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LYAPUNOV-TYPE INEQUALITIES FOR THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

MUSTAFA FAHRI AKTAŞ, DEVRIM ÇAKMAK

Dedicated to the memory of Aydın Tiryaki (June 1956 - May 2016)

Abstract. In this article, we establish new Lyapunov-type inequalities for third-order linear differential equations

$$y''' + q(t)y = 0$$

under the three-point boundary conditions

$$y(a) = y(b) = y(c) = 0$$

and

$$y(a) = y''(d) = y(b) = 0$$

by bounding Green's functions $G(t, s)$ corresponding to appropriate boundary conditions. Thus, we obtain the best constants of Lyapunov-type inequalities for three-point boundary value problems for third-order linear differential equations in the literature.

1. Introduction

Lyapunov [18] obtained the remarkable result: If $q \in C([0, \infty), \mathbb{R})$ and y is a nontrivial solution of

$$y'' + q(t)y = 0 \tag{1.1}$$

under the Dirichlet boundary conditions

$$y(a) = y(b) = 0 \tag{1.2}$$

and $y(t) \not\equiv 0$ for $t \in (a, b)$, then

$$\frac{4}{b-a} \leq \int_a^b |q(s)| ds \,. \tag{1.3}$$

Thus, this inequality provides a lower bound for the distance between two consecutive zeros of y. The inequality (1.3) is the best possible in the sense that if the constant 4 in the left hand side of (1.3) is replaced by any larger constant, then there exists an example of (1.1) for which (1.3) no longer holds (see [14, p. 345], [16, p. 267]). In this paper, our aim is to obtain the best constants of Lyapunov-type inequalities for third-order linear differential equations with three-point boundary conditions. The above result of Lyapunov has found many applications in areas

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like eigenvalue problems, stability, oscillation theory, disconjugacy, etc. Since then, there have been several results to generalize the above linear equation in many directions; see the references.

There are various methods used to obtain Lyapunov-type inequalities for different types of boundary value problems. One of the most useful methods is as follows: Nehari [21] started with the Green's function of the problem (1.1) with (1.2), which is

$$G(t,s) = -\begin{cases} \frac{(t-a)(b-s)}{b-a}, & a \le s \le t, \\ \frac{(s-a)(b-t)}{b-a}, & t \le s \le b, \end{cases} \tag{1.4}$$

and he wrote

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds. \tag{1.5}$$

Then by choosing $t = t_0$, where $|y(t)|$ is maximized and canceling out $|y(t_0)|$ on both sides, he obtained

$$1 \le \max_{a \le t \le b} \int_a^b |G(t,s)||q(s)|ds. \tag{1.6}$$

Note that if we take the absolute maximum value of the function $|G(t,s)|$ for all $t, s \in [a,b]$ in (1.6), then we obtain the inequality (1.3). Following the ideas of these papers, this method has been applied in a huge number of works to different second and higher order ordinary differential equations with different types of boundary conditions. We see that by bounding the Green's function $G(t,s)$ in various ways, we can obtain the best constants in the Lyapunov-type inequalities in other differential equations with associated boundary conditions as well. Thus, we obtain the best constants of the Lyapunov-type inequalities for three-point boundary value problems for third-order linear differential equations by using the absolute maximum values of the Green's functions $G(t,s)$ in the literature.

In this article, we consider the third-order linear differential equation of the form

$$y''' + q(t)y = 0, \tag{1.7}$$

where $q \in C([0,\infty),\mathbb{R})$ and $y(t)$ is a real solution of (1.7) satisfying the three-point boundary conditions

$$y(a) = y(b) = y(c) = 0 \tag{1.8}$$

and

$$y(a) = y''(d) = y(b) = 0, \tag{1.9}$$

$a, b, c, d \in \mathbb{R}$ with $a < b < c$ and $a \le d \le b$ are three points and $y(t) \not\equiv 0$ for $t \in (a,b) \cup (b,c)$ and $t \in (a,b)$, respectively.

Some of the recent studies about Lyapunov-type inequalities for third and higher order boundary value problems are as follows: In 1999, Parhi and Panigrahi [22] established the inequalities similar to the classical Lyapunov inequality (1.3) for the third-order linear differential equation (1.7) under the three-point boundary conditions (1.8) and (1.9) as follows.

Theorem 1.1 ([22, Theorem 2]). *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.8)*, then*

$$\frac{4}{(c-a)^2} < \int_a^c |q(s)|\, ds\,. \tag{1.10}$$

Theorem 1.2 ([22, Theorem 1]). *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.9), *then*

$$\frac{4}{(b-a)^2} < \int_a^b |q(s)|\,ds\,. \tag{1.11}$$

In 2010, Yang et al. [25] extended the inequality (1.11) for the third-order linear differential equation

$$(r_2(t)(r_1(t)y'))')' + q(t)y = 0. \tag{1.12}$$

Theorem 1.3 ([25, Theorem 1]). *If $y(t)$ is a nontrivial solution of* (1.12) *satisfying the conditions*

$$y(a) = y(b) = 0, \tag{1.13}$$

$$(r_1(t)y'(t))' \text{ has a zero } d \in [a,b], \tag{1.14}$$

then

$$\begin{aligned} &\min_{a\le t_0\le b}\Big[\Big(\int_a^{t_0} r_1(s)ds\int_a^{t_0} r_2(s)ds\Big)^{-1} + \Big(\int_{t_0}^b r_1(s)ds\int_{t_0}^b r_2(s)ds\Big)^{-1}\Big] \\ &< 2\int_a^b |q(s)|\,ds\,, \end{aligned} \tag{1.15}$$

where $|y(t_0)| = \max\{|y(t)| : a \le t \le b\}$.

