

$$= \sum_{j=1}^{N} ie^{i\lambda_j(x+y)} X_j \Gamma_j F_j T_j Y_j \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sum_{j=1}^{N} i \int_{-\infty}^{\infty} K(x,z) e^{i\lambda_j(x+z)} Z_j \Gamma_j F_j T_j Y_j \, dz.$$

Define the matrix, $\Gamma_j F_j T_j = \Lambda_j$, where

$$\Lambda_j := \begin{bmatrix} \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+1)} & \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+2)} & \dots & \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+n_j)} \\ \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+2)} & \binom{2}{1} \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+3)} & \dots & 0 \\ \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+3)} & \binom{3}{2} \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+4)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}(t)}{k!} t_{j(k+n_j)} & 0 & \dots & 0 \end{bmatrix}.$$

With the help of the entries from the matrix $\Lambda_j$, we have the norming constants $c_{jm}$ associated with bound state pole $\lambda_j$ of multiplicity $n_j$ as in (4.1). $\square$

Next we consider the relationship of the norming constants for any $t$ and the initial value of the norming constants at $t = 0$ that are dependent on the matrix $A_j$ which is defined as

$$A_j := \begin{bmatrix} -\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -\lambda_j \end{bmatrix}. \tag{4.8}$$

**Theorem 4.2.** *The time evolution of the norming constants $c_{jk}$ is*

$$\begin{aligned} &\begin{bmatrix} c_{j(n_j-1)}(t) & c_{j(n_j-2)}(t) & \dots & c_{j0}(t) \end{bmatrix} \\ &= \begin{bmatrix} c_{j(n_j-1)}(0) & c_{j(n_j-2)}(0) & \dots & c_{j0}(0) \end{bmatrix} e^{-4iA_j^2 t}. \end{aligned} \tag{4.9}$$

*Proof.* First rewrite (4.1) in matrix form as

$$C_j = \Gamma_j P_j, \tag{4.10}$$

where

$$C_j := \begin{bmatrix} c_{j(n_j-1)} & \dots & c_{j0} \end{bmatrix}, \quad \Gamma_j := \begin{bmatrix} \gamma_{j(n_j-1)}(t) & \dots & \gamma_{j0}(t) \end{bmatrix},$$

$$P_j := \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{t_{jn_j} i^{-1}}{(n_j-1)!} \\ 0 & 0 & \dots & \frac{t_{jn_j} i^0}{(n_j-2)!} & \frac{t_{j(n_j-1)} i^{-1}}{(n_j-2)!} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \frac{t_{jn_j} i^{n_j-3}}{1!} & \dots & \frac{t_{j3} i^0}{1!} & \frac{t_{j2} i^{-1}}{1!} \\ \frac{t_{jn_j} i^{n_j-2}}{0!} & \frac{t_{j(n_j-1)} i^{n_j-3}}{0!} & \dots & \frac{t_{j2} i^0}{0!} & \frac{t_{j1} i^{-1}}{0!} \end{bmatrix} \tag{4.11}$$

and the entries of $C_j$ and $\Gamma_j$ are dependent on $t$. Since $t_{jk}$ is independent of time for all $k$ values the time derivative of (4.10) becomes

$$(C_j)_t = (\Gamma_j)_t P_j. \tag{4.12}$$

Therefore,

$$\begin{aligned} &\begin{bmatrix} (c_{j(n_j-1)})_t & (c_{j(n_j-2)})_t & \dots & (c_{j0})_t \end{bmatrix} \\ &= \begin{bmatrix} (\gamma_{j(n_j-1)}(t))_t & (\gamma_{j(n_j-2)}(t))_t & \dots & (\gamma_{j0}(t))_t \end{bmatrix} P_j. \end{aligned}$$

Rewriting (4.1) in matrix form as

$$(\Gamma_j)_t = \Gamma_j M_j, \tag{4.13}$$

where the matrix $M_j$ is defined as

$$M_j := \begin{bmatrix} 4i\lambda_j^2 & 0 & 0 & \dots & 0 & 0 \\ (n_j-1)8i\lambda_j & 4i\lambda_j & 0 & \dots & 0 & 0 \\ (n_j-2)(n_j-1)4i & (n_j-2)8i\lambda_j & 4i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4i\lambda_j & 0 \\ 0 & 0 & 0 & \dots & 8i\lambda_j & 4i\lambda_j \end{bmatrix}$$

and substituting (4.13) into (4.12) we obtain

$$(C_j)_t = \Gamma_j M_j P_j. \tag{4.14}$$

Since a permutation of columns converts $P_j$ into an upper triangular matrix with nonzero diagonal entries, it is invertible. Then (4.14) is equivalent to

