Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 149, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMMETRIC SUPERLINEAR PROBLEMS UNDER STRONG RESONANCE CONDITIONS

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Abstract. We study the existence and multiplicity of solutions of the problem

$$\begin{aligned} -\Delta u &= -\lambda_1 u^- + g(x,u), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$), u^- denotes the negative part of $u\colon \Omega \to \mathbb{R}$, λ_1 is the first eigenvalue of the N-dimensional Laplacian with Dirichlet boundary conditions in Ω, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(x,0) = 0$ for all $x \in \Omega$. We assume that the nonlinearity $g(x,s)$ has a strong resonant behavior for large negative values of s and is superlinear, but subcritical, for large positive values of s. Because of the lack of compactness in this kind of problem, we establish conditions under which the associated energy functional satisfies the Palais-Smale condition. We prove the existence of three nontrivial solutions of problem (1) as a consequence of Ekeland's Variational Principle and a variant of the mountain pass theorem due to Pucci and Serrin [14].

1. Introduction

Let Ω denote a bounded, connected, open subset of $\mathbb{R}^N$, for $N \geqslant 2$, with smooth boundary $\partial\Omega$. We are interested in the existence and multiplicity of solutions of the semilinear elliptic boundary value problem (BVP):

$$\begin{aligned} -\Delta u &= -\lambda_1 u^- + g(x,u), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where u^- denotes the negative part of $u\colon \Omega \to \mathbb{R}$, λ_1 is the first eigenvalue of the N-dimensional Laplacian with Dirichlet boundary conditions in Ω, and $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and its primitive

$$G(x,s) = \int_0^s g(x,\xi)d\xi, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in \mathbb{R}, \tag{1.2}$$

satisfy the following conditions:

(A1) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g(x,0) = 0$ for all $x \in \overline{\Omega}$.
(A2) $\lim_{s\to-\infty} g(x,s) = 0$, uniformly for a.e. $x \in \Omega$.

2010 *Mathematics Subject Classification.* 35J20.
Key words and phrases. Strong resonance; Palais-Smale condition; Ekeland's principle.

Submitted February 19, 2017. Published June 23, 2017.

(A3) There exists a constant σ such that $1 \leqslant \sigma < (N+2)/(N-2)$ for $N \geqslant 3$, or $1 \leqslant \sigma < \infty$ for $N = 2$, and

$$\lim_{s\to+\infty} \frac{g(x,s)}{s^\sigma} = 0,$$

uniformly for a.e. $x \in \Omega$.

(A4) There are constants $\mu > \max\left\{2, \frac{2N\sigma}{N+2}\right\}$ and $s_o > 0$ such that

$$0 < \mu G(x,s) \leqslant s g(x,s), \quad \text{for } s \geqslant s_o \text{ and } x \in \overline{\Omega}.$$

(A5) $\lim_{s\to-\infty} G(x,s) \equiv G_{-\infty}$, uniformly in x, where $G_{-\infty} \in \mathbb{R}$.

Writing

$$q(x,s) = -\lambda_1 s^- + g(x,s), \quad \text{for } (x,s) \in \overline{\Omega}\times\mathbb{R}, \tag{1.3}$$

we assume further that

(A6) $q \in C^1(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ and $q(x,0) = 0$; and

(A7) $\frac{\partial q}{\partial s}(x,0) = a$, for all $x \in \overline{\Omega}$, where $a > \lambda_1$.

We determine conditions under which the BVP in (1.1) has nontrivial solutions. By a solution of (1.1) we mean a weak solution; i.e, a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u\cdot\nabla v\,dx + \lambda_1\int_\Omega u^- v\,dx - \int_\Omega g(x,u)v\,dx = 0, \quad \text{for all } v \in H_0^1(\Omega), \tag{1.4}$$

where $H_0^1(\Omega)$ is the Sobolev space obtained through completion of $C_c^\infty(\Omega)$ with respect to the metric induced by the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \quad \text{for all } u \in H_0^1(\Omega).$$

The weak solutions of (1.1) are the critical points of the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u^-)^2\,dx - \int_\Omega G(x,u(x))\,dx, \tag{1.5}$$

for $u \in H_0^1(\Omega)$. Indeed, the functional J given in (1.5) is in $C^1(H_0^1(\Omega),\mathbb{R})$ with Fréchet derivative at every $u \in H_0^1(\Omega)$ given by

$$J'(u)v = \int_\Omega \nabla u\cdot\nabla v\,dx + \lambda_1\int_\Omega u^- v\,dx - \int_\Omega g(x,u)v\,dx, \quad \text{for all } v \in H_0^1(\Omega). \tag{1.6}$$

Thus, comparing (1.4) with (1.6), we see that critical points of J are weak solutions of (1.1).

In many problems, the following condition, known as the Palais-Smale condition, is usually needed to prove the existence of critical points of a functional.

Definition 1.1 (Palais-Smale Sequence)**.** Let $J \in C^1(X,\mathbb{R})$, where X is a Banach space with norm $\|\cdot\|$. A sequence (u_m) in X satisfying

$$J(u_m) \to c \quad \text{and} \quad \|J'(u_m)\| \to 0 \quad \text{as } m \to \infty,$$

is said to be a Palais-Smale sequence for J at c.

If (u_m) is a sequence satisfying

(i) $|J(u_m)| \leqslant M$ for all $m = 1, 2, 3, \dots$ and some $M > 0$;

(ii) $\|J'(u_m)\| \to 0$ as $m \to \infty$,

we say that (u_m) is a Palais-Smale sequence for J.

Definition 1.2 (Palais-Smale Condition)**.** A functional $J \in C^1(X, \mathbb{R})$, where X is a Banach space with norm $\|\cdot\|$, is said to satisfy the the Palais-Smale condition at c, denoted $(\mathrm{PS})_c$, if every Palais-Smale sequence for J at c has a convergent subsequence. In particular, if J has a Palais-Smale sequence at c, and J satisfies the $(\mathrm{PS})_c$ condition, then c is a critical value of J.

We say that J satisfies the (PS) condition if every (PS) sequence for J has a convergent subsequence.

It follows from condition (A2) and (1.3) that

$$\lim_{s\to-\infty} \frac{q(x,s)}{s} = \lambda_1, \quad \text{for all } x \in \overline{\Omega}. \tag{1.7}$$

The condition in (1.7) makes the BVP in (1.1) into a problem at resonance. Existence for problems at resonance is sometimes obtained by imposing a Landesman-Lazer type condition on the nonlinearity. The authors of this article obtained existence and multiplicity for the BVP (1.1) in [15] for the case in which

$$\lim_{s\to-\infty} g(x,s) = g_{-\infty}(x)$$

exists for all $x \in \overline{\Omega}$, and

$$\int_\Omega g_{-\infty}(x)\varphi_1(x)\,dx > 0, \tag{1.8}$$

where φ_1 is an eigenfunction of the N-dimensional Laplacian over Ω corresponding to the eigenvalue λ_1, with $\varphi_1(x) > 0$ for all $x \in \Omega$. In the case in which the Landesman-Lazer condition (1.8) holds, the authors were able to prove that the functional J defined in (1.5) satisfies the (PS) condition.

Note that the assumption in (A2) prevents condition (1.8) from holding true. So that, a Landesman-Lazer type condition does not hold for the problem at hand. As a consequence, we will not be able to prove that the functional J satisfies the (PS) condition. We will, however, be able to show that J satisfies the $(\mathrm{PS})_c$ condition at values of c that are not in an exceptional set, Λ. In the case in which conditions (A1)–(A5) hold true, we will prove in the next section that the functional $J \in C^1(H_0^1(\Omega), \mathbb{R})$ given in (1.5) satisfies the $(\mathrm{PS})_c$ condition provided that

$$c \neq -G_{-\infty}|\Omega|, \tag{1.9}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω; thus, the exceptional set in this case is

$$\Lambda = \{-G_{-\infty}|\Omega|\}. \tag{1.10}$$

It is not hard to see that the functional J defined in (1.5) does not satisfy the $(\mathrm{PS})_c$ condition at $c = c_{-\infty} \equiv -G_{-\infty}|\Omega|$. Indeed, the sequence of functions (u_m) given by

$$u_m = -m\varphi_1, \quad \text{for } m = 1, 2, 3, \dots,$$

is a $(\mathrm{PS})_{c_{-\infty}}$ sequence, as a consequence of assumptions (A2) and (A5). However,

$$\|u_{m+1} - u_m\| = \|\varphi_1\|, \quad \text{for all } m = 1, 2, 3, \dots;$$

so that (u_m) has no convergent subsequence.

This lack of compactness is typical of problems at *strong resonance*. The term strong resonance refers to the situation described by the assumptions in (A2) and (A5) and was introduced by Bartolo, Benci and Fortunato in [3]. In [3], the authors consider problems similar to (1.1) in which g is bounded, and in which the exceptional set is a singleton as in (1.10); more precisely, the authors of [3] consider the class of BVPs of the form

$$\begin{aligned} -\Delta u &= q_k(u), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

where

$$q_k(s) = \lambda_k s - g(s),$$

with λ_k an eigenvalue of the Laplacian, and $g\colon \mathbb{R} \to \mathbb{R}$ a bounded, continuous function with

$$\lim_{|s|\to\infty} sg(s) = 0.$$

Furthermore, the authors of [3] assume that the function

$$G(s) = \int_{-\infty}^{s} g(\xi)\, d\xi$$

is defined for all $s \in \mathbb{R}$, and satisfies $G(s) \geqslant 0$ for all $s \in \mathbb{R}$, and

$$\lim_{s\to\infty} G(s) = 0.$$

The authors of [3] proved existence of weak solutions of BVP (1.11) by introducing a compactness condition (Condition (C)) that replaces the (PS) condition, and using the new condition to prove a variant of the deformation lemma.

In [7] and [8], Costa and Silva are able to obtain some of the existence and multiplicity results of Bartolo, Benci and Fortunato [3] by establishing that the associated functional J satisfies the $(\mathrm{PS})_c$ condition for values of c that are not in an exceptional set. More recently, Hirano, Li and Wang [12] have used Morse Theory to obtain multiplicity results for this type of problems with strong resonance. In [12], the exceptional set, Λ, consists of a finite number of values. They are able to compute critical groups around the values in Λ; that is, critical groups are computed at values where the (PS) condition fails. These critical groups are then incorporated into a new version of the Morse inequality, which allowed the authors of [12] to obtain multiplicity results.

In all the articles cited so far, the nonlinearity g is assumed to be bounded. In the present work, we relax that assumption by allowing $g(x, s)$ to grow superlinearly, but subcritically, in s, for positive values of s (see (A3) and (A4)), while $g(x, s)$ is bounded for negative values of s (see (A2)).

For additional information on problems at strong resonance in the context of critical point theory, the reader is referred to the works of Arcoya and Costa [2], Li [13], and Chang and Liu [6], and the bibliographies found in those papers.

After establishing that the functional J defined in (1.5) satisfies the $(\mathrm{PS})_c$ condition for $c \neq -G_{-\infty}|\Omega|$ in Section 2, under assumptions (A1)–(A6), we then proceed to show in Section 3 that J has a local minimizer distinct from 0, provided that (A5) holds with $G_{-\infty} \leqslant 0$, and (A7) also holds. In subsequent sections, we introduce an additional condition on the nonlinearity that will allow us to prove the existence of more critical point of J. In particular, we will assume the following:

(A8) there exists $s_1 > 0$ such that $g(x, s_1) = 0$ for all $x \in \Omega$.