In 2013, Kiselak [17] extended the Lyapunov-type inequalities from linear differential equation to the third-order half-linear differential equation

$$\begin{aligned} &\Big(\frac{1}{r_2(t)}\Big|\Big(\frac{1}{r_1(t)}|y'(t)|^{p_1-1}y'(t)\Big)'\Big|^{p_2-1}\Big(\frac{1}{r_1(t)}|y'(t)|^{p_1-1}y'(t)\Big)'\Big)' \\ &+ q(t)|y(t)|^{p_3-1}y(t) = 0, \end{aligned} \tag{1.16}$$

where $0 < p_1, p_2$ and $p_3 = p_1p_2$.

Theorem 1.4 ([17, Theorem 2.1]). *If $y(t)$ is a nontrivial solution of* (1.16) *satisfying the conditions* (1.13), *and*

$$\Big(\frac{1}{r_1(t)}|y'(t)|^{p_1-1}y'(t)\Big)' \text{ has a zero } d \in [a,b], \tag{1.17}$$

then

$$\begin{aligned} &\min_{a\le t_0\le b}\Big[\Big(\int_a^{t_0} r_1^{1/p_1}(s)ds\Big(\int_a^{t_0} r_2^{1/p_2}(s)ds\Big)^{1/p_1}\Big)^{-1} \\ &+ \Big(\int_{t_0}^b r_1^{1/p_1}(s)ds\Big(\int_{t_0}^b r_2^{1/p_2}(s)ds\Big)^{1/p_1}\Big)^{-1}\Big] < 2\Big(\int_a^b |q(s)|ds\Big)^{1/p_3}, \end{aligned} \tag{1.18}$$

where $|y(t_0)| = \max\{|y(t)| : a \le t \le b\}$.

Theorem 1.5 ([17, Theorem 2.2]). *If $y(t)$ is a nontrivial solution of the problem* (1.16) *with* (1.8), *then*

$$\begin{aligned} &\min_{t_0\in[a,c]}\Big[\Big(\int_a^{t_0} r_1^{1/p_1}(s)ds(\int_a^{t_0} r_2^{1/p_2}(s)ds)^{1/p_1}\Big)^{-1} \\ &+ \Big(\int_{t_0}^c r_1^{1/p_1}(s)ds\Big(\int_{t_0}^c r_2^{1/p_2}(s)ds\Big)^{1/p_1}\Big)^{-1}\Big] \\ &< 2\Big(\int_a^c |q(s)|ds\Big)^{1/p_3}, \end{aligned} \tag{1.19}$$

where $|y(t_0)| = \max\{|y(t)| : a \le t \le c\}$.

In 2014, Dhar and Kong [13] obtained the following Lyapunov-type inequalities for third-order half-linear differential equation (1.16).

Theorem 1.6 ([13, Theorem 2.5]). *If $y(t)$ is a nontrivial solution of* (1.16) *with* (1.13) *and* (1.17), *then*

$$\begin{aligned} &2^{p_2+p_3}\Big(\int_a^b r_1^{-1/p_1}(s)ds\Big)^{-p_3}\Big(\int_a^b r_2^{-1/p_2}(s)ds\Big)^{-p_2} \\ &< \int_a^d q^-(s)ds + \int_d^b q^+(s)ds, \end{aligned} \tag{1.20}$$

where

$$q^-(t) = \max\{-q(t), 0\}, \tag{1.21}$$

$$q^+(t) = \max\{q(t), 0\}. \tag{1.22}$$

From (1.21) and (1.22), it is easy to see that $-q^-(t) \le q(t) \le q^+(t)$, $q^-(t) \le |q(t)|, q^+(t) \le |q(t)|$, and $|q(t)| = q^+(t) + q^-(t)$ for $t \in [a, b]$.

