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# ASYMPTOTIC POWER TYPE BEHAVIOR OF SOLUTIONS TO A NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract. This article concerns a general fractional differential equation of order between 1 and 2. We consider the cases where the nonlinear term contains or does not contain other (lower order) fractional derivatives (of Riemann-Liouville type). Moreover, the nonlinearity involves also a nonlinear non-local in time term. The case where this non-local term has a singular kernel is treated as well. It is proved, in all these situations, that solutions approach power type functions at infinity.

## 1. Introduction

We consider the initial value problem

$$\begin{gathered}(D_{0^+}^{\alpha+1}y)(t) = f\Big(t, (D_{0+}^{\beta}y)(t), \int_0^t k(t,s,(D_{0+}^{\gamma}y)(s))ds\Big), \quad t>0,\\ (I_{0^+}^{1-\alpha}y)(0^+) = a_1, \quad (D_{0+}^{\alpha}y)(0^+) = a_2, \quad a_1,a_2\in\mathbb{R},\end{gathered} \tag{1.1}$$

where $D_{0^+}^{\alpha+1}$, $D_{0^+}^{\beta}$ and $D_{0^+}^{\gamma}$ are the Riemann-Liouville fractional derivatives of orders $\alpha+1$, $\beta$ and $\gamma$, respectively, $0\le\beta\le\alpha<1$ and $0\le\gamma\le\alpha<1$. The definition of the Riemann-Liouville fractional derivative is given in the next section. Notice that $D_{0^+}^{\alpha+1} = DD_{0^+}^{\alpha} = (D_{0^+}^{\alpha})'$, $0<\alpha<1$.

We study the asymptotic behavior of solutions of this nonlinear fractional integro-differential problem. Different types of the nonlinear function $f$ and the kernel $k$ are discussed. In this regard, we consider the case of fractional and non-fractional source terms and also the case of singular kernels.

It is of great importance to have an idea about the behavior of solutions for large values of the time variable. Unfortunately, relatively few problems only can be solved explicitly. Therefore there is a need to find analytical techniques which allow us to explore the behavior of solutions without solving the differential equations. The study of asymptotically linear solutions to linear and nonlinear ordinary differential equations is important in many fields like fluid mechanics, differential geometry, bidimensional gravity, Jacobi fields, etc. see e.g. [17].

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In many cases, the main idea to study the asymptotic behavior of solutions is to establish sufficient reasonable conditions ensuring comparison or similarity with the long-time behavior of solutions of simpler differential equations. This important issue has attracted many researchers, see [13, 16, 21, 22, 25].

Recently, some papers discussed the issue of asymptotic behavior for some types of fractional differential equations, see [5, 7, 9, 12, 18, 20]. In 2004, Momani, et al. [19] discussed the Lyapunov stability and asymptotic stability for solutions of the fractional integro-differential equation

$$(D_{a^+}^{\alpha}y)(t) = f(t, y(t)) + \int_a^t k(t, s, y(s))ds, \quad 0 < \alpha \leq 1, \ t \geq a, \tag{1.2}$$

with the initial condition $(I_{a^+}^{1-\alpha}y)(a^+) = c_0 \in \mathbb{R}$. The assumptions

$$\begin{gathered} |f(t, y(t))| \leq \gamma(t)|y|, \\ \int_s^t k(\sigma, s, y(s))d\sigma \leq \delta(t)|y|, \quad s \in [a, t], \end{gathered}$$

where $\gamma(t)$ and $\delta(t)$ are continuous nonnegative functions and

$$\sup_{t \geq a} \int_a^t (t-s)^{\alpha-1}[\gamma(s) + \delta(s)]ds < \infty,$$

were imposed. The authors proved that every solution $y(t)$ of (1.2) satisfies

$$|y(t)| \leq \frac{|c_0|}{\Gamma(\alpha)}(t-a)^{\alpha-1} \exp\Big\{\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}[\gamma(s) + \delta(s)]ds\Big\} < \infty,$$

and if

$$\int_a^t (t-s)^{\alpha-1}[\gamma(s) + \delta(s)]ds = O((t-a)^{\alpha-1}),$$

then $|y(t)| \leq C_0(t-a)^{\alpha-1}$ where $C_0$ is a positive constant, and hence the solution of (1.2) is asymptotically stable.