In Section 4, we prove that if, in addition to (A1)–(A5), with $G_{-\infty} \leqslant 0$, (A6) and (A7), we also assume (A8), then J has a second local minimizer distinct from 0. Finally, in Section 5, we prove the existence of a third nontrivial critical point of J

by means of a variant of the mountain-pass theorem proved by Pucci and Serrin in [14].

2. Proof of the Palais-Smale Condition

In this section we prove that the functional J defined in (1.5), where g and its primitive G satisfy the conditions in (A1)–(A5), satisfies the $(\mathrm{PS})_c$ condition provided that $c \neq -G_{-\infty}|\Omega|$.

Proposition 2.1. *Assume that g and G satisfy* (A1)–(A5)*, and define J as in* (1.5)*. Then, J satisfies the $(PS)_c$ for $c \neq -G_{-\infty}|\Omega|$.*

Proof. Assume that $c \neq -G_{-\infty}|\Omega|$ and let (u_m) be a sequence in $H_0^1(\Omega)$ satisfying

$$J(u_m) \to c \quad \text{and} \quad \|J'(u_m)\| \to 0 \quad \text{as } m \to \infty. \tag{2.1}$$

Thus, according to (1.5) and (1.6),

$$\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx - \int_\Omega G(x,u_m(x))\,dx \to c, \quad \text{as } m \to \infty, \tag{2.2}$$

and

$$\Big|\int_\Omega \nabla u_m \cdot \nabla v\,dx + \lambda_1 \int_\Omega u_m^- v\,dx - \int_\Omega g(x,u_m)v\,dx\Big| \leqslant \varepsilon_m \|v\|, \tag{2.3}$$

for all m and all $v \in H_0^1(\Omega)$, where (ε_m) is a sequence of positive numbers that tends to 0 as $m \to \infty$.

We will show that (u_m) has a subsequence that converges in $H_0^1(\Omega)$. It follows from (A2) that there exists $s_1 > 0$ such that

$$-1 \leqslant g(x,s) \leqslant 1, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}. \tag{2.4}$$

Consequently,

$$-|s| \leqslant sg(x,s) \leqslant |s|, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}, \tag{2.5}$$

and

$$-C_1 - |s| \leqslant G(x,s) \leqslant C_1 + |s|, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}, \tag{2.6}$$

for some positive constant C_1. Combining (2.5) and (2.6), and using the continuity of g, we can find a positive constant C_2 such that

$$-C_2 - 3|s| \leqslant sg(x,s) - 2G(x,s) \leqslant C_2 + 3|s|, \quad \text{for } s \leqslant 0, \text{ and all } x \in \overline{\Omega}. \tag{2.7}$$

Similarly, we obtain from (A3) that there exists a positive constant C_3 such that

$$|g(x,s)| \leqslant C_3 + |s|^\sigma, \quad \text{for } s \geqslant 0 \text{ and } x \in \overline{\Omega}. \tag{2.8}$$

Finally, we obtain from (A4) that there exist positive constants C_4 and C_5 such that

$$G(x,s) \geqslant C_4 s^\mu - C_5, \quad \text{for } s \geqslant 0 \text{ and } x \in \overline{\Omega}. \tag{2.9}$$

Now, it follows from (2.2) that there exists a positive constant C_6 such that

$$\Big|\int_\Omega |\nabla u_m|^2\,dx - \lambda_1 \int_\Omega (u_m^-)^2\,dx - \int_\Omega 2G(x,u_m(x))\,dx\Big| \leqslant C_6, \quad \text{for all } m. \tag{2.10}$$

Taking $v = u_m$ in (2.3), we obtain

$$\Big|\int_\Omega |\nabla u_m|^2\,dx - \lambda_1 \int_\Omega (u_m^-)^2\,dx - \int_\Omega g(x,u_m(x))u_m(x)\,dx\Big| \leqslant \varepsilon_m \|u_m\|, \tag{2.11}$$

for all m.

Combining (2.10) and (2.11) we then obtain that

$$\Big|\int_\Omega [g(x,u_m(x))u_m(x) - 2G(x,u_m(x))]\,dx\Big| \leqslant C_6 + \varepsilon_m\|u_m\|, \quad \text{for all } m. \tag{2.12}$$

Next, define the sets

$$\Omega_m^- = \{x\in\Omega \mid u_m(x)<0\};\quad \Omega_m^+ = \{x\in\Omega \mid u_m(x)\geqslant 0\};$$
$$\Omega_m^o = \{x\in\Omega \mid 0\leqslant u_m(x)\leqslant s_o\};\quad \Omega_m^{s_o} = \{x\in\Omega \mid u_m(x)> s_o\}.$$

Then, using the estimate in (2.7),

$$\Big|\int_{\Omega_m^-} [g(x,u_m(x))u_m(x) - 2G(x,u_m(x))]\,dx\Big| \leqslant C + 3\|u_m^-\|_{L^1}, \quad \text{for all } m. \tag{2.13}$$

Note: From this point on in this paper, the symbol C will be used to represent any positive constant. Thus, C might represent different constants in various estimates, even within the same inequality.

It follows from (2.12) and (2.13) that

$$\Big|\int_{\Omega_m^+} [g(x,u_m(x))u_m(x) - 2G(x,u_m(x))]\,dx\Big| \leqslant C + \varepsilon_m\|u_m\| + 3\|u_m^-\|_{L^1}, \tag{2.14}$$

for all m.

Using the continuity of g and G we deduce the existence of a positive constant C such that

$$\int_{\Omega_m^o} [g(x,u_m(x))u_m(x) - 2G(x,u_m(x))]\,dx \leqslant C, \quad \text{for all } m. \tag{2.15}$$

On the other hand, using (A4) we obtain that

$$(\mu-2)\int_{\Omega_m^{s_o}} G(x,u_m(x))\,dx \leqslant \int_{\Omega_m^{s_o}} [g(x,u_m(x))u_m(x) - 2G(x,u_m(x))]\,dx,$$

for all m; so that, using this estimate in conjunction with (2.15), (2.13), (2.12) and the assumption that $\mu>2$, we obtain that

$$\int_{\Omega_m^{s_o}} G(x,u_m(x))\,dx \leqslant C + 3\|u_m^-\|_{L^1} + \varepsilon_m\|u_m\|, \quad \text{for all } m. \tag{2.16}$$

Noting that $\Omega_m^+ = \Omega_m^o \cup \Omega_m^{s_o}$, we obtain from (2.16) that

$$\Big|\int_{\Omega_m^+} G(x,u_m(x))\,dx\Big| \leqslant C + 3\|u_m^-\|_{L^1} + \varepsilon_m\|u_m\|, \quad \text{for all } m. \tag{2.17}$$

Next, we take $v=-u_m^-$ in (2.3) to obtain

$$\Big|\int_\Omega |\nabla u_m^-|^2\,dx - \lambda_1\int_\Omega (u_m^-)^2\,dx - \int_{\Omega_m^-} g(x,u_m)u_m\,dx\Big| \leqslant \varepsilon_m\|u_m^-\|, \tag{2.18}$$

We get from (2.5) and (2.6) that

$$\Big|\int_{\Omega_m^-} g(x,u_m(x))u_m(x)\,dx\Big| \leqslant C + \|u_m^-\|_{L^1}, \quad \text{for all } m, \tag{2.19}$$

and

$$\Big|\int_{\Omega_m^-} G(x,u_m(x))\,dx\Big| \leqslant C + \|u_m^-\|_{L^1}, \quad \text{for all } m. \tag{2.20}$$

Taking $v = u_m^+$ in (2.3) we then get

$$\Big|\int_\Omega |\nabla u_m^+|^2\,dx - \int_{\Omega_m^+} g(x,u_m)u_m\,dx\Big| \leqslant \varepsilon_m\|u_m^+\|, \quad \text{for all } m. \tag{2.21}$$

It follows from (2.21), (2.17) and (2.14) that

$$\int_\Omega |\nabla u_m^+|^2\,dx \leqslant C + \varepsilon_m\|u_m^+\| + 2\varepsilon_m\|u_m\| + 6\|u_m^-\|_{L^1}, \quad \text{for all } m,$$

which can be rewritten as

$$\int_\Omega |\nabla u_m^+|^2\,dx \leqslant C + 3\varepsilon_m\|u_m^+\| + 2\varepsilon_m\|u_m^-\| + 6\|u_m^-\|_{L^1}, \quad \text{for all } m, \tag{2.22}$$

by the triangle inequality.

We claim that, if (1.9) holds true, then (u_m^-) is bounded. We argue by contradiction. Suppose, passing to a subsequence if necessary, that

$$\|u_m^-\| \to \infty, \quad \text{as } m \to \infty. \tag{2.23}$$

It follows from (2.22), the Cauchy-Schwarz inequality, and the Poincaré inequality that

$$\|u_m^+\| \leqslant C + C\sqrt{1 + \|u_m^-\|}, \quad \text{for all } m. \tag{2.24}$$

Combining (2.24) and (2.23) we then deduce that

$$\lim_{m\to\infty} \frac{\|u_m^+\|}{\|u_m^-\|} = 0. \tag{2.25}$$

Next, define

$$v_m = -\frac{u_m^-}{\|u_m^-\|}, \quad \text{for all } m; \tag{2.26}$$

so that $\|v_m\| = 1$ for all m. We may therefore extract a subsequence (v_{m_k}) of (v_m) such that

$$v_{m_k} \rightharpoonup \overline{v} \text{ (weakly) as } k \to \infty, \tag{2.27}$$

for some $\overline{v} \in H_0^1(\Omega)$. We may also assume, passing to further subsequences if necessary, that

$$v_{m_k} \to \overline{v} \text{in } L^2(\Omega) \quad \text{as } k \to \infty, \tag{2.28}$$

$$v_{m_k}(x) \to \overline{v}(x) \quad \text{for a.e. } x \in \Omega \text{ as } k \to \infty. \tag{2.29}$$

Now, it follows from (2.3) and the fact that $u_{m_k} = u_{m_k}^+ - u_{m_k}^-$ that

$$\begin{aligned} &\Big| -\int_\Omega \nabla u_{m_k}^- \cdot \nabla v\,dx + \lambda_1 \int_\Omega u_{m_k}^- v\,dx\Big| \\ &\leqslant \varepsilon_{m_k}\|v\| + \int_\Omega |\nabla u_{m_k}^+ \cdot \nabla v|\,dx + \int_\Omega |g(x,u_{m_k}(x))||v|\,dx, \end{aligned} \tag{2.30}$$

for all k and all $v \in H_0^1(\Omega)$. Using the Cauchy-Schwarz inequality, we can rewrite the estimate in (2.30) as

$$\begin{aligned} &\Big|\int_\Omega \nabla u_{m_k}^- \cdot \nabla v\,dx - \lambda_1 \int_\Omega u_{m_k}^- v\,dx\Big| \\ &\leqslant \varepsilon_{m_k}\|v\| + \|u_{m_k}^+\|\|v\| + \int_\Omega |g(x,u_{m_k}(x))||v|\,dx, \end{aligned} \tag{2.31}$$

for all k and all $v \in H_0^1(\Omega)$.

Next, we estimate the last integral on the right-hand side of (2.31) by first writing

$$\begin{aligned} &\int_\Omega |g(x,u_{m_k}(x))||v|\,dx \\ &= \int_{\Omega^-_{m_k}} |g(x,u_{m_k}(x))||v|\,dx + \int_{\Omega^+_{m_k}} |g(x,u_{m_k}(x))||v|\,dx, \end{aligned} \tag{2.32}$$

for all k and all $v \in H^1_0(\Omega)$.