Theorem 1.7 ([13, Theorem 2.6]). *If $y(t)$ is a nontrivial solution of* (1.16) *with* (1.8), *then*

$$\begin{aligned} &2^{p_2+p_3}\Big(\int_a^b r_1^{-1/p_1}(s)ds\Big)^{-p_3}\Big(\int_a^b r_2^{-1/p_2}(s)ds\Big)^{-p_2} \\ &< \max_{a\le d\le b}\Big[\int_a^d q^-(s)ds + \int_d^b q^+(s)ds\Big], \end{aligned} \tag{1.23}$$

where $d \in [a, b]$ is given in (1.17), *or*

$$\begin{aligned} &2^{p_2+p_3}\Big(\int_b^c r_1^{-1/p_1}(s)ds\Big)^{-p_3}\Big(\int_b^c r_2^{-1/p_2}(s)ds\Big)^{-p_2} \\ &< \max_{b\le d\le c}\Big[\int_b^d q^-(s)ds + \int_d^c q^+(s)ds\Big], \end{aligned} \tag{1.24}$$

where $d \in [b, c]$ is given in (1.17). *As a result,*

$$\begin{aligned} &2^{p_2+p_3}\Big(\int_a^c r_1^{-1/p_1}(s)ds\Big)^{-p_3}\Big(\int_a^c r_2^{-1/p_2}(s)ds\Big)^{-p_2} \\ &< \max_{a\le d\le c}\Big[\int_a^d q^-(s)ds + \int_d^c q^+(s)ds\Big], \end{aligned} \tag{1.25}$$

where $d \in [a, c]$, $q^-(t)$, and $q^+(t)$ are given in (1.17), (1.21) *and* (1.22), *respectively.*

In 2016, Dhar and Kong [12] obtained the following result for third-order linear differential equation (1.7).

Theorem 1.8 ([12, Theorem 2.1]). *If $y(t)$ is a nontrivial solution of* (1.7) *with* (1.8), *then one of the following holds:*

(a) $2 < \int_a^c (s-a)(c-s)q^-(s)ds$
(b) $2 < \int_a^c (s-a)(c-s)q^+(s)ds$
(c) $2 < \int_a^b (s-a)(b-s)q^-(s)ds + \int_b^c (s-b)(c-s)q^+(s)ds$,

where $q^-(t)$ and $q^+(t)$ are given in (1.21) *and* (1.22), *respectively.*

In 2003, Yang [26] obtained the Lyapunov-type inequalities for the following $(2n+1)$-th order differential equations

$$y^{(2n+1)} + q(t)y = 0 \tag{1.26}$$

for $n \in \mathbb{N}$ and n-th order differential equations

$$y^{(n)} + q(t)y = 0 \tag{1.27}$$

for $n \geq 2$, $n \in \mathbb{N}$, as follows.

Theorem 1.9 ([26, Theorem 1])**.** *If $y(t)$ is a nontrivial solution of* (1.26) *satisfying the conditions*

$$y^{(i)}(a) = y^{(i)}(b) = 0 \tag{1.28}$$

for $i = 0, 1, \dots, n-1$ and

$$y^{(2n)}(t) \text{ has a zero } d \in (a,b), \tag{1.29}$$

then

$$\frac{n! 2^{n+1}}{(b-a)^{2n}} < \int_a^b |q(s)| ds\,. \tag{1.30}$$

Theorem 1.10 ([26, Theorem 2])**.** *If $y(t)$ is a solution of* (1.27) *satisfying the conditions*

$$y(a) = y(t_2) = \dots = y(t_{n-1}) = y(b) = 0, \tag{1.31}$$

where $a = t_1 < t_2 < \dots < t_{n-1} < t_n = b$ $y(t) \neq 0$ for $t \in (t_k, t_{k+1})$, $k = 1, 2, \dots, n-1$, then

$$\frac{(n-2)! n^{n-1}}{(b-a)^{n-1}(n-1)^{n-2}} < \int_a^b |q(s)|\, ds\,. \tag{1.32}$$

In 2010, Çakmak [8] obtained the following Lyapunov-type inequality for problem (1.27) with (1.31) by fixing the fault in Theorem 1.10 given by Yang [26].

Theorem 1.11 ([8, Theorem 1])**.** *If $y(t)$ is a nontrivial solution of* (1.27) *with* (1.31)*, then*

$$\frac{(n-2)! n^{n}}{(b-a)^{n-1}(n-1)^{n-1}} < \int_a^b |q(s)|\, ds\,. \tag{1.33}$$

Recently, Dhar and Kong [11] obtained Lyapunov-type inequalities for odd-order linear differential equations

$$y^{(2n+1)} + (-1)^{n-1} q(t)y = 0 \tag{1.34}$$

for $n \in \mathbb{N}$.

Theorem 1.12 ([11, Theorem 2.1])**.** *If $y(t)$ is a nontrivial solution of* (1.34) *satisfying the conditions*

$$y^{(i+1)}(a) = y^{(i+1)}(c) = 0 \tag{1.35}$$

for $i = 0, 1, \dots, n-1$ and

$$y(b) = 0 \quad \text{for } b \in [a,c], \tag{1.36}$$

then

$$\frac{(2n-1)! 2^{2n}}{(c-a)^{2n} S_n} < \int_a^c |q(s)|\, ds\,, \tag{1.37}$$

where

$$S_n = \sum_{j=0}^{n-1}\sum_{k=0}^{j} 2^{2k-2j}\binom{n-1+j}{j}\binom{j}{k}B(n+1,n+k-j), \tag{1.38}$$

$$B(\alpha,\beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz \quad \text{the Beta function for } \alpha,\beta>0\,. \tag{1.39}$$

Theorem 1.13 ([11, Theorem 2.2]). *Assume that $y(t)$ is a nontrivial solution of* (1.34) *with* (1.35).