Furati and Tatar [10] considered (1.2) subject to the initial condition

$$\lim_{t \to a^+}(t^{1-\alpha}y(t)) = b, \quad b \in \mathbb{R}, \ 0 < \alpha < 1, \ a = 0,$$

and showed that solutions decay polynomially for some nonlinear functions $f$ and $k$. When $k \equiv 0$, they proved in [11] that solutions of the problem exist globally and decay as a power function in the space $C_{1-\alpha}^{\alpha}[0, \infty)$ defined in (3.1), see Section 2. In 2007, the same authors considered in [9] the equation (1.2) and found uniform bounds for solutions and also provided sufficient conditions assuring decay of power type for the solutions.

In 2015, Medveď and Pospíšil considered in the paper [18] a more general case when the right-hand side depends on Caputo fractional derivatives of the solution. They proved that there exists a constant $b \in \mathbb{R}$ such that any global solution of the initial value problem

$$\begin{gathered} ({}^CD_{a^+}^{\alpha}x)(t) = f\Big(t, x(t), x'(t), \dots, x^{(n-1)}(t), ({}^CD_{a^+}^{\alpha_1}x)(t), \dots, ({}^CD_{a^+}^{\alpha_m}x)(t)\Big), \\ x^{(i)}(a) = c_i, \quad i = 0, 1, \dots, n-1, \ n \in \mathbb{N}, \end{gathered}$$

where $t \geq a$ and $n - 1 < \alpha_j < \alpha < n$, $j = 1, 2, \dots, m$, $m \in \mathbb{N}$, is asymptotic to $bt^r$ with $r = \max\{n-1, \alpha_m\}$.

To the best of our knowledge, there are no similar investigations on the asymptotic behavior of solutions for fractional integro-differential equations of type (1.1).

There is a great volume of literature on the well-posedness for various classes of fractional differential and integro-differential equations; see [1, 2, 3, 4, 6, 14, 27, 28, 29]. In fact most of the analytical investigations are on existence and uniqueness. Several nonlinearities of the form

$$f(t,y),\quad f(t,y,D^{\beta}_{0+}y),\quad f\Big(t,y,D^{\beta}_{0+}y,\int_0^t k(s,t,D^{\gamma}_{0+}y(s))ds\Big),$$

(with different kinds of fractional derivatives) or even more general ones have been treated. The local existence has been proved under much weaker conditions than those for the asymptotic behavior. For our purpose here, the local existence holds under the simple continuity of the nonlinearities. In this paper we will be concerned mainly with the asymptotic properties of solutions. Therefore, the local existence (which we will assume throughout this document) justifies our investigations. There is no need for uniqueness as our results will apply for all possible solutions.

The rest of this paper is organized as follows. In Section 2 we present the used notations, underlying function spaces, background material and some preliminary results. It contains, in particular, the definitions and basic properties of the fractional integrals and derivatives used in this paper. Some useful lemmas and inequalities that will be used later in our proofs are listed there. The asymptotic behavior of solutions for fractional integro-differential equations of type (1.1) is studied in detail in Section 3. Finally, we illustrate our findings by an example in the last section, Section 4.

## 2. Preliminaries

In this section we briefly introduce some basic definitions, notions and properties from the theory of fractional calculus.

**Definition 2.1** ([15])**.** Let $-\infty \le a < b \le \infty$. The space $L^p(a,b)$ $(1 \le p \le \infty)$ consists of all (Lebesgue) real-valued measurable functions $f$ on $(a,b)$ for which $\|f\|_p < \infty$, where

$$\|f\|_p = \Big(\int_a^b |f(s)|^p ds\Big)^{1/p},\quad 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname{ess\,sup}_{a\le t\le b}|f(t)|,$$

and $\operatorname{ess\,sup}|f(t)|$ is the essential supremum of the function $|f(t)|$.