To estimate the first integral on the right-hand side of (2.32), we use (2.4), the Cauchy-Schwarz inequality, and the Poincaré inequality to get that

$$\Big|\int_{\Omega^-_{m_k}} |g(x,u_{m_k}(x))||v|\,dx\Big| \leqslant C\|v\|, \quad \text{for all } k \text{ and all } v \in H^1_0(\Omega). \tag{2.33}$$

To estimate the second integral in the right-hand side of (2.32), apply Hölder's inequality with $p = 2N/(N+2)$ and $q = 2N/(N-2)$ for $N \geqslant 3$. If $N = 2$, take $1 \leqslant p \leqslant \mu/\sigma$, which can be done because (A4) implies that $p\sigma < \mu$. Then,

$$\Big|\int_{\Omega^+_{m_k}} |g(x,u_{m_k})||v|\,dx\Big| \leqslant \Big(\int_{\Omega^+_{m_k}} |g(x,u_{m_k})|^p\Big)^{1/p}\Big(\int_\Omega |v|^q\Big)^{1/q};$$

so that, in view of (2.8) and the Sobolev embedding theorem,

$$\Big|\int_{\Omega^+_{m_k}} |g(x,u_{m_k})||v|\,dx\Big| \leqslant C\Big(\int_\Omega (C + |u^+_{m_k}|^\sigma)^p\,dx\Big)^{1/p}\|v\|, \tag{2.34}$$

for all k and all $v \in H^1_0(\Omega)$. We then obtain from (2.34) and Minkowski's inequality that

$$\Big|\int_{\Omega^+_{m_k}} |g(x,u_{m_k})||v|\,dx\Big| \leqslant C(1 + \|u^+_{m_k}\|^\sigma_{L^{p\sigma}})\|v\|, \tag{2.35}$$

for all k and all $v \in H^1_0(\Omega)$.

Combining (2.32) with the estimates in (2.33) and (2.35), we then obtain that

$$\int_\Omega |g(x,u_{m_k}(x))||v|\,dx \leqslant C(1 + \|u^+_{m_k}\|^\sigma_{L^{p\sigma}})\|v\|, \tag{2.36}$$

for all k and all $v \in H^1_0(\Omega)$.

Finally, combining the estimates in (2.31) and (2.36),

$$\Big|\int_\Omega \nabla u^-_{m_k}\cdot\nabla v\,dx - \lambda_1\int_\Omega u^-_{m_k} v\,dx\Big| \leqslant C(1 + \|u^+_{m_k}\| + \|u^+_{m_k}\|^\sigma_{L^{p\sigma}})\|v\|, \tag{2.37}$$

for all k and all $v \in H^1_0(\Omega)$.

Next, divide on both sides of (2.37) by $\|u^-_{m_k}\|$ and use (2.26) to get

$$\begin{aligned} &\Big|\int_\Omega \nabla v_{m_k}\cdot\nabla v\,dx - \lambda_1\int_\Omega v_{m_k} v\,dx\Big| \\ &\leqslant C\Big(\frac{1}{\|u^-_{m_k}\|} + \frac{\|u^+_{m_k}\|}{\|u^-_{m_k}\|} + \frac{\|u^+_{m_k}\|^\sigma_{L^{p\sigma}}}{\|u^-_{m_k}\|}\Big)\|v\|, \end{aligned} \tag{2.38}$$

for all k and all $v \in H^1_0(\Omega)$.

We will show next that

$$\lim_{k\to\infty} \frac{\|u^+_{m_k}\|^\sigma_{L^{p\sigma}}}{\|u^-_{m_k}\|} = 0. \tag{2.39}$$

Using the estimates in (2.9) and (2.17) we obtain

$$\int_\Omega (u_{m_k}^+)^\mu \, dx \leqslant C + C\|u_{m_k}^-\| + \varepsilon_{m_k}\|u_{m_k}\|, \quad \text{for all } k, \tag{2.40}$$

where we have also used the Cauchy-Schwarz and Poincaré inequalities. It then follows from (2.40) that

$$\|u_{m_k}^+\|_{L^\mu} \leqslant C(1 + \|u_{m_k}^-\|^{1/\mu} + \|u_{m_k}\|^{1/\mu}), \quad \text{for all } k. \tag{2.41}$$

Next, dividing on both sides of (2.41) by $\|u_{m_k}^-\|^{1/\sigma}$ and using the fact that

$$\|u_{m_k}\| \leqslant \|u_{m_k}^+\| + \|u_{m_k}^-\|$$

we obtain

$$\frac{\|u_{m_k}^+\|_{L^\mu}}{\|u_{m_k}^-\|^{1/\sigma}} \leqslant C\Big(\frac{1}{\|u_{m_k}^-\|^{1/\sigma}} + \frac{\|u_{m_k}^-\|^{1/\mu}}{\|u_{m_k}^-\|^{1/\sigma}} + \frac{\|u_{m_k}^+\|^{1/\mu}}{\|u_{m_k}^-\|^{1/\sigma}}\Big), \quad \text{for all } k,$$

which we can rewrite as

$$\frac{\|u_{m_k}^+\|_{L^\mu}}{\|u_{m_k}^-\|^{1/\sigma}} \leqslant C\Big(\frac{1}{\|u_{m_k}^-\|^{1/\sigma}} + \frac{1}{\|u_{m_k}^-\|^{1/\sigma - 1/\mu}} + \Big(\frac{\|u_{m_k}^+\|}{\|u_{m_k}^-\|}\Big)^{1/\mu} \frac{1}{\|u_{m_k}^-\|^{1/\sigma - 1/\mu}}\Big), \tag{2.42}$$

for all k. Now, in view of (A4) we see that $\mu > \sigma$; we then obtain from (2.23) and (2.25), in conjunction with (2.42), that

$$\lim_{k\to\infty} \frac{\|u_{m_k}^+\|_{L^\mu}}{\|u_{m_k}^-\|^{1/\sigma}} = 0. \tag{2.43}$$

Next, using the condition $\mu > p\sigma$ in (A4) to apply Hölder's inequality with $p_1 = \mu/p\sigma$ and p_2 its conjugate exponent we obtain

$$\|u_{m_k}^+\|_{L^{p\sigma}}^{p\sigma} = \int_\Omega (u_{m_k}^+)^{p\sigma} \, dx \leqslant \big(\int_\Omega (u_{m_k}^+)^\mu \, dx\big)^{p\sigma/\mu} |\Omega|^{1/p_2};$$

so that,

$$\|u_{m_k}^+\|_{L^{p\sigma}}^\sigma \leqslant C\|u_{m_k}^+\|_{L^\mu}^\sigma, \quad \text{for all } k,$$

and, dividing on both sides by $\|u_{m_k}^-\|$,

$$\frac{\|u_{m_k}^+\|_{L^{p\sigma}}^\sigma}{\|u_{m_k}^-\|} \leqslant C\Big(\frac{\|u_{m_k}^+\|_{L^\mu}}{\|u_{m_k}^-\|^{1/\sigma}}\Big)^\sigma, \quad \text{for all } k. \tag{2.44}$$

It then follows from (2.43) and (2.44) that

$$\lim_{k\to\infty} \frac{\|u_{m_k}^+\|_{L^{p\sigma}}^\sigma}{\|u_{m_k}^-\|} = 0,$$

which is (2.39).

Using (2.23), (2.25) and (2.39), we obtain from (2.38) that

$$\lim_{k\to\infty} \Big| \int_\Omega \nabla v_{m_k} \cdot \nabla v \, dx - \lambda_1 \int_\Omega v_{m_k} v \, dx \Big| = 0, \quad \text{for all } v \in H_0^1(\Omega). \tag{2.45}$$

It then follows from (2.26), (2.27) and (2.45) that

$$\int_\Omega \nabla \overline{v} \cdot \nabla v \, dx - \lambda_1 \int_\Omega \overline{v} v \, dx = 0, \quad \text{for all } v \in H_0^1(\Omega);$$

so that, $\overline{v}$ is a weak solution of the BVP

$$\begin{gathered} -\Delta u = \lambda_1 u, \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.46}$$

Now, it follows from (2.18) that

$$\Big|\int_\Omega |\nabla u_{m_k}^-|^2\,dx - \lambda_1\int_\Omega (u_{m_k}^-)^2\,dx\Big| \leqslant \varepsilon_{m_k}\|u_{m_k}^-\| + \Big|\int_{\Omega_{m_k}^-} g(x,u_{m_k})u_{m_k}\,dx\Big|, \tag{2.47}$$

for all k; where, according to (2.19),

$$\Big|\int_{\Omega_m^-} g(x,u_{m_k}(x))u_{m_k}(x)\,dx\Big| \leqslant C(1+\|u_{m_k}^-\|), \quad \text{for all } k. \tag{2.48}$$

Thus, combining (2.47) and (2.48),

$$\Big|\int_\Omega |\nabla u_{m_k}^-|^2\,dx - \lambda_1\int_\Omega (u_{m_k}^-)^2\,dx\Big| \leqslant C(1+\|u_{m_k}^-\|), \quad \text{for all } k. \tag{2.49}$$

Next, divide on both sides of (2.49) by $\|u_{m_k}^-\|^2$ and use (2.26) to obtain

$$\Big|1-\lambda_1\int_\Omega (v_{m_k})^2\,dx\Big| \leqslant C\Big(\frac{1}{\|u_{m_k}^-\|^2}+\frac{1}{\|u_{m_k}^-\|}\Big), \quad \text{for all } k. \tag{2.50}$$

It then follows from (2.23), (2.28) and (2.50) that

$$\lambda_1\int_\Omega (\overline{v})^2\,dx = 1,$$

from which we conclude that $\overline{v}$ is a nontrivial solution of BVP (2.46). Consequently, since $v_m \leqslant 0$ for all m, according to (2.26), we obtain that

$$\overline{v} = -\varphi_1, \tag{2.51}$$

where φ_1 is the eigenfunction for the BVP (2.46) corresponding to the eigenvalue λ_1 with

$$\varphi_1 > 0 \text{ in } \Omega \quad \text{and} \quad \|\varphi_1\| = 1.$$

We therefore obtain from (2.51) that

$$\overline{v} < 0 \quad \text{in } \Omega. \tag{2.52}$$

Furthermore,

$$\frac{\partial\overline{v}}{\partial\nu} > 0 \quad \text{on } \partial\Omega, \tag{2.53}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. We can then conclude from (2.25), (2.29), in conjunction with (2.52) and (2.53), that

$$u_{m_k}(x) \to -\infty \quad \text{for a.e. } x\in\Omega. \tag{2.54}$$

Thus, using (A5) and the Lebesgue dominated convergence theorem, we obtain from (2.54) that

$$\lim_{k\to\infty}\int_\Omega G(x,u_{m_k}(x))\,dx = G_{-\infty}|\Omega|. \tag{2.55}$$

It then follows from (2.55) and the first assertion in (2.1) that

$$\lim_{k\to\infty}\Big(\frac12\int_\Omega |\nabla u_{m_k}|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_{m_k}^-)^2\,dx\Big) = c + G_{-\infty}|\Omega|. \tag{2.56}$$

Next, we go back to the estimate in (2.3) and set $v = u_{m_k}^+$ to obtain

$$\Big|\int_\Omega |\nabla u_{m_k}^+|^2\,dx - \int_\Omega g(x,u_{m_k}^+)u_{m_k}^+\,dx\Big| \leqslant \varepsilon_{m_k}\|u_{m_k}^+\|, \quad \text{for all } k,$$

or, dividing by $\|u_{m_k}^+\|$,

$$\Big|\|u_{m_k}^+\| - \int_\Omega g(x, u_{m_k}^+)\frac{u_{m_k}^+}{\|u_{m_k}^+\|}\,dx\Big| \leqslant \varepsilon_{m_k}, \quad \text{for all } k. \tag{2.57}$$

Now, it follows from (2.54) that

$$u_{m_k}^+ \to 0 \text{ a.e. as } k \to \infty.$$

Therefore, it follows from the assumption that $g(x, 0) = 0$ in (A1), together with the Lebesgue dominated convergence theorem and the estimate in (2.57), that

$$\lim_{k\to\infty} \|u_{m_k}^+\| = 0. \tag{2.58}$$

Next, set $V = \operatorname{span}\{\varphi_1\}$ and $W = V^\perp$; so that, $H_0^1(\Omega) = V \oplus W$.