(a) *If $y(b)=0$ for $b\in(a,c)$ and $y(t)\neq 0$ for $t\in[a,b)\cup(b,c]$, then one of the following holds:*
 (i) $\frac{(2n-1)!2^{2n}}{(c-a)^{2n}S_n} < \int_a^c q^-(s)ds$
 (ii) $\frac{(2n-1)!2^{2n}}{(c-a)^{2n}S_n} < \int_a^c q^+(s)ds$
 (iii) $\frac{(2n-1)!2^{2n}}{(c-a)^{2n}S_n} < \int_a^b q^-(s)ds + \int_b^c q^+(s)ds.$

(b) *If $y(a)=0$ and $y(t)\neq 0$ for $t\in(a,c]$, then*
$$\frac{(2n-1)!2^{2n}}{(c-a)^{2n}S_n} < \int_a^c q^+(s)ds. \tag{1.40}$$

(c) *If $y(c)=0$ and $y(t)\neq 0$ for $t\in[a,c)$, then*
$$\frac{(2n-1)!2^{2n}}{(c-a)^{2n}S_n} < \int_a^c q^-(s)ds, \tag{1.41}$$
where $q^-(t)$, $q^+(t)$, and S_n are given in (1.21), (1.22), *and* (1.38), *respectively.*

In this paper, we use Green's functions to obtain the best constants of Lyapunov-type inequalities for the problems (1.7) with (1.8) or (1.9) in the literature. In addition, we obtain lower bounds for the distance between two points of a solution of the problems (1.7) with (1.8) or (1.9).

2. Some preliminary lemmas

We state important lemmas which we will use in the proofs of our main results. In the following lemma, we construct Green's function for the third order nonhomogeneous differential equation

$$y''' = g(t) \tag{2.1}$$

with the three-point boundary conditions (1.8) inspired by Murty and Sivasundaram [20] as follows.

Lemma 2.1. *If $y(t)$ is a solution of* (2.1) *satisfying $y(a)=y(b)=y(c)=0$ with $a<b<c$ and $y(t)\not\equiv 0$ for $t\in(a,b)\cup(b,c)$, then*

$$y(t) = \int_a^c G_c(t,s)g(s)ds, \tag{2.2}$$

where, for $t \in [a,b]$,

$$G_c(t,s)=\begin{cases} G_{c1}(t,s)=\frac{(s-a)^2(b-t)(c-t)}{2(b-a)(c-a)}, & a\le s<t\le b<c\\ G_{c2}(t,s)=\frac{(t-a)^2(b-s)(c-s)}{2(b-a)(c-a)} & \\ +\frac{(s-t)(t-a)[(b-s)(c-a)+(c-s)(b-a)]}{2(b-a)(c-a)}, & a\le t\le s<b<c\\ G_{c3}(t,s)=\frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)}, & a<t<b<s<c \end{cases} \tag{2.3}$$

and for $t\in[b,c]$,

$$G_c(t,s)=\begin{cases} G_{c4}(t,s)=-\frac{(s-a)^2(t-b)(c-t)}{2(b-a)(c-a)}, & a<s<b<t<c\\ G_{c5}(t,s)=-\frac{(c-t)^2(s-b)(s-a)}{2(c-a)(c-b)} & \\ -\frac{(t-s)(c-t)[(s-b)(c-a)+(s-a)(c-b)]}{2(c-a)(c-b)}, & a<b<s\le t\le c\\ G_{c6}(t,s)=-\frac{(c-s)^2(t-b)(t-a)}{2(c-a)(c-b)}, & a<b\le t<s\le c. \end{cases} \tag{2.4}$$

Proof. Integrating (1.7) from a to t to find y, we obtain

$$y''(t)=d_2+\int_a^t g(s)\,ds, \tag{2.5}$$

$$y'(t)=d_1+d_2(t-a)+\int_a^t (t-s)g(s)\,ds, \tag{2.6}$$

$$y(t)=d_0+d_1(t-a)+d_2\frac{(t-a)^2}{2}+\int_a^t\frac{(t-s)^2}{2}g(s)\,ds\,. \tag{2.7}$$

Thus, the general solution of (1.7) is (2.7).

Now, by using the boundary conditions (1.8), we find the constants d_0, d_1, and d_2. Thus, $y(a)=0$ implies $d_0=0$ and $y(b)=y(c)=0$ imply

$$d_1=\int_a^c\frac{(c-s)^2(b-a)}{2(c-a)(c-b)}g(s)\,ds-\int_a^b\frac{(b-s)^2(c-a)}{2(c-b)(b-a)}g(s)\,ds, \tag{2.8}$$

$$d_2=\int_a^b\frac{(b-s)^2}{(c-b)(b-a)}g(s)\,ds-\int_a^c\frac{(c-s)^2}{(c-a)(c-b)}g(s)\,ds. \tag{2.9}$$