**Definition 2.2** ([15])**.** We denote by $C[a,b]$ and $C^n[a,b]$, $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, the spaces of continuous and $n$-times continuously differentiable functions on $[a,b]$, with the norms

$$\|f\|_C = \max_{t\in[a,b]}|f(t)|,$$
$$\|f\|_{C^n} = \sum_{i=0}^n \|f^{(i)}\|_C = \sum_{i=0}^n \max_{t\in[a,b]}|f^{(i)}(t)|,\quad n\in\mathbb{N}_0,$$

respectively, where $C[a,b] = C^0[a,b]$.

**Definition 2.3** ([15])**.** We denote by $C_\gamma[a,b]$, $0 \le \gamma < 1$, the weighted space of continuous functions

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} : (t-a)^\gamma f(t) \in C[a,b]\}, \tag{2.1}$$

with the norm

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_C,$$

In particular, $C[a,b] = C_0[a,b]$.

**Definition 2.4** ([15])**.** For $n \in \mathbb{N}$ and $0 \le \gamma < 1$, we denote by $C^n_\gamma[a,b]$, the following weighted space of continuously differentiable functions up to order $n-1$ with $n$-th derivative in $C_\gamma[a,b]$,

$$C^n_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} : f \in C^{n-1}[a,b],\ f^{(n)} \in C_\gamma[a,b]\},$$

with the norm

$$\|f\|_{C^n_\gamma} = \sum_{k=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_{C_\gamma}\,.$$

In particular, $C_\gamma[a,b] = C^0_\gamma[a,b]$.

Next we introduce some definitions, notation and properties of the Riemann-Liouville fractional derivative.

**Definition 2.5.** The Riemann-Liouville left-sided fractional integral of order $\alpha > 0$ is defined by

$$(I^\alpha_{a+}u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)ds, \quad a < t < b,$$

provided the right-hand side exists. We define $I^0_{a+}u = u$. The function $\Gamma$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0$.

**Definition 2.6.** The Riemann-Liouville left-sided fractional derivative of order $\alpha \ge 0$, is defined by

$$(D^\alpha_{a^+}u)(t) = D^n(I^{n-\alpha}_{a^+}u)(t),\ t > a,$$

where $D^n = \frac{d^n}{dt^n}$, $n = [\alpha] + 1$, $[\alpha]$ is the integral part of $\alpha$. In particular, when $\alpha = m \in \mathbb{N}_0$, it follows from the definition that $D^m_{a^+}u = D^m u$.

The next lemma shows that the Riemann-Liouville fractional integral and derivative of the power functions yield power functions multiplied by certain coefficients and with the order of the fractional derivative added or subtracted from the power.

**Lemma 2.7** ([15])**.** *If $\alpha \ge 0$, $\beta > 0$, then*

$$\big(I^\alpha_{a^+}(s-a)^{\beta-1}\big)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, \quad t > a,$$

$$\big(D^\alpha_{a^+}(s-a)^{\beta-1}\big)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad t > a.$$

The Riemann fractional integration operator $I^\alpha_{a+}$ has the semigroup property expressed in the following lemma.

**Lemma 2.8** ([15]). *Let $\alpha > 0$, $\beta > 0$ and $0 \le \gamma < 1$. Then*

$$I^{\alpha}_{a+} I^{\beta}_{a+} u = I^{\alpha+\beta}_{a+} u,$$

*almost everywhere in $[a,b]$ for $u \in L^p(a,b)$ and holds at any point in $(a,b]$ if $u \in C_\gamma[a,b]$. When $u \in C[a,b]$, this relation is valid at every point in $[a,b]$.*

**Lemma 2.9** ([15]). *Let $0 < \beta \le \alpha$ and $0 \le \gamma < 1$. If $u \in C_\gamma[a,b]$, then*

$$D^{\beta}_{a^+} I^{\alpha}_{a^+} u = I^{\alpha-\beta}_{a^+} u$$

*at every point in $(a,b]$.*

The following result is about the composition $I^{\alpha}_{a+} D^{\alpha}_{a^+}$ of the Riemann-Liouville fractional integration and differentiation operators.