Write $u_{m_k}^- = v_k + w_k$, for each k, where $v_k \in V$ and $w_k \in W$. Once again, use the estimate in (2.3), this time with $v = w_k$, to obtain

$$\Big|\int_\Omega |\nabla w_k|^2\,dx - \lambda_1\int_\Omega w_k^2\,dx - \int_\Omega g(x, u_{m_k}(x))w_k\,dx\Big| \leqslant \varepsilon_{m_k}\|w_k\|, \tag{2.59}$$

for all k.

Now, since $w_k \in W$, we have that

$$\lambda_2\int_\Omega w_k^2\,dx \leqslant \int_\Omega |\nabla w_k|^2\,dx, \quad \text{for all } k, \tag{2.60}$$

where λ_2 denotes the second eigenvalue of the N-dimensional Laplacian over Ω with Dirichlet boundary conditions. Consequently,

$$\big(1 - \frac{\lambda_1}{\lambda_2}\big)\|w_k\|^2 \leqslant \int_\Omega |\nabla w_k|^2\,dx - \lambda_1\int_\Omega w_k^2\,dx, \quad \text{for all } k. \tag{2.61}$$

Thus, setting $\alpha = 1 - \frac{\lambda_1}{\lambda_2}$ in (2.61), we obtain from (2.61) and (2.59) that

$$\alpha\|w_k\|^2 \leqslant \varepsilon_{m_k}\|w_k\| + \Big|\int_\Omega g(x, u_{m_k}(x))w_k\,dx\Big|, \quad \text{for all } k, \tag{2.62}$$

where $\alpha > 0$.

Next, we divide on both sides of (2.62) by $\|w_k\|$ to get

$$\alpha\|w_k\| \leqslant \varepsilon_{m_k} + \Big|\int_\Omega g(x, u_{m_k}(x))\frac{w_k}{\|w_k\|}\,dx\Big|, \quad \text{for all } k. \tag{2.63}$$

Now, it follows from (2.63), (2.54), assumption (A2), and the Lebesgue dominated convergence theorem that

$$\lim_{k\to\infty} \|w_k\| = 0. \tag{2.64}$$

Next, we observe that

$$\int_\Omega |\nabla u_{m_k}|^2\,dx = \int_\Omega |\nabla u_{m_k}^+|^2\,dx + \int_\Omega |\nabla v_k|^2\,dx + \int_\Omega |\nabla w_k|^2\,dx, \quad \text{for all } k,$$

and

$$\int_\Omega (u_{m_k}^-)^2\,dx = \int_\Omega v_k^2\,dx + \int_\Omega w_k^2\,dx, \quad \text{for all } k;$$

consequently,

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla u_{m_k}|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_{m_k}^-)^2\,dx \\ &= \frac{1}{2}\|u_{m_k}^+\|^2 + \frac{1}{2}\int_\Omega |\nabla w_k|^2\,dx - \frac{\lambda_1}{2}\int_\Omega w_k^2\,dx, \end{aligned} \tag{2.65}$$

for all k, where we have used the fact that $v_k \in V$ for all k. It follows from (2.58), (2.64), (2.60) and (2.65) that

$$\lim_{k\to\infty} \Big(\frac{1}{2}\int_\Omega |\nabla u_{m_k}|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_{m_k}^-)^2\,dx\Big) = 0. \tag{2.66}$$

Combining (2.56) and (2.66) we obtain

$$G_{-\infty}|\Omega| + c = 0,$$

which is in direct contradiction with (1.9). We therefore conclude that (u_m^-) is bounded.

Since, (u_m^-) is bounded, it follows from (2.22) that (u_m^+) is also bounded. Consequently, (u_m) is bounded.

We will next proceed to show that (u_m) has a subsequence that converges strongly in $H_0^1(\Omega)$. To see why this is the case, first write the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ defined in (1.5) in the form

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \mathrm{Q}(x,u(x))\,dx, \quad \text{for all } u \in H_0^1(\Omega),$$

where

$$\mathrm{Q}(x,s) = \int_0^s q(x,\xi)\,d\xi, \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in \mathbb{R},$$

where q is as given in (1.3). It follows from (1.3) and the assumptions in (A2) and (A3), that $q(x,s)$ has subcritical growth in s, uniformly in $x \in \overline{\Omega}$; so that, the derivative map of J, $\nabla J\colon H_0^1(\Omega) \to H_0^1(\Omega)$, is of the form

$$\nabla J = I - \nabla \mathcal{Q}, \tag{2.67}$$

where $\nabla\mathcal{Q}\colon H_0^1(\Omega) \to H_0^1(\Omega)$, given by

$$\langle \nabla\mathcal{Q}(u), v\rangle = \int_\Omega g(x,u(x))v(x)\,dx, \quad \text{for } u,v \in H_0^1(\Omega),$$

is a compact operator.

Now, it follows from the second condition in (2.1) and (2.67) that

$$u_m - \nabla\mathcal{Q}(u_m) \to 0, \quad \text{as } m \to \infty. \tag{2.68}$$

Since we have already seen that the $(\mathrm{PS})_c$ sequence (u_m) is bounded, we can extract a subsequence, (u_{m_k}), of (u_m) that converges weakly to some $u \in H_0^1(\Omega)$. Therefore, given that the map $\nabla\mathcal{Q}\colon H_0^1(\Omega) \to H_0^1(\Omega)$ is compact, we have that

$$\lim_{k\to\infty} \nabla\mathcal{Q}(u_{m_k}) = \nabla\mathcal{Q}(u). \tag{2.69}$$

Thus, combining (2.68) and (2.69), we obtain that

$$\lim_{k\to\infty} u_{m_k} = \nabla\mathcal{Q}(u).$$

We have therefore shown that (u_m) has a subsequence that converges strongly in $H_0^1(\Omega)$, and the proof of the fact that J satisfies that $(\mathrm{PS})_c$ condition, provided that $c \neq -G_{-\infty}|\Omega|$, is now complete. $\square$

3. Existence of a local minimizer

Assume that g and G satisfy conditions (A1)–(A7) hold. In this section, we will use Ekeland's Variational Principle and a cutoff technique similar to that used by Chang, Li and Liu in [5] to prove the existence of a nontrivial solution of problem (1.1) for the case in which $G_{-\infty} \leqslant 0$ in (A5).

To do that, we first define $\widetilde{g} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ by

$$\widetilde{g}(x,s) = \begin{cases} g(x,s), & \text{for } s < 0, \\ 0, & \text{for } s \geq 0. \end{cases} \tag{3.1}$$

Define a corresponding functional $\widetilde{J} : H_0^1(\Omega) \to \mathbb{R}$ by

$$\widetilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u^-)^2\,dx - \int_\Omega \widetilde{G}(x,u)\,dx, \quad u \in H_0^1(\Omega), \tag{3.2}$$

where

$$\widetilde{G}(x,s) = \int_0^s \widetilde{g}(x,\xi)\,d\xi, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in \mathbb{R}. \tag{3.3}$$

Then, $\widetilde{J} \in C^1(H_0^1(\Omega), \mathbb{R})$. We claim that $\widetilde{J}$ is bounded below. In fact, by condition (A5) and (3.1), it follows that

$$|\widetilde{G}(x,s)| \leqslant M_o, \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in \mathbb{R}, \tag{3.4}$$

for some $M_o > 0$. Then, using (3.1) and (3.4), we can write

$$\begin{aligned} \widetilde{J}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_1}{2}\int_\Omega (u^-)^2 dx - \int_\Omega \widetilde{G}(x,u) dx, \\ &\geqslant \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 - \frac{\lambda_1}{2}\|u^-\|_{L^2} - M_o|\Omega|, \end{aligned} \tag{3.5}$$

for all $u \in H_0^1(\Omega)$. It then follows from (3.5) and the Poincaré inequality that

$$\widetilde{J}(u) \geqslant -M_o|\Omega|, \quad \text{for all } u \in H_0^1(\Omega);$$

so that $\widetilde{J}$ is bounded below. Thus, the infimum of $\widetilde{J}$ over $H_0^1(\Omega)$ exists; we can, therefore define

$$c_1 = \inf_{u \in H_0^1(\Omega)} \widetilde{J}(u). \tag{3.6}$$

Notice that, since $\widetilde{J}(0) = 0$, we must have $c_1 \leqslant 0$. In fact, we presently show that, if (A6) and (A7) hold, then

$$c_1 < 0. \tag{3.7}$$

To do this, first use (1.3) and (A7) to compute

$$\lim_{s \to 0^-} \frac{g(x,s)}{s} = a - \lambda_1;$$

so that

$$\lim_{s \to 0^-} \frac{g(x,s)}{s} > 0,$$

for all $x \in \overline{\Omega}$, by the assumption on a in (A7). Consequently, there exists $s_1 < 0$ such that

$$g(x,s) < 0, \quad \text{for } s_1 < s < 0,$$

and all $x \in \overline{\Omega}$. It then follows from the definition of $\widetilde{G}$ in (3.3) that

$$\widetilde{G}(x,s) > 0 \quad \text{for } s_1 < s < 0, \text{ and all } x \in \overline{\Omega}. \tag{3.8}$$

Next, let $\varepsilon > 0$ be small enough so that

$$s_1 < -\varepsilon\varphi_1(x) < 0, \quad \text{for all } x \in \Omega. \tag{3.9}$$

We then have that

$$\int_\Omega \widetilde{G}(x, -\varepsilon\varphi(x))\,dx > 0, \tag{3.10}$$

by (3.8) and (3.9). It then follows from the definition of $\widetilde{J}$ in (3.2) and (3.10) that

$$\widetilde{J}(-\varepsilon\varphi_1) = -\int_\Omega \widetilde{G}(x, -\varepsilon\varphi(x))\,dx < 0.$$

Consequently, in view of the definition of c_1 in (3.6), we obtain that $c_1 < 0$, which is (3.7).

We now use (3.6) and a consequence of Ekeland's Variational Principle (see [10, Theorem 4.4]) to obtain, for each positive integer m, $u_m \in H_0^1(\Omega)$ such that

$$\widetilde{J}(u_m) \leqslant \inf_{u\in H_0^1(\Omega)} \widetilde{J}(u) + \frac{1}{m}, \quad \text{for all } m, \tag{3.11}$$

and

$$\|\widetilde{J}'(u_m)\| \le \frac{1}{m}, \quad \text{for all } m;$$

we therefore obtain a $(\mathrm{PS})_c$ sequence for $c = c_1$. Consequently, if $\widetilde{J}$ happens to satisfy the $(\mathrm{PS})_c$ condition at $c = c_1$, we would conclude that c_1 is a critical value of $\widetilde{J}$. We will show shortly that this is the case if we assume that $G_{-\infty}$ given in (A5) satisfies

$$G_{-\infty} \leqslant 0. \tag{3.12}$$

We will first establish that $\widetilde{J}$ satisfies the $(PS)_c$ provided that $c \neq -G_{-\infty}|\Omega|$.