Substituting the constants d_0, d_1, and d_2 in the general solution (2.7), we obtain

$$\begin{aligned} y(t)=&\int_a^t\Big(\frac{(c-s)^2(t-a)(b-t)}{2(c-a)(c-b)}-\frac{(b-s)^2(t-a)(c-t)}{2(b-a)(c-b)}+\frac{(t-s)^2}{2}\Big)g(s)ds\\ &+\int_t^b(\frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)}-\frac{(b-s)^2(t-a)(c-t)}{2(b-a)(c-b)})g(s)ds\\ &+\int_b^c\frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)}g(s)ds \end{aligned} \tag{2.10}$$

for $t\in[a,b]$ and

$$\begin{aligned} y(t)=&\int_a^b\Big(\frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)}-\frac{(b-s)^2(t-a)(c-t)}{2(b-a)(c-b)}+\frac{(t-s)^2}{2}\Big)g(s)ds\\ &+\int_b^t(\frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)}+\frac{(t-s)^2}{2})g(s)ds \end{aligned}$$

$$+\int_t^c \frac{(c-s)^2(b-t)(t-a)}{2(c-a)(c-b)} g(s)ds$$

for $t\in[b,c]$. This completes the proof. □

Consider the function $G_c(t,s)$ for $t\in[a,b]$. It is easy to see that

$$\begin{aligned} 0\le G_{c1}(t,s)\le G_{c1}(s) &= \frac{(s-a)^2(b-s)(c-s)}{2(b-a)(c-a)} \\ \le G_{c1}^*(s) &= \frac{(s-a)^2(c-s)^2}{2(b-a)(c-a)} \end{aligned} \tag{2.11}$$

for $a\le s<t\le b<c$. Since the function $G_{c1}^*(s)$ takes the maximum value at $\frac{a+c}{2}$, i.e.

$$G_{c1}^*(s)\le \max_{a\le s\le b} G_{c1}^*(s) = G_{c1}^*(\frac{a+c}{2}) = \frac{(c-a)^3}{32(b-a)}. \tag{2.12}$$

Thus, $0\le G_{c1}(t,s)\le G_{c1}^*(s)\le \frac{(c-a)^3}{32(b-a)}$ for $a\le s<t\le b$. Now, we consider $0\le G_{c2}(t,s)$ for $a\le t\le s<b<c$. Let $g_{c1}(t,s)=\frac{(t-a)^2(b-s)(c-s)}{2(b-a)(c-a)}$ and $g_{c2}(t,s)=\frac{(s-t)(t-a)[(b-s)(c-a)+(c-s)(b-a)]}{2(b-a)(c-a)}$. We know that

$$\max_{a\le t\le s<b} G_{c2}(t,s)\le \max_{a\le t\le s<b} g_{c1}(t,s)+\max_{a\le t\le s<b} g_{c2}(t,s). \tag{2.13}$$

Thus, we find the maximum value of the functions $g_{c1}(t,s)$ and $g_{c2}(t,s)$. It is easy to see that from (2.12),

$$0\le g_{c1}(t,s)\le G_{c1}^*(s)\le \frac{(c-a)^3}{32(b-a)}$$

for $a\le t\le s<b$. Now, we find the absolute maximum of $g_{c2}(t,s)$. $g_{c2}(t,s)$ takes its maximum value at the point

$$(t_0,s_0)=\Big(\frac{4a^2-ab-ac-2bc}{3(2a-b-c)},\frac{ab+ac+2a^2-4bc}{3(2a-b-c)}\Big),$$

and its maximum value is

$$g_{c2}(\frac{4a^2-ab-ac-2bc}{3(2a-b-c)},\frac{ab+ac+2a^2-4bc}{3(2a-b-c)})=\frac{4}{27}(\frac{(c-a)(b-a)}{2a-b-c})^2.$$

Thus, we have

$$G_c(t,s)\le \min\Big\{\frac{(c-a)^3}{32(b-a)},\frac{(c-a)^3}{32(b-a)}+\frac{4}{27}\Big(\frac{(c-a)(b-a)}{2a-b-c}\Big)^2\Big\}=\frac{(c-a)^3}{32(b-a)} \tag{2.14}$$

for $a\le t,\ s<b$ [1, 26]. Similarly, we obtain

$$0\le G_{c3}(t,s)\le G_{c3}(s)=\frac{(c-s)^2(s-a)^2}{2(c-a)(c-b)}\le \frac{(c-a)^3}{32(c-b)} \tag{2.15}$$

for $a<t<b<s<c$. Therefore, we have

$$G_c(t,s)\le \begin{cases} \frac{(c-a)^3}{32(b-a)}, & a\le t,s\le b,\\ \frac{(c-a)^3}{32(c-b)}, & t<b<s<c \end{cases} \tag{2.16}$$

for $t\in[a,b]$. Similarly, it is easy to see that we have

$$|G_c(t,s)|\le \begin{cases} \frac{(c-a)^3}{32(b-a)}, & a<s<b<t,\\ \frac{(c-a)^3}{32(c-b)}, & b\le t,s\le c, \end{cases} \tag{2.17}$$

for $t \in [b, c]$.