**Lemma 2.10** ([15]). *Let $\alpha > 0$, $0 \le \gamma < 1$, $n = [\alpha] + 1$. If $u \in C_\gamma\ [a,b]$ and $I^{n-\alpha}_{a^+} u \in C^n_\gamma[a,b]$, then*

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} u)(t) = u(t) - \sum_{i=1}^{n} \frac{(D^{n-i} I^{n-\alpha}_{a^+} u)(a)}{\Gamma(\alpha - i + 1)} (t-a)^{\alpha - i}$$

*for all $t \in (a,b]$. In particular, if $0 < \alpha < 1$, $u \in C_\gamma\ [a,b]$ and $I^{1-\alpha}_{a^+} u \in C^1_\gamma[a,b]$, then*

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} u)(t) = u(t) - \frac{(I^{1-\alpha}_{a^+} u)(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}, \tag{2.2}$$

*for all $t \in (a,b]$.*

For more details about fractional integrals and fractional derivatives, the reader is referred to the books [24, 26, 15].

Let $S \subset \mathbb{R}$. For two functions $f, g : S \to \mathbb{R}\backslash\{0\}$, we write $f \propto g$ if $g/f$ is nondecreasing on $S$.

Next, we mention two lemmas, due to Pinto [23], about some useful nonlinear integral inequalities.

**Lemma 2.11** ([23, Theorem 1]). *Let $u, \lambda_i$, $i = 1, \dots, n$ be continuous and nonnegative functions on $I = [a,b]$ and the functions $\omega_i$, $i = 1, \dots, n$ be continuous nonnegative and nondecreasing on $[0,\infty)$ such that $\omega_1 \propto \omega_2 \propto \dots \propto \omega_n$. Assume further that $c$ is a positive constant. If*

$$u(t) \le c + \sum_{i=1}^{n} \int_a^t \lambda_i(s) \omega_i(u(s)) ds, \ t \in [a,b],$$

*then, for $t \in [a, b_1]$,*

$$u(t) \le W_n^{-1}(W_n(c_{n-1}) + \int_a^t \lambda_n(s) ds),$$

*where*

1. *$W_i(v) = \int_{v_i}^{v} \frac{d\tau}{\omega_i(\tau)}, v > 0, v_i > 0, i = 1, \dots, n$ and $W_i^{-1}$ is the inverse function of $W_i$.*
2. *The constants $c_i$ are given by $c_0 = c$ and $c_i = W_i^{-1}(W_i(c_{i-1}) + \int_a^{b_1} \lambda_i(s) ds)$, $i = 1, \dots, n-1$.*

(3) *The number $b_1 \in [a,b]$ is the largest number such that*
$$\int_a^{b_1} \lambda_i(s)ds \le \int_{c_{i-1}}^{\infty} \frac{d\tau}{\omega_i(\tau)}, \quad i = 1,\dots,n.$$

**Lemma 2.12** ([23, Theorem 4])**.** *Let $u$, $\lambda_i, \omega_i$, $i = 1,2,3$ and $c$ be as in Lemma 2.11. If*
$$u(t) \le c + \int_a^t \lambda_1(s)\omega_1(u(s))ds + \int_a^t \lambda_2(s)\omega_2\Big(\int_a^s \lambda_3(\tau)\omega_3(u(\tau))d\tau\Big)ds,$$
*then, for $t \in [a,b_1]$,*
$$u(t) \le W_3^{-1}(W_3(c_2) + \int_a^t \lambda_3(s)ds),$$
*where $W_i, W_i^{-1}$, $i = 1,2,3$ and $c_0, c_1, c_2$ are the same as in Lemma 2.11.*

## 3. Main Results

According to the types of the functions $f$ and $k$, we consider the case of fractional and non-fractional source terms and also the case of singular kernels. We discuss the asymptotic behavior of solutions for the problem (1.1) in the sense of the following definition.

**Definition 3.1.** By a solution $y$ of (1.1), we mean a function $y : (0,b] \to \mathbb{R}$, that is continuable (continuous on $(0,+\infty)$), satisfying the equation and the initial conditions in (1.1) and is in the space $C_{1-\alpha}^{\alpha+1}[0,b]$, $0 < b \le \infty$, defined by
$$C_{1-\alpha}^{\alpha+1}[0,b] = \{y : (0,b] \to \mathbb{R} : y \in C_{1-\alpha}[0,b],\ D_{0^+}^{\alpha+1} y \in C_{1-\alpha}[0,b]\}, \tag{3.1}$$
where the space $C_{1-\alpha}[0,b]$ is defined in (2.1).