Proposition 3.1. *Assume that g and G satisfy* (A1), (A2) *and* (A5)*, and define $\widetilde{J}$ as in* (3.2)*, where $\widetilde{G}$ is given in* (3.3) *and* (3.1)*. Then, $\widetilde{J}$ satisfies the $(PS)_c$ condition for $c \neq -G_{-\infty}|\Omega|$.*

Proof. Assume that $c \neq -G_{-\infty}|\Omega|$ and let (u_m) be a $(PS)_c$ sequence for $\widetilde{J}$; that is,

$$\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx - \int_\Omega \widetilde{G}(x, u_m)\,dx \to c, \quad \text{as } m \to +\infty, \tag{3.13}$$

and,

$$\Big|\int_\Omega \nabla u_m \cdot \nabla\varphi\,dx + \lambda_1\int_\Omega u_m^-\varphi\,dx - \int_\Omega \widetilde{g}(x, u_m)\varphi\,dx\Big| \le \varepsilon_m\|\varphi\|, \tag{3.14}$$

for all m and all $\varphi \in H_0^1(\Omega)$, where (ε_m) is a sequence of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$. Write $u_m = u_m^+ - u_m^-$. We will show that (u_m^+) and (u_m^-) are bounded sequences.

First, let's see that (u_m^+) is bounded. Setting $\varphi = u_m^+$ in (3.14) we have

$$\Big|\int_\Omega |\nabla u_m^+|^2\,dx - \int_\Omega \widetilde{g}(x, u_m)u_m^+\,dx\Big| \le \varepsilon_m\|u_m^+\| \quad \text{for all } m. \tag{3.15}$$

By (3.1) and the assumption in (A2), it can be shown that $\widetilde{g}(x, u_m)$ is bounded for all $x \in \overline{\Omega}$. Then, using Hölder and Poincaré's inequalities, we obtain that

$$\Big|\int_\Omega \widetilde{g}(x, u_m)u_m^+\,dx\Big| \le C\|u_m^+\|, \tag{3.16}$$

for some constant $C > 0$. Then, from (3.15) and (3.16), we obtain that

$$\|u_m^+\|^2 \leq (C + \varepsilon_m)\,\|u_m^+\|, \quad \text{for all } m,$$

which shows that (u_m^+) is a bounded sequence.

Next, let us show that (u_m^-) is a bounded sequence. Suppose that this is not the case; then, passing to a subsequence if necessary, we may assume that

$$\|u_m^-\| \to \infty \quad \text{as } m \to \infty. \tag{3.17}$$

Define

$$v_m = -\frac{u_m^-}{\|u_m^-\|}, \quad \text{for all } m. \tag{3.18}$$

Then, since $\|v_m\| = 1$ for all m, passing to a further subsequences if necessary, we may assume that there is $v \in H_0^1(\Omega)$ such that

$$v_m \rightharpoonup v \quad \text{(weakly) in } H_0^1(\Omega), \text{ as } m \to \infty; \tag{3.19}$$

$$v_m \to v \quad \text{in } L^2(\Omega), \quad \text{as } m \to \infty; \tag{3.20}$$

$$v_m(x) \to v(x) \quad \text{for a.e. } x \text{in } \Omega, \text{ as } m \to \infty. \tag{3.21}$$

Now, writing $u_m = u_m^+ - u_m^-$ in (3.14) we have

$$\Big| \int_\Omega \nabla u_m^+ \cdot \nabla\varphi\,dx - \int_\Omega \nabla u_m^- \cdot \nabla\varphi\,dx + \lambda_1 \int_\Omega u_m^-\varphi\,dx - \int_\Omega \widetilde{g}(x,u_m)\varphi\,dx \Big| \leqslant \varepsilon_m\|\varphi\|,$$

for all $\varphi \in H_0^1(\Omega)$ and all m, from which we obtain that

$$\Big| -\int_\Omega \nabla u_m^- \cdot \nabla\varphi\,dx + \lambda_1 \int_\Omega u_m^-\varphi\,dx - \int_\Omega \widetilde{g}(x,u_m)\varphi\,dx \Big| \leqslant (\varepsilon_m + C\|u_m^+\|)\|\varphi\|, \tag{3.22}$$

for all $\varphi \in H_0^1(\Omega)$, all m, and some constant $C > 0$, by the Cauchy-Schwarz and Poincaré inequalities.

Now, we divide both sides of (3.22) by $\|u_m^-\|$ and use (3.18) to obtain

$$\Big| \int_\Omega \nabla v_m \cdot \nabla\varphi\,dx - \lambda_1 \int_\Omega v_m\varphi\,dx - \int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^-\|}\varphi\,dx \Big| \leqslant \Big(\frac{\varepsilon_m + C\|u_m^+\|}{\|u_m^-\|}\Big)\|\varphi\|, \tag{3.23}$$

for all $\varphi \in H_0^1(\Omega)$ and all m. Since $\widetilde{g}$ is bounded, by condition (A2) and (3.1), we obtain from (3.17) that

$$\lim_{m\to+\infty} \frac{\widetilde{g}(x,u_m(x))}{\|u_m^-\|} = 0, \quad \text{for a. e. } x \in \overline{\Omega}.$$

It then follows from the Lebesgue dominated convergence theorem that

$$\lim_{m\to+\infty} \int_\Omega \frac{\widetilde{g}(x,u_m(x))}{\|u_m^-\|}\varphi\,dx = 0, \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{3.24}$$

Therefore, using (3.19), (3.20), (3.24), (3.17), the fact that the sequence (u_m^+) is bounded, and letting $m \to \infty$ in (3.23), we obtain

$$\int_\Omega \nabla v \cdot \nabla\varphi\,dx - \lambda_1 \int_\Omega v\varphi\,dx = 0, \quad \text{for all } \varphi \in H_0^1(\Omega);$$

so that, v is a weak solution of the BVP

$$\begin{gathered} -\Delta u = \lambda_1 u, \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.25}$$

Next, we set $\varphi = v_m$ in (3.23) to get

$$\Big|1 - \lambda_1 \int_\Omega v_m^2\,dx - \int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^-\|} v_m\,dx\Big| \leqslant \frac{\varepsilon_m + C\|u_m^+\|}{\|u_m^-\|}, \quad \text{for all } m, \tag{3.26}$$

where we have also used the definition of v_m in (3.18).

Now, using the Cauchy-Schwarz and Poincaré inequalities, we obtain that

$$\Big|\int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^-\|} v_m\,dx\Big| \leqslant \frac{C}{\|u_m^-\|}, \quad \text{for all } m, \tag{3.27}$$

for some positive constant C, since $\widetilde{g}$ is bounded. We then get from (3.27) and (3.17) that

$$\lim_{m\to\infty} \int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^-\|} v_m\,dx = 0. \tag{3.28}$$

Thus, letting $m \to \infty$ in (3.26) and using (3.20), (3.28), and (3.17), we obtain that

$$\lambda_1 \int_\Omega v^2 dx = 1,$$

which shows that v is a nontrivial solution of (3.25).

Now, it follows from (3.18) and (3.21) that

$$v(x) \leqslant 0, \quad \text{for a. e. } x \in \Omega.$$

Consequently, since v is nontrivial, it must be the case that

$$v = -\varphi_1, \tag{3.29}$$

where φ_1 is the eigenfunction of the BVP problem (3.25) corresponding to the eigenvalue λ_1 with $\varphi_1 > 0$, $\|\varphi_1\| = 1$. Thus, $v < 0 \in \Omega$ and $\partial v/\partial\nu > 0$ on $\partial\Omega$, where ν is the outward unit normal vector to $\partial\Omega$.

Next, we write $u_m = u_m^+ - u_m^-$ and use (3.18) to get

$$\frac{u_m}{\|u_m^-\|} = \frac{u_m^+}{\|u_m^-\|} + v_m, \quad \text{for all } m;$$

so that, by the fact that (u_m^+) is bounded and (3.17), we may assume, passing to a further subsequence if necessary, that

$$\frac{u_m(x)}{\|u_m^-\|} \to -\varphi_1(x), \quad \text{for a. e. } x \in \Omega, \quad \text{as } m \to \infty, \tag{3.30}$$

where we have also used (3.21) and (3.29). It then follows from (3.30) that

$$u_m(x) \to -\infty \quad \text{for a. e. } x \in \Omega, \text{ as } m \to \infty. \tag{3.31}$$

Then, using condition (A5) and the Lebesgue dominated convergence theorem, we conclude from (3.13) that

$$\lim_{m\to\infty} \Big(\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = c + G_{-\infty}|\Omega|. \tag{3.32}$$

Next, we divide both sides of (3.15) by $\|u_m^+\|$ to obtain

$$\Big|\|u_m^+\| - \int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^+\|} u_m^+\,dx\Big| \leqslant \varepsilon_m, \quad \text{for all } m. \tag{3.33}$$

Notice that

$$\int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^+\|} u_m^+\,dx = \int_\Omega \widetilde{g}(x,u_m^+)\frac{u_m^+}{\|u_m^+\|}\,dx, \quad \text{for all } m;$$

so that, using the Cauchy-Schwarz and Poincaré inequalities,

$$\Big|\int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^+\|} u_m^+\,dx\Big| \leqslant C\sqrt{\int_\Omega \widetilde{g}(x,u_m^+(x))^2\,dx}, \quad \text{for all } m, \tag{3.34}$$

and some positive constant C. Now, it follows from (3.31) that

$$u_m^+(x) \to 0 \quad \text{for a.e } x \in \Omega, \quad \text{as } m \to \infty;$$

consequently, using the assumption (A1) along with the Lebesgue dominated convergence theorem, we obtain from (3.34) that

$$\lim_{m\to\infty} \int_\Omega \frac{\widetilde{g}(x,u_m)}{\|u_m^+\|} u_m^+\,dx = 0. \tag{3.35}$$

Therefore, letting m tend to ∞ in (3.33) and using (3.35) we obtain that

$$\lim_{m\to\infty} \|u_m^+\| = 0. \tag{3.36}$$

Thus, combining (3.32) and (3.36) we can then write

$$\lim_{m\to\infty} \Big(\frac{1}{2}\int_\Omega |\nabla u_m^-|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = c + G_{-\infty}|\Omega|. \tag{3.37}$$

We may now proceed as in the proof of Proposition 2.1 in Section 2 to show that

$$\lim_{m\to\infty} \Big(\frac{1}{2}\int_\Omega |\nabla u_m^-|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = 0.$$

Hence, in view of (3.37), we obtain that $c + G_{-\infty}|\Omega| = 0$, which contradicts the assumption that $c \neq -G_{-\infty}|\Omega|$. We therefore conclude that (u_m^-) must be a bounded sequence. Thus, since we have already seen that (u_m^+) is bounded, we see that (u_m) is bounded.