Now, we give another important lemma. In the following lemma, we construct Green's function for the third order nonhomogeneous differential equation (2.1) with the three-point boundary conditions (1.9) inspired by Moorti and Garner [19] as follows.

Lemma 2.2 ([19, Table 1]). *If $y(t)$ is a solution of (2.1) satisfying $y(a) = y''(d) = y(b) = 0$ with $a \le d \le b$ and $y(t) \not\equiv 0$ for $t \in (a, b)$, then*

$$y(t) = \int_a^b G_d(t,s)g(s)\,ds, \tag{2.18}$$

holds, where for $d < s$,

$$G_d(t,s) = \begin{cases} G_{d1}(t,s) = \frac{(t-a)(b-s)^2}{2(b-a)}, & a \le t \le s \le b, \\ G_{d2}(t,s) = \frac{(s-a)(b-t)^2}{2(b-a)} & \\ +\frac{(t-s)(b-t)(b+s-2a)}{2(b-a)}, & a \le s < t \le b, \end{cases} \tag{2.19}$$

and for $s \le d$,

$$G_d(t,s) = \begin{cases} G_{d3}(t,s) = -\frac{(t-a)^2(b-s)}{2(b-a)} & \\ -\frac{(s-t)(t-a)(2b-a-s)}{2(b-a)}, & a \le t \le s \le b, \\ G_{d4}(t,s) = -\frac{(s-a)^2(b-t)}{2(b-a)}, & a \le s < t \le b\,. \end{cases} \tag{2.20}$$

Consider the function $G_d(t,s)$ for $d < s$. It is easy to see that

$$0 \le G_{d1}(t,s) \le G_{d1}(s) = \frac{(s-a)(b-s)^2}{2(b-a)} \tag{2.21}$$

for $a \le t \le s \le b$. Since the function $G_{d1}(s)$ takes the maximum value at $\frac{2a+b}{3}$, i.e.

$$G_{d1}(s) \le \max_{a \le s \le b} G_{d1}(s) = G_{d1}(\frac{2a+b}{3}) = \frac{2(b-a)^2}{27}. \tag{2.22}$$

Thus, $0 \le G_{d1}(t,s) \le G_{d1}(s) \le \frac{2(b-a)^2}{27}$ for $a \le t \le s \le b$. Now, we consider $0 \le G_{d2}(t,s)$ for $a \le s < t \le b$. Let $g_{d1}(t,s) = \frac{(s-a)(b-t)^2}{2(b-a)}$ and $g_{d2}(t,s) = \frac{(t-s)(b-t)(b+s-2a)}{2(b-a)}$. We know that

$$\max_{a \le s < t \le b} G_{d2}(t,s) \le \max_{a \le s < t \le b} g_{d1}(t,s) + \max_{a \le s < t \le b} g_{d2}(t,s). \tag{2.23}$$

Thus, we find the maximum value of the functions $g_{d1}(t,s)$ and $g_{d2}(t,s)$. It is easy to see that from (2.22), $0 \le g_{d1}(t,s) \le G_{d1}(s) \le \frac{2(b-a)^2}{27}$ for $a \le s < t \le b$. Now, we find the absolute maximum of $g_{d2}(t,s)$. $g_{d2}(t,s)$ takes its maximum value at the point

$$(t_0, s_0) = (\frac{2a+b}{3}, \frac{4a-b}{3}),$$

and its maximum value is

$$g_{d2}(\frac{2a+b}{3}, \frac{4a-b}{3}) = \frac{4(b-a)^2}{27}.$$

Thus,

$$G_d(t,s) \le \min\big\{\frac{2(b-a)^2}{27}, \frac{2(b-a)^2}{27} + \frac{4(b-a)^2}{27}\big\} = \frac{2(b-a)^2}{27} \tag{2.24}$$

for $d < s$. Similarly, it is easy to see that we have

$$|G_d(t,s)| \le \frac{2(b-a)^2}{27} \tag{2.25}$$

for $s \le d$.

Remark 2.3. It is easy to see that if we take $d = a$ or $d = b$ in Lemma 2.2, the problems (1.7) with (1.9) become two-point boundary value problems.

3. Main results

Now, we give one of main results of this paper.

Theorem 3.1. *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.8)*, then*

$$C \le \int_a^c |q(s)|\,ds\,, \tag{3.1}$$

where

$$C = \min\Big\{\frac{32(c-b)}{(c-a)^3}, \frac{32(b-a)}{(c-a)^3}\Big\}.$$

Proof. Let $y(a) = y(b) = y(c) = 0$ where $a, b, c \in \mathbb{R}$ with $a < b < c$ are three points, and y is not identically zero on $(a,b) \cup (b,c)$. From (2.2), (2.16), and (2.17), we obtain

$$\begin{aligned} |y(t)| &\le \int_a^c |G_c(t,s)||y'''(s)|ds \qquad (3.2)\\ &\le \int_a^b \frac{(c-a)^3}{32(b-a)}|y'''(s)|ds + \int_b^c \frac{(c-a)^3}{32(c-b)}|y'''(s)|ds \le \frac{1}{C}\int_a^c |y'''(s)|ds. \qquad (3.3) \end{aligned}$$