We assume that the functions $f$ and $k$ satisfy the hypotheses

(A1) $f(t,u,v)$ is a $C_{1-\alpha}$ function in $D = \{(t,u,v) : t \ge 0,\ u,\ v \in \mathbb{R}\}$.
(A2) $k(t,s,u)$ is continuous in $E = \{(t,s,u) : 0 \le s < t < \infty,\ u \in \mathbb{R}\}$.

Before presenting our main results we need to define the following classes of functions:

**Definition 3.2.** We say that a function $h : [0,\infty) \to [0,\infty)$ is of type $\mathcal{H}_\sigma$ if $h \in C[0,\infty)$ and $t^\sigma h(t) \in L^1(1,\infty)$, $\sigma \ge 0$.

**Definition 3.3.** We say that a function $g$ is of type $\mathcal{G}$ if it is continuous nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$ with $g(v) \le u g(\frac{v}{u})$, $u \ge 1, v > 0$ and $\int_{t_0}^t \frac{d\tau}{g(\tau)} \to \infty$ as $t \to \infty$ for any $t_0 > 0$.

The above classes are not empty. Examples showing this fact are given in the next subsections. We will need to deal with the limit of the ratio of the Riemann-Liouville fractional integral $I_{a^+}^{\alpha+1}$ of a function and the power function $t^\alpha$ as $t \to \infty$. This is treated in the next lemma.

**Lemma 3.4.** *Let $f \in L^1(a,\infty)$, $a \ge 0$. Suppose that $u$ and $v$ are real-valued functions defined on $[a,\infty)$, then*
$$\lim_{t\to\infty} \frac{1}{t^\alpha} \int_a^t (t-s)^\alpha f(s,u(s),v(s))ds = \int_a^\infty f(s,u(s),v(s))ds\,.$$

*Proof.* It is sufficient to prove that

$$\lim_{t\to\infty} \Big|\frac{1}{t^\alpha}\int_a^t (t-s)^\alpha f(s,u(s),v(s))d\tau) - \int_a^\infty f(s,u(s),v(s))ds\Big| = 0\,.$$

Note that

$$\begin{aligned}
&\Big|\frac{1}{t^\alpha}\int_a^t (t-s)^\alpha f(s,u(s),v(s)) - \int_a^\infty f(s,u(s),v(s))ds\Big| \\
&= \Big|\int_a^t (1-\frac{s}{t})^\alpha f(s,u(s),v(s))ds - \int_a^\infty f(s,u(s),v(s))ds\Big| \\
&= \Big|\int_a^\infty (\chi_{[a,t]}(s)(1-\frac{s}{t})^\alpha - 1) f(s,u(s),v(s))ds\Big| \\
&\le \int_a^\infty \big|\chi_{[a,t]}(s)(1-\frac{s}{t})^\alpha - 1\big|\big|f(s,u(s),v(s))\big|ds,
\end{aligned}$$

where

$$\chi_{[a,t]}(s) = \begin{cases} 1, & s\in[a,t] \\ 0, & s\notin[a,t]. \end{cases}$$

Since

$$\lim_{t\to\infty} \chi_{[a,t]}(s)(1-\frac{s}{t})^\alpha = 1,$$

by the Dominated Convergence Theorem [8] we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\Big|\frac{1}{t^\alpha}\int_a^t (t-s)^\alpha f(s,u(s),v(s)) - \int_a^\infty f(s,u(s),v(s))ds\Big| \\
&\le \lim_{t\to\infty}\int_a^\infty \big|\chi_{[a,t]}(s)(1-\frac{s}{t})^\alpha - 1\big||f(s,u(s),v(s))|ds \\
&= \int_a^\infty \lim_{t\to\infty}\big|\chi_{[a,t]}(s)(1-\frac{s}{t})^\alpha - 1\big||f(s,u(s),v(s))|ds = 0,
\end{aligned}$$

which is the desired result. □

The following lemmas will be needed in the next subsections.

**Lemma 3.5.** *Let