$$(C_j)_t = \Gamma_j P_j P_j^{-1} M_j P_j. \tag{4.15}$$

Using (4.10), (4.15) can be written as

$$(C_j)_t = C_j P_j^{-1} M_j P_j.$$

Solving the above differential equation gives the expression

$$C_j(t) = C_j(0)e^{P_j^{-1} M_j P_j t}. \tag{4.16}$$

Upon calculation we see that

$$P_j^{-1} M_j P_j = -4iA_j^2. \tag{4.17}$$

Hence (4.9) follows from (4.17) in (4.16). □

## 5. Conclusion

The goal of this paper is to describe a method to solve the NLS equation with bound states with multiplicity using the inverse scattering transform. Recall that both Zakharov and Shabat in [11] and Olmedilla [10] attempted to deal with non-simple bound state poles, but were not able to obtain the results for arbitrary multiplicity. In this paper we established expressions for the dependency constants $\gamma_{js}(t)$ and the norming constants $c_{js}(t)$ and the time evolution of each given their initial values in Theorems 3.2 and 4.2. This result has been used [4, 5], however the proof is never provided in the literature.

The newly established expressions now enable us to create the Marchenko integral equation which will give an exact solution to the NLS equation. We begin with the formulation of the Marchenko integral equation for a set of $N$ bound state poles, each of arbitrary order $n_j$. Using the representation for the time evolution of the norming constants in (4.9) to construct the Marchenko integral equation kernel associated with bound state poles of higher multiplicities results in a compact, easily usable form. From [6] the inner most summation in (2.1) can be written as

$$\sum_{m=0}^{n_j-1} c_{jm}(t)\frac{y^m}{m!}e^{i\lambda_j y} = -i\int_{-\infty}^{\infty} C_j(\lambda - iA_j)^{-1}B_j e^{i\lambda y}\frac{d\lambda}{2\pi}, \tag{5.1}$$

with the matrix $A_j$ defined as in (4.8), the matrix $C_j$ defined as in (4.11), and the matrix $B_j$ defined as

$$B_j := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Defining the matrices $A$, $B$, and $C$ as

$$A := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_N \end{bmatrix}, \quad B := [\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix},$$
$$C := \begin{bmatrix} C_1 & C_2 & \dots & C_N \end{bmatrix},$$

we can then rewrite the kernel of the integral equation as

$$\Omega(y,t) = \int_{-\infty}^{\infty} R(\lambda,t) e^{i\lambda y} \frac{d\lambda}{2\pi} + Ce^{-Ay-4iA^2t}B.$$

An immediate consequence from [6] is that the exact solution to the NLS equation for the $n$-soliton solution with bound states $\lambda_j$ with multiplicity $n_j$ can be written as

$$u(x,t) = -2B^\dagger e^{-A^\dagger x} G(x,t)^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger,$$

where the dagger denotes the matrix adjoint and $G(x,t)$ is defined as

$$\begin{aligned} G(x,t) := I + \Big( \int_x^\infty e^{-A^\dagger s + 4i(A^\dagger)^2 t} C^\dagger C e^{-Ax} G(x,t)^{-1} e^{-Ax+4i(A^\dagger)^2 t} dz \Big) \\ \times \Big( \int_x^\infty e^{-Az} BB^\dagger e^{-A^\dagger x} dz \Big). \end{aligned}$$

## References


[1] M. J. Ablowitz, H. Segur; *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.

[2] M. J. Ablowitz, B. Prinari, A. D. Trubatch; *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Note Series vol. 302, Cambridge University Press, Cambridge, 2004.

[3] M. J. Ablowitz; *Nonlinear dispersive waves*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2011.

[4] T. Aktosun, T. Busse, F. Demontis, C.van der Mee; *Exact solutions to the nonlinear Schrödinger equation*, In: Topics in Operator Theory, Vol. 2: Systems and mathematical physics (J. A. Ball, V. Bolotnikov, J. W. Helton, L. Rodman, and I. M. Spitkovsky (eds.)), pp. 1-12, Birkhäuser, Basel, 2010.

[5] T.Aktosun, T. Busse, F. Demontis, C. van der Mee; *Symmetries for exact solutions to the nonlinear Schrödinger equation*, J. Phys. A **43** (2010), 025202.

[6] T. Aktosun, F. Demontis, C. van der Mee; *Exact solutions to the focusing nonlinear Schrödinger equation*, Inverse Problems **23** (2007), 2171–2195.

[7] T. Busse; *Generalized inverse scattering transform for the nonlinear Schrödinger equation*, Ph.D. Thesis, University of Texas at Arlington, 2008.

[8] F. Demontis; *Direct and inverse scattering of the matrix Zakharov-Shabat system*, Lambert Academic Publishing, Saarbrücken Germany (2012); also: Ph.D. thesis, University of Cagliari, Italy, 2007.

[9] A. Hasegawa and F. Tappert; *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion*, Applied Physics Letters **23**(3) (1973), 142–144.

[10] E. Olmedilla; *Multiple pole solutions of the nonlinear Schrödinger equation*, Phys. D. **25** (1987), 330–346.

[11] V. E. Zakharov, A. B. Shabat; *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP, **34**, 62–69 (1972).

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