From (1.7) and inequality (3.3), we obtain

$$|y'''(t)| = |q(t)||y(t)| \le \frac{|q(t)|}{C}\int_a^c |y'''(s)|ds. \tag{3.4}$$

Integrating from a to c both sides of (3.4), we obtain

$$\int_a^c |y'''(s)|ds \le \frac{1}{C}\int_a^c |y'''(s)|ds \int_a^c |q(s)|ds. \tag{3.5}$$

Next, we prove that

$$0 < \int_a^c |y'''(s)|ds. \tag{3.6}$$

If (3.6) is not true, then we have

$$\int_a^c |y'''(s)|ds = 0. \tag{3.7}$$

It follows from (3.3) and (3.7) that $y(t) \equiv 0$ for $t \in (a,c)$, which contradicts with (1.8) since $y(t) \ne 0$ for all $t \in (a,c)$. Thus, by using (3.6) in (3.5), we obtain inequality (3.1). □

Remark 3.2. It is easy to see that in the special cases, inequality (3.1) is sharper than (1.10), (1.19), (1.25), (1.32), (1.33), and (1.37) in the sense that they follow from (3.1), but not conversely. Therefore, our result improves Theorems 1.1, 1.5, 1.7, 1.8, and 1.10–1.13, in the special cases. In fact, the Lyapunov-type inequality (3.1) is the best possibility for problem (1.7) with (1.8) in the sense that the constant 32 in the left hand side of (3.1) cannot be replaced by any larger constant.

Note that we can rewrite Green's functions $G_c(t,s)$ given in (2.3) and (2.4) as follows: for $t \in [a,b]$,

$$0 \le G_c(t,s) \le \begin{cases} G_{c1}(s) = \frac{(s-a)^2(b-s)(c-s)}{2(b-a)(c-a)}, & a \le s < t \le b \\ G_{c2}(s) = \frac{(s-a)^2(b-s)(c-s)}{2(b-a)(c-a)} \\ +\frac{(s-a)^2[(b-s)(c-a)+(c-s)(b-a)]}{2(b-a)(c-a)}, & a \le t \le s < b \\ G_{c3}(s) = \frac{(c-s)^2(s-a)^2}{2(c-a)(c-b)}, & b < s < c \end{cases} \tag{3.8}$$

and for $t \in [b,c]$,

$$|G_c(t,s)| \le \begin{cases} G_{c4}(s) = \frac{(s-a)^2(c-s)^2}{2(b-a)(c-a)}, & a < s < b \\ G_{c5}(s) = \frac{(c-s)^2(s-b)(s-a)}{2(c-a)(c-b)} \\ +\frac{(c-s)^2[(s-b)(c-a)+(s-a)(c-b)]}{2(c-a)(c-b)}, & b < s \le t \le c \\ G_{c6}(s) = \frac{(c-s)^2(s-b)(s-a)}{2(c-a)(c-b)}, & b \le t < s \le c\,. \end{cases} \tag{3.9}$$

Thus, for $t \in [a,b]$, we have

$$0 \le G_c(t,s) \le \max_{a \le s \le c} \big\{G_{c1}(s), G_{c2}(s), G_{c3}(s)\big\} = \max_{a \le s \le c} \big\{G_{c2}(s), G_{c3}(s)\big\} \tag{3.10}$$

and for $t \in [b,c]$,

$$|G_c(t,s)| \le \max_{a \le s \le c} \big\{G_{c4}(s), G_{c5}(s), G_{c6}(s)\big\} = \max_{a \le s \le c} \big\{G_{c4}(s), G_{c5}(s)\big\}. \tag{3.11}$$

Thus, from (3.10) and (3.11), we obtain

$$|G_c(t,s)| \le G_{c*}(s) = \max_{a \le s \le c} \big\{G_{c2}(s), G_{c3}(s)\big\} \text{ or } \max_{a \le s \le c} \big\{G_{c4}(s), G_{c5}(s)\big\} \tag{3.12}$$

for $s \in [a,c]$ [1, 26].

The proof of the following result proceeds as in Theorem 3.1 by using (3.12) instead of (3.2) and hence it is omitted.

Theorem 3.3. *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.8)*, then*

$$1 \le \int_a^c G_{c*}(s)|q(s)|\,ds\,, \tag{3.13}$$

where $G_{c*}(s)$ is given in (3.12).

Theorem 3.4. *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.8)*, then*

$$1 \le \int_a^c |G_c(t_0,s)||q(s)|\,ds\,, \tag{3.14}$$

where $|y(t_0)| = \max\{|y(t)| : a \le t \le c\}$.