We have therefore shown that any $(\mathrm{PS})_c$ sequence with $c \neq -G_{-\infty}|\Omega|$ must be bounded. The remainder of this proof now proceeds as in the proof of Proposition 2.1 presented in Section 2, using in this case the fact that $\widetilde{g}$ is bounded. □

Now, if we assume that the value $G_{-\infty}$ given in (A5) satisfies the condition in (3.12), then we would have that $-G_{-\infty}|\Omega| \geqslant 0$. Consequently, in view of (3.7), we see that the value of c_1 given in (3.6) is such that

$$c_1 < -G_{-\infty}|\Omega|;$$

therefore, $\widetilde{J}$ satisfies the $(\mathrm{PS})_c$ condition at $c = c_1$. Hence, by the discussion preceding the statement of Proposition 3.1, c_1 is a critical value of $\widetilde{J}$. Thus, there exists $u_1 \in H_0^1(\Omega)$ that is a global minimizer for $\widetilde{J}$. We note that $u_1 \not\equiv 0$ in Ω by (3.7).

Now, since the function $\widetilde{g}$ defined in (3.1) is locally Lipschitz (refer to assumption in (A6)), it follows that u_1 is a classical solution of the problem

$$\begin{gathered} -\Delta u = -\lambda_1 u^- + \widetilde{g}(x,u), \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.38}$$

(see Agmon [1]).

Let $\Omega_+ = \{x \in \Omega \mid u_1(x) > 0\}$. Then, by the definition of $\widetilde{g}$ in (3.1), u_1 solves the BVP

$$\begin{gathered} -\Delta u = 0, \quad \text{in } \Omega_+; \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.39}$$

which has only the trivial solution $u \equiv 0$. Thus, $\Omega_+ = \emptyset$ and therefore $u_1 \leqslant 0$ in Ω.

Before we state the main result of this section, though, we will discuss some properties of the critical point u_1.

Since we have already seen that $u_1 \leqslant 0$ in Ω, it follows from the definition of $\widetilde{g}$ in (3.1) that u_1 is also a solution of the BVP (1.1); consequently, u_1 is also a critical point of J. We will show shortly that u_1 is a local minimizer for J.

Since u_1 is a solution of the BVP (3.38), then u_1 is also a solution of the BVP

$$\begin{gathered} -\Delta u - p(x)u = \lambda_1 u_1 - g^-(x, u_1(x)), \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.40}$$

where

$$p(x) = \begin{cases} \frac{g^+(x,u_1(x))}{u_1(x)}, & \text{if } u_1(x) < 0; \\ 0, & \text{if } u_1(x) = 0. \end{cases} \tag{3.41}$$

Now, it follows from (3.40) and the fact that $u_1(x) \leqslant 0$ for all $x \in \Omega$ that u_1 solves

$$\begin{gathered} -\Delta u - p(x)u \leqslant 0, \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.42}$$

Thus, since $p(x) \leqslant 0$, according to (3.41), we can apply the Hopf's Maximum Principle (see, for instance, [11, Theorem 4, p. 333]) to conclude that

$$u_1(x) < 0, \quad \text{for all } x \in \Omega, \tag{3.43}$$

since u_1 is nontrivial, and

$$\frac{\partial u_1}{\partial \nu}(x) > 0, \quad \text{for } x \in \partial\Omega, \tag{3.44}$$

where ν denotes the outward unit normal vector to the surface $\partial\Omega$. We can then use (3.43) and (3.44), and the assumption that Ω is bounded to show that there exists $\delta > 0$ such that, if $u \in C^1(\overline{\Omega}) \cap H_0^1(\Omega)$ is such that

$$\|u - u_1\|_{C^1(\overline{\Omega})} < \delta,$$

then

$$u(x) < 0, \quad \text{for all } x \in \Omega.$$

Consequently, if u is in a δ-neighborhood of u_1 in the $C^1(\overline{\Omega})$ topology, then

$$J(u) = \widetilde{J}(u) \geqslant \widetilde{J}(u_1) = J(u_1);$$

so that u_1 is a local minimizer of J in the $C^1(\overline{\Omega})$ topology. It then follows from a result of Brezis and Nirenberg in [4] that u_1 is also a local minimizer for J in the $H_0^1(\Omega)$ topology. We have therefore demonstrated the following theorem.

Theorem 3.2. *Assume that g and G satisfy conditions* (A1)–(A4). *Assume also that (A6) and (A7) are satisfied. If (A5) holds true with $G_{-\infty} \leqslant 0$, then the BVP* (1.1) *has a nontrivial solution, u_1, that is a local minimizer of the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ defined in* (1.5).

In the next section, we will provide additional conditions on the nonlinearity, g, that will allow us to show that the functional J defined in (1.5) has another local minimizer.

4. Existence of a second local minimizer

In addition to (A1)–(A5), with $G_{-\infty} \leqslant 0$, and (A6)–(A7), we will assume

(A8) there exists $s_1 > 0$ such that $g(x, s_1) = 0$ for all $x \in \Omega$.

In this case, we consider the truncated nonlinearity $\overline{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$\overline{g}(x,s) = \begin{cases} g(x,s), & \text{for } 0 \leqslant s \leqslant s_1, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.1}$$

The corresponding primitive,

$$\overline{G}(x,s) = \int_0^s \overline{g}(x,\xi)\,d\xi, \quad \text{for all } x \in \Omega \text{ and } s \in \mathbb{R},$$

is then given by

$$\overline{G}(x,s) = \begin{cases} 0, & \text{if } s \leqslant 0; \\ G(x,s), & \text{if } 0 < s \leqslant s_1; \\ G(x,s_1), & \text{if } s > s_1, \end{cases} \tag{4.2}$$

where G is as given in (1.2).

In view of the definitions of $\overline{g}$ and $\overline{G}$ in (4.1) and (4.2), respectively, we see that $\overline{g}$ and $\overline{G}$ are bounded functions. Thus there exist positive constants M_1 and M_2 such that

$$\begin{aligned} |\overline{g}(x,s)| &\leqslant M_1, \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in \mathbb{R}, \\ |\overline{G}(x,s)| &\leqslant M_2, \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in \mathbb{R}. \end{aligned} \tag{4.3}$$

The corresponding truncated functional, $\overline{J}\colon H_0^1(\Omega) \to \mathbb{R}$ is then given by

$$\overline{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u^-)^2\,dx - \int_\Omega \overline{G}(x,u)\,dx, \quad u \in H_0^1(\Omega), \tag{4.4}$$

where $\overline{G}$ is given in (4.2). We then get that $\overline{J}$ is Fréchet differentiable with continuous derivative given by

$$\langle \nabla\overline{J}(u), \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\varphi\,dx + \lambda_1 \int_\Omega u^- \varphi\,dx - \int_\Omega \overline{g}(x,u)\varphi\,dx, \tag{4.5}$$

for all u and φ in $H_0^1(\Omega)$.

Next, we show that $\overline{J}$ satisfies the $(\mathrm{PS})_c$ condition for $c \notin \overline{\Lambda}$, where the exceptional set, $\overline{\Lambda}$, in this case is

$$\overline{\Lambda} = \{0\}.$$

Proposition 4.1. *Assume that g and G satisfy* (A1), (A2) *and* (A8)*, and define $\overline{J}$ as in* (4.4)*, where $\overline{G}$ is given in* (4.2)*. Then, $\overline{J}$ satisfies the $(PS)_c$ for $c \neq 0$.*

Proof. Let (u_m) be a $(\mathrm{PS})_c$ sequence for $\overline{J}$, where

$$c \neq 0; \tag{4.6}$$

so that, according to (4.4) and (4.5)

$$\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx - \int_\Omega \overline{G}(x,u_m)\,dx \to c, \quad \text{as } m \to \infty, \tag{4.7}$$

and

$$\Big|\int_\Omega \nabla u_m \cdot \nabla\varphi\,dx + \lambda_1 \int_\Omega u_m^-\varphi\,dx - \int_\Omega \overline{g}(x,u_m)\varphi\,dx\Big| \leqslant \varepsilon_m \|\varphi\|, \tag{4.8}$$

for all m and all $\varphi \in H_0^1(\Omega)$, where (ε_m) is a sequence of positive numbers that tends to 0 as $m \to \infty$.

Writing $u_m = u_m^+ - u_m^-$ for all m, and taking $\varphi = u_m^+$ in (4.8) we obtain

$$\Big| \|u_m^+\|^2 - \int_\Omega \overline{g}(x, u_m^+) u_m^+ \, dx \Big| \leqslant \varepsilon_m \|u_m^+\|, \quad \text{for all } m. \tag{4.9}$$

Using (4.3) we then estimate

$$\Big| \int_\Omega \overline{g}(x, u_m^+) u_m^+ \, dx \Big| \leqslant C \|u_m^+\|, \quad \text{for all } m, \tag{4.10}$$

where we have also used the Cauchy-Schwarz and Poincaré inequalities. Combing (4.9) and (4.10) we then get that

$$\|u_m^+\|^2 \leqslant C \|u_m^+\|, \quad \text{for all } m, \tag{4.11}$$

where we have used the fact that $\varepsilon_m \to 0$ as $m \to \infty$.

It follows from (4.11) that the sequence (u_m^+) is bounded in $H_0^1(\Omega)$.

Next, we show that (u_m^-) is also bounded in $H_0^1(\Omega)$. If this is not the case, we may assume, passing to a subsequence if necessary, that

$$\|u_m^-\| \to \infty \quad \text{as } m \to \infty. \tag{4.12}$$

Define

$$v_m = -\frac{u_m^-}{\|u_m^-\|}, \quad \text{for all } m. \tag{4.13}$$

Then, since

$$\|v_m\| = 1, \quad \text{for all } m, \tag{4.14}$$

passing to a further subsequences if necessary, we may assume that there is $v \in H_0^1(\Omega)$ such that

$$v_m \rightharpoonup v \quad \text{(weakly) in } H_0^1(\Omega), \text{ as } m \to \infty; \tag{4.15}$$

$$v_m \to v \quad \text{in } L^2(\Omega), \quad \text{as } m \to \infty; \tag{4.16}$$

$$v_m(x) \to v(x) \quad \text{for a.e. } x \text{in } \Omega, \quad \text{as } m \to \infty. \tag{4.17}$$

Now, writing $u_m = u_m^+ - u_m^-$ in (4.8) we have

$$\Big| \int_\Omega \nabla u_m^- \cdot \nabla\varphi \, dx + \lambda_1 \int_\Omega u_m^- \varphi \, dx - \int_\Omega \overline{g}(x, u_m)\varphi \, dx \Big| \leqslant (\varepsilon_m + C\|u_m^+\|)\|\varphi\|, \tag{4.18}$$

for all m and all $\varphi \in H_0^1(\Omega)$, where we have also used the Cauchy-Schwarz and Poincaré inequalities.

Next, divide both sides of (4.18) by $\|u_m^-\|$ and use (4.13) to obtain

$$\Big| \int_\Omega \nabla v_m \cdot \nabla\varphi \, dx - \lambda_1 \int_\Omega v_m \varphi \, dx - \int_\Omega \frac{\overline{g}(x, u_m)}{\|u_m^-\|}\varphi \, dx \Big| \leqslant \Big(\frac{\varepsilon_m + C\|u_m^+\|}{\|u_m^-\|} \Big)\|\varphi\|, \tag{4.19}$$

for all $\varphi \in H_0^1(\Omega)$ and all m.

Using the estimate in (4.3) and (4.12) we obtain

$$\lim_{m\to\infty} \Big| \int_\Omega \frac{\overline{g}(x, u_m)}{\|u_m^-\|}\varphi \, dx \Big| = 0, \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{4.20}$$

Combining (4.19) and (4.20) we then get

$$\lim_{m\to\infty} \Big| \int_\Omega \nabla v_m \cdot \nabla\varphi \, dx - \lambda_1 \int_\Omega v_m \varphi \, dx \Big| = 0, \quad \text{for all } \varphi \in H^1(\Omega), \tag{4.21}$$

where we have used (4.12) and the facts that (u_m^+) is bounded in $H_0^1(\Omega)$ and $\varepsilon_m \to 0$ as $m \to \infty$.

Now, it follows from (4.15), (4.16) and (4.21) that

$$\int_\Omega \nabla v \cdot \nabla \varphi \, dx - \lambda_1 \int_\Omega v\varphi \, dx = 0, \quad \text{for all } \varphi \in H_0^1(\Omega); \tag{4.22}$$

so that, v is a weak solution of the BVP

$$\begin{gathered} -\Delta u = \lambda_1 u, \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Next, we take $\varphi = v_m$ in (4.19) and use (4.14) to obtain

$$\Big|1 - \lambda_1 \int_\Omega v_m^2 \, dx - \int_\Omega \frac{\overline{g}(x, u_m)}{\|u_m^-\|} v_m \, dx\Big| \leqslant \frac{\varepsilon_m + C\|u_m^+\|}{\|u_m^-\|}, \quad \text{for all } m, \tag{4.23}$$

where, by (4.3), (4.12) and (4.16),

$$\lim_{m\to\infty} \Big| \int_\Omega \frac{\overline{g}(x, u_m)}{\|u_m^-\|} v_m \, dx \Big| = 0. \tag{4.24}$$

Thus, using (4.12), and the facts that (u_m^+) is a bounded sequence and $\varepsilon_m \to 0$ as $m \to \infty$, we obtain from (4.23) and (4.24) that

$$\lim_{m\to\infty} \Big|1 - \lambda_1 \int_\Omega v_m^2 \, dx\Big| = 0;$$

so that, in view of (4.16),

$$\lambda_1 \int_\Omega v^2 \, dx = 1, \tag{4.25}$$

from which we conclude that $v \not\equiv 0$. Thus, v is an eigenfunction of $-\Delta$ with Dirichlet boundary conditions over Ω. It follows from this observation and (4.22), in conjunction with (4.25), that $\|v\| = 1$. Consequently, we obtain from the definition of v_m in (4.13) and from (4.17) that

$$v = -\varphi_1. \tag{4.26}$$

Recall that we have chosen φ_1 so that $\varphi_1 > 0$ in Ω and $\|\varphi_1\| = 1$.

Next, writing $u_m = u_m^+ - u_m^-$ and using (4.13), we obtain

$$\frac{u_m}{\|u_m^-\|} = \frac{u_m^+}{\|u_m^-\|} + v_m, \quad \text{for all } m;$$

so that, by the fact that (u_m^+) is bounded and (4.12), we may assume, passing to a further subsequence if necessary, that

$$\frac{u_m(x)}{\|u_m^-\|} \to -\varphi_1(x), \quad \text{for a. e. } x \in \Omega, \quad \text{as } m \to \infty, \tag{4.27}$$

where we have also used (4.17) and (4.26). It then follows from (4.27) that

$$u_m(x) \to -\infty \quad \text{for a. e. } x \in \Omega, \text{ as } m \to \infty. \tag{4.28}$$

Then, using the definition of $\overline{G}$ in (4.2) and the Lebesgue dominated convergence theorem, we conclude from (4.28) that

$$\lim_{m\to\infty} \int_\Omega \overline{G}(x, u_m(x)) \, dx = 0;$$

so that, in conjunction with (4.7),

$$\lim_{m\to\infty}\Big(\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = c. \tag{4.29}$$

We may now proceed as in the proof of Proposition 2.1 and show that

$$\lim_{m\to\infty}\Big(\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = 0. \tag{4.30}$$

Note that (4.29) and (4.30) are in contradiction with (4.6). Consequently, (u_m^-) is also bounded in $H_0^1(\Omega)$. Therefore, as in the last portion of the proof of Proposition 2.1, we can show that (u_m) has a convergent subsequence. We have therefore established the fact that $\overline{J}$ satisfies the $(\mathrm{PS})_c$ condition, provided that $c \neq 0$. □

It follows from the definition of $\overline{J}$ in (4.5) and the estimate in (4.4) that $\overline{J}$ is bounded from below in $H_0^1(\Omega)$. Indeed, we obtain the estimate

$$\overline{J}(u) \geqslant \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 - \frac{\lambda_1}{2}\|u^-\|_{L^2}^2 - M_2|\Omega|, \quad \text{for all } u \in H_0^1(\Omega);$$

so that, using the Poincaré inequality,

$$\overline{J}(u) \geqslant \frac{1}{2}\|u^+\|^2 - M_2|\Omega|, \quad \text{for all } u \in H_0^1(\Omega),$$

from which we obtain that

$$\overline{J}(u) \geqslant -M_2|\Omega|, \quad \text{for all } u \in H_0^1(\Omega).$$

Set

$$c_2 = \inf_{v\in H_0^1(\Omega)} \overline{J}(v). \tag{4.31}$$

We will show that, if (A6) and (A7) hold, then

$$c_2 < 0. \tag{4.32}$$

Indeed, for

$$0 < t < \frac{s_1}{\max_{x\in\overline{\Omega}} \varphi_1(x)},$$

compute

$$\overline{J}(t\varphi_1) = J(t\varphi_1) = \frac{t^2}{2} - \int_\Omega G(x, t\varphi_1(x))\,dx,$$

so that

$$\frac{d}{dt}\left[\overline{J}(t\varphi_1)\right] = t - \int_\Omega g(x, t\varphi_1(x))\varphi_1(x)\,dx,$$
$$\frac{d^2}{dt^2}\left[\overline{J}(t\varphi_1)\right] = 1 - \int_\Omega \frac{\partial g}{\partial s}(x, t\varphi_1(x))(\varphi_1(x))^2\,dx.$$

It then follows from (4) and (A6) and (A7) that

$$\lim_{t\to 0^+}\frac{d^2}{dt^2}[\overline{J}(t\varphi_1)] = 1 - a\int_\Omega \varphi_1^2\,dx,$$

or

$$\lim_{t\to 0^+}\frac{d^2}{dt^2}[\overline{J}(t\varphi_1)] = 1 - \frac{a}{\lambda_1} < 0,$$

since $a > \lambda_1$ according to (A7). Consequently, there exists $t_1 > 0$ such that

$$\overline{J}(t_1\varphi_1) < 0.$$

Thus, in view of (4.31), the assertion in (4.32) follows.

In view of (4.32) and the result in Proposition 4.1 we see that $\overline{J}$ satisfies the $(\mathrm{PS})_{c_2}$ condition. Thus, given the definition of c_2 in (4.31), the argument invoking Ekeland's Variational Principle leading to Theorem 3.2 in Section 3 can now be used to obtain a minimizer, u_2, of $\overline{J}$. Furthermore, as was done in Section 3, we can use the Maximum Principle to conclude that

$$0 < u_2(x) < s_1, \quad \text{for all } x \in \Omega;$$

so that, u_2 is also a critical point of J. Indeed, u_2 is a local minimizer of J by the Brézis and Nirenberg result in [4]. We have therefore established the following multiplicity result.

Theorem 4.2. *Assume that g and G satisfy conditions* (A1)–(A5)*, with $G_{-\infty} \leqslant 0$, and* (A6)–(A8)*. Let $J\colon H_0^1(\Omega) \to \mathbb{R}$ be the C^1 functional defined in* (1.5)*. In addition to the local minimizer, u_1, of J given by Theorem 3.2, which is a negative solution of the BVP* (1.1)*, there exists another local minimizer, u_2, of J that yields a positive solution of the BVP* (1.1)*.*

5. Existence of a third nontrivial critical point

In the previous section we saw that, if g and G satisfy conditions (A1)–(A5), with $G_{-\infty} \leqslant 0$, and (A6)–(A8)), then the functional J defined in (1.5) has two local minimizers distinct from 0. In this section we prove the existence of a third, nontrivial, critical point of J. This will follow from following variant of the Mountain-Pass Theorem first proved by Pucci and Serrin in 1985, [14].

Theorem 5.1 ([14, Theorem 1])**.** *Let X be a real Banach space with norm $\|\cdot\|$ and $J : X \to \mathbb{R}$ be a C^1 functional. Let u_o and u_1 be distinct points in X. Assume that there are real numbers r and R such that*

$$0 < r < \|u_1 - u_o\| < R,$$

and a real number a such that

$$J(u_o) \leqslant a \quad J(u_1) \leqslant a,$$
$$J(v) \geqslant a \quad \text{for all } v \text{ such that } r < \|v - u_o\| < R.$$

Put

$$\Gamma = \{\gamma \in C([0,1], X) | \gamma(0) = u_o, \gamma(1) = u_1\}, \tag{5.1}$$

and let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)). \tag{5.2}$$

Assume further that any sequence $(u_n)_{n=1}^{\infty}$ in X such that

$$J(u_n) \to c \quad \text{as } n \to \infty,$$
$$J'(u_n) \to 0, \quad \text{as } n \to \infty$$

possesses a convergent subsequence. Then, there exists a critical point $\overline{u}$ in X different from u_o and u_1, corresponding to the critical value c given in (5.2)*.*

In [14], Pucci and Serrin apply the result in Theorem 5.1 to the case in which u_o and u_1 are two distinct local minima of the functional J. Thus, according to Theorem 5.1, if the functional J satisfies the $(\mathrm{PS})_c$ condition, where c is as given in (5.2), we would obtain a third critical point of J distinct from the two minimizers.

We summarize this observation in the following corollary to the Pucci-Serrin result in Theorem 5.1.

Corollary 5.2. [14, Corollary 1] *Suppose that J has two distinct local minimizers, u_o and u_1. Let c be as given in* (5.2) *and suppose that J satisfies the $(PS)_c$ condition. Then, J possesses a third critical point.*

According to Proposition 2.1, we will be able to apply Corollary 5.2 to our problem provided we can show that $c \neq -G_{-\infty}|\Omega|$. Since we are also assuming that $G_{-\infty} \leqslant 0$, we will be able to obtain a third nontrivial critical point of J if we can prove that c given by (5.2) is negative. This can be achieved if we can show that there is some path γ in Γ defined in (5.1), connecting the two local minimizers, such that $J(\gamma(t)) < 0$ for all $t \in [0,1]$. To do this, we borrow an idea used by Courant [9] in the proof of the so called Finite Dimensional Mountain-Pass Theorem. With these observations in mind, we are ready to prove the main result of this section.

Theorem 5.3. *Let J satisfies the conditions* (A1)–(A5)*, with $G_{-\infty} \leqslant 0$, and* (A6)–(A8)*. Let $J\colon H_0^1(\Omega) \to \mathbb{R}$ be the C^1 functional defined in* (1.5)*, and u_1 and u_2 be the two local minimizers of J given by Theorem 4.2. Then, J has a third nontrivial critical point. Consequently, the BVP* (1.1) *has three nontrivial weak solutions.*

Proof. Let u_1 and u_2 denote the two local minimizers of J given by Theorem 4.2. We then have that u_1 and u_2 are nontrivial and $u_1 \neq u_2$. Furthermore,

$$J(u_1) < 0 \quad \text{and} \quad J(u_2) < 0. \tag{5.3}$$

Define

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) | \gamma(0) = u_1, \gamma(1) = u_2\}, \tag{5.4}$$

and put

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)). \tag{5.5}$$

Arguing by contradiction, assume that u_1, u_2 and 0 are the only critical points of J. Then, certainly, there is a path $\gamma_o \in \Gamma$ that does not contain 0. Then, u_1 and u_2 are the only critical points of J along γ_o. Set

$$\gamma_o([0,1]) = M \cup N,$$

where $M = \{\gamma_o(t) \in H_0^1(\Omega) : t \in [0,1] \text{ and } J(\gamma_o(t)) \geqslant 0\}$ and $N = \gamma_o([0,1]) \backslash M$. We will consider two cases $M = \emptyset$ and $M \neq \emptyset$ separately.