Proof. Let $y(a) = y(b) = y(c) = 0$ where $a,b,c \in \mathbb{R}$ with $a < b < c$ are three points, and y is not identically zero on $(a,b) \cup (b,c)$. Pick $t_0 \in (a,c)$ so that $|y(t_0)| = \max\{|y(t)| : a \le t \le c\}$. From (1.7) and (2.2), we obtain

$$\begin{aligned} |y(t_0)| &= \Big|\int_a^c G_c(t_0,s)[-q(s)y(s)]ds\Big| \\ &\le \int_a^c |G_c(t_0,s)||q(s)||y(s)|ds \end{aligned} \tag{3.15}$$

and hence

$$|y(t_0)| \leq |y(t_0)| \int_a^c |G_c(t_0,s)||q(s)|ds. \tag{3.16}$$

Dividing both sides by $|y(t_0)|$, we obtain inequality (3.14). □

Now, we give other main results of this paper under three-point boundary conditions (1.9). The proofs of following results are similar to that of Theorems 3.1–3.4 and hence they are omitted.

Theorem 3.5. *If $y(t)$ is a nontrivial solution of problem* (1.7) *with* (1.9)*, then*

$$\frac{27}{2(b-a)^2} \leq \int_a^b |q(s)|\,ds\,. \tag{3.17}$$

Remark 3.6. It is easy to see in the special cases that the inequality (3.17) is sharper than (1.11), (1.15), (1.18), (1.20), and (1.30) in the sense that they follow from (3.17), but not conversely. Therefore, our result improves Theorems 1.2–1.4, 1.6, and 1.9 in the special cases. In fact, the Lyapunov-type inequality (3.17) is the best possibility for the problem (1.7) with (1.9) in the sense that the constant $\frac{27}{2}$ in the left hand side of (3.17) cannot be replaced by any larger constant.

From (2.19) and (2.20), it is easy to see that for $d < s$,

$$G_d(t,s) \leq \begin{cases} G_{d1}(s) = \frac{(s-a)(b-s)^2}{2(b-a)}, & a \leq t \leq s \\ G_{d2}(s) = \frac{(s-a)(b-s)^2}{2(b-a)} + \frac{(b-s)^2(b+s-2a)}{2(b-a)}, & a \leq s \leq t \end{cases} \tag{3.18}$$

and for $s \leq d$,

$$|G_d(t,s)| \leq \begin{cases} G_{d3}(s) = \frac{(s-a)^2(b-s)}{2(b-a)} + \frac{(s-a)^2(2b-a-s)}{2(b-a)}, & a \leq t \leq s \leq b \\ G_{d4}(s) = \frac{(s-a)^2(b-s)}{2(b-a)}, & a \leq s < t \leq b\,. \end{cases} \tag{3.19}$$

Therefore, we have

$$|G_d(t,s)| \leq G_{d*}(s) = \max_{a \leq s \leq b} \{G_{d2}(s), G_{d3}(s)\}. \tag{3.20}$$

Theorem 3.7. *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.9)*, then*

$$1 \leq \int_a^b G_{d*}(s)|q(s)|\,ds\,, \tag{3.21}$$

where $G_{d*}(t)$ is given in (3.20).

Theorem 3.8. *If $y(t)$ is a nontrivial solution of the problem* (1.7) *with* (1.9)*, then*

$$1 \leq \int_a^b |G_d(t_0,s)||q(s)|\,ds\,, \tag{3.22}$$

where $|y(t_0)| = \max\{|y(t)| : a \leq t \leq b\}$.

We may adopt a different point of view and use (3.1) or (3.17) to obtain an extension of the following oscillation criterion due originally to Liapounoff (cf. [4]): $y''(t)$ and $y''(t)y^{-1}(t)$ are continuous for $a \leq t \leq b$, with $y(a) = y(b) = 0$, then

$$\frac{4}{b-a} < \int_a^b |y''(s)y^{-1}(s)|ds. \tag{3.23}$$

Thus, (3.1) or (3.17) leads to the following extension: If $y'''(t)$ and $y'''(t)y^{-1}(t)$ are continuous for $a \le t \le c$ or $a \le t \le b$, $y(t)$ has three points including a, b, c, and d, then

$$C \le \int_a^c |y'''(s)y^{-1}(s)|ds \text{ or } \frac{27}{2(b-a)^2} \le \int_a^b |y'''(s)y^{-1}(s)|\,ds. \tag{3.24}$$

Now, we give another application of the obtained Lyapunov-type inequalities for the eigenvalue problem

$$y''' + \lambda h(t)y = 0 \tag{3.25}$$

under three points boundary conditions (1.8) or (1.9). Thus, if there exists a nontrivial solution $y(t)$ of linear homogeneous problem (3.25), then we have

$$\frac{C}{\int_a^c |h(s)|ds} \le \lambda \quad \text{or} \quad \frac{27}{2(b-a)^2\int_a^b |h(s)|ds} \le \lambda. \tag{3.26}$$

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Mustafa Fahri Aktaş
Gazi University, Faculty of Sciences, Department of Mathematics, 06500 Teknikokullar, Ankara, Turkey
E-mail address: mfahri@gazi.edu.tr

Devrim Çakmak
Gazi University, Faculty of Education, Department of Mathematics Education, 06500 Teknikokullar, Ankara, Turkey
E-mail address: dcakmak@gazi.edu.tr