If $M = \emptyset$, then $J(\gamma_o(t)) < 0$ for all $t \in [0,1]$. Hence, in view of the definition of c in (5.5), $c < 0$. Consequently, by Proposition 2.1 and the assumption that $G_{-\infty} \leqslant 0$, J satisfies the $(PS)_c$ condition. Thus, we can apply Corollary 5.2 with $X = H_0^1(\Omega)$ to conclude that J has a critical point, $\overline{u}$, distinct from u_1 and u_2. Furthermore, $\overline{u} \neq 0$, since $c < 0$. This would contradict the initial assumption that there are no critical point of J other than 0, u_1 and u_2.

Next, assume that $M \neq \emptyset$. Observe that $u_1, u_2 \notin M$ in view of (5.3). Then, M contains no critical points of J; consequently, $\nabla J(\gamma_o(t)) \neq 0$ for all $t \in [0,1]$ such that $\gamma_o(t) \in M$, where ∇J is the gradient of J obtained by the Riesz Representation Theorem for Hilbert spaces. In particular, since M is compact, there exists $\delta > 0$ such that

$$\|\nabla J(u)\| \geq \delta, \quad \text{for all } u \in M. \tag{5.6}$$

It then follows from that assumption that J is C^1 that there exists $\varepsilon > 0$ such that

$$\|\nabla J(v)\| \geq \frac{\delta}{2} \quad \text{for all } v \in M_\varepsilon = \{v \in H_0^1(\Omega) | \operatorname{dist}(v, M) < \varepsilon\}. \tag{5.7}$$

Notice that M_ε is an open neighborhood of M which does not contain u_1 and u_2 because they are local minimizers of J.

Next, let $\rho \in C_c(H_0^1(\Omega), \mathbb{R})$ denote a function with $\operatorname{supp}(\rho) \subset M_\varepsilon$ such that $\rho(u) = A$, for $u \in M$, and $0 \leq \rho(u) \leq A$ for all $u \in H_0^1(\Omega)$, where A a positive constant to be chosen shortly. Define the following deformation $\eta\colon H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega)$ given by

$$\eta(u,t) = u - t\rho(u)\nabla J(u), \quad \text{for } u \in H_0^1(\Omega) \quad \text{and } t \in \mathbb{R}. \tag{5.8}$$

It then follows that

$$\begin{aligned} \frac{d}{dt} J(\eta(u,t)) &= \langle \nabla J(\eta(u,t)), \eta_t(u,t) \rangle \\ &= \langle \nabla J(\eta(u,t)), -\rho(u)\nabla J(\eta(u,t)) \rangle \\ &= -\rho(u)\|\nabla J(\eta(u,t))\|^2, \end{aligned} \tag{5.9}$$

for $u \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. In particular, it follows from (5.9) and (5.8) that

$$\frac{d}{dt} J(\eta(u,t))\big|_{t=0} = -\rho(u)\|\nabla J(u)\|^2, \quad \text{for all } u \in H_0^1(\Omega). \tag{5.10}$$

Note that, for points $u \in H_0^1(\Omega)$ such that $\rho(u) > 0$, it follows from (5.6) and (5.7) that

$$\rho(u)\|\nabla J(\eta(u,t))\|^2 > \rho(u)\frac{\delta^2}{4} > 0. \tag{5.11}$$

Hence, by (5.10) and the assumption that $J \in C^1(H_0^1(\Omega), \mathbb{R})$, we obtain that, for each $u \in H_0^1(\Omega)$ such that $\rho(u) > 0$, there exists a neighborhood U of u and $T_u > 0$ such that

$$\frac{d}{dt} J(\eta(u,t)) < -\frac{\rho(u)}{2}\|\nabla J(u)\|^2, \quad \text{for } u \in U \text{ and } |t| < T_u. \tag{5.12}$$

On the other hand, if $\rho(u) = 0$, it follows from the definition of η in (5.8) that $\eta(u,t) = u$ for all $t \in \mathbb{R}$ so that

$$\frac{d}{dt}[\eta(u,t)] = 0, \quad \text{for } t \in \mathbb{R}.$$

Therefore,

$$\frac{d}{dt} J(\eta(u,t)) = 0, \quad \text{for all } t \in [0,1],\ u \in X \text{ with } \rho(u) = 0. \tag{5.13}$$

Since $\operatorname{supp}(\rho)$ is compact, it follows from (5.11), (5.12) and (5.13) that there exists $T > 0$ such that

$$\frac{d}{dt} J(\eta(u,t)) \leq -\frac{\rho(u)}{8}\delta^2, \quad \text{for } u \in H_0^1(\Omega) \text{ and } |t| < T.$$

Consequently,

$$J(\eta(u,T)) \leq J(u) - \frac{\rho(u)}{8}\delta^2 T, \quad \text{for all } u \in H_0^1(\Omega). \tag{5.14}$$

Next, define $\gamma(t) = \eta(\gamma_o(t), T)$, for all $t \in [0,1]$. Note that $\gamma \in \Gamma$. Indeed, since $u_1 \notin M_\varepsilon$ and $u_2 \notin M_\varepsilon$, we have, by the properties of ρ and the definition of η in (5.8), that

$$\eta(u_i, T) = u_i, \quad \text{for } i = 1, 2.$$

Now, if $v \in N$, it follows from (5.14) that

$$J(\eta(v,T)) < 0, \quad \text{for all } u \in N. \tag{5.15}$$

On the other hand, for $v \in M$, using (5.14), we obtain that

$$J(\eta(v,T)) \leq L - \frac{A}{8}\delta^2 T, \quad \text{for all } u \in M, \tag{5.16}$$

where we have set $L = \max_{t\in[0,1]} J(\gamma_o(t))$. Choose $A = \frac{12L}{\delta^2 T}$. Then, it follows from (5.16) that

$$J(\eta(v,T)) \leqslant -\frac{L}{2} < 0, \quad \text{for } v \in M. \tag{5.17}$$

Therefore, by combining (5.15), (5.17) and the choice of A, we conclude that

$$J(\eta(\gamma_o(t), T)) < 0, \quad \text{for all } t \in [0,1].$$

Therefore, the path $\gamma = \eta(\gamma_o, T)$ connecting the two local minimizers u_1 and u_2 is such that

$$J(\gamma(t)) < 0, \quad \text{for all } t \in [0,1].$$

Consequently, by the definition of c in (5.5) and (5.4), $c < 0$. It then follows, by the result of Proposition 2.1 and the assumption that $G_{-\infty} \leqslant 0$, that J satisfies the $(PS)_c$ condition. We can therefore apply Corollary 5.2, with $X = H_0^1(\Omega)$, to obtain a critical point $\overline{u}$ of J different from u_1 and u_2 and such that $\overline{u} \neq 0$. However, this contradicts the assumption that 0, u_1 and u_2 are the only critical points of J. We therefore obtain the existence of a third nontrivial critical point of J. □

Remark: We provide an example of a function g that satisfies the conditions (A1)–(A8), and to which the results in Theorem 5.3 will apply. In this example we assume that $N \geqslant 3$.

Let s_1 denote a positive real number; r be a real number satisfying

$$\max\Big(\frac{2}{N-2}, 1\Big) < r < \frac{N+2}{2(N-2)};$$

and a a real number with $a > \lambda_1$. To construct g, let $g_0 \colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following conditions:

(a) $g_0(x,0) = 0$ and $g_0(x,s) = 0$ for all $x \in \Omega$ and $s \geqslant s_1$;

(b) the derivative of g_0 has a jumping discontinuity at 0 prescribed by

$$\lim_{s\to 0^-} \frac{g_0(x,s)}{s} = a - \lambda_1 \quad \text{and} \qquad \lim_{s\to 0^+} \frac{g_0(x,s)}{s} = a + s_1^{r-1},$$

uniformly for a.e $x \in \Omega$;

(c) $\lim_{s\to -\infty} g_0(x,s) = 0$;

(d) $\lim_{s\to -\infty} G_0(x,s) = G_{-\infty}$ where $G_0(x,s) = \int_0^s g(x,\xi)d\xi$, for $x \in \Omega$, $s \in \mathbb{R}$.

Then, the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$g(x,s) = g_0(x,s) + (s^+)^r - s_1^{r-1}s^+, \quad \text{for } x \in \Omega, \text{ and } \ s \in \mathbb{R},$$

where s^+ denotes the positive part of s, satisfies the conditions (A1)–(A8).

Acknowledgments. The authors would like to thank Professor Alfonso Castro for very helpful conversations during the time they worked on the problems discussed in this paper. The authors also appreciate the referee's careful reading of the manuscript and very helpful suggestions and corrections.

References

[1] S. Agmon; *The L^p approach to the Dirichlet problem*, Ann. Scuola Norm. Sup. Pisa **13** (1959), 405–448.
[2] D. Arcoya and D. G. Costa; *Nontrivial solutions for a strong resonant problem*, Differential and Integral Equations **1** (1995), 151–159.
[3] P. Bartolo, V. Benci, D Fortunato; *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Analysis, Theory, Methods and Applications **7** (1983), 981–1012.
[4] H. Brézis, L. Nirenberg; *H^1 versus C^1 local minimizers*, C. R. Acad. Sci. Paris **317** (1993), 465–475.
[5] K. C. Chang, S. J. Li, J. Liu; *Remarks on multiple solutions for asymptotically linear elliptic boundary value problems*, Topological Methods in Nonlinear Analysis **3** (1994), 43–58.
[6] K. C. Chang, J.Q. Liu; *A strong resonant problem*, Chinese Ann. Math. **11** (1990), 191–210.
[7] D. G. Costa and E. Silva, *The Palais–Smale condition versus coercivity*, Nonlinear Analysis: Theory, Methods and Applications **16** (1991), 371–381.
[8] D. G. Costa and E. Silva; *Existence of solutions for a class of resonant elliptic problems*, Journal of Mathematics Analysis and Applications **175** (1993), 411–424.
[9] Richard Courant; *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, first ed., Interscience Publishers, Inc., New York, N.Y., 1950.
[10] D. G. de Figueiredo; *The ekeland variational principle with applications and detours*, Tata Institute of Fundamental Research, Bombay, 1989.
[11] L. C. Evans; *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998.
[12] N. Hirano, S. Li, Z-Q. Wang; *Morse Theory without (PS) condition at isolated values and strong resonance problems*, Calculus of Variations **10** (2000), 223–247.
[13] Shujie Li; *A new Morse theory and strong resonance problems*, Topological Methods in Nonlinear Analysis **21** (2003), 81–100.
[14] Patrizia Pucci, James Serrin; *A mountain pass theorem*, Journal of Differential Equations **60** (1985), 142–149.
[15] L. Recôva, A. Rumbos; *An asymmetric superlinear elliptic problem at resonance*, Nonlinear Analysis **112** (2015), 181–198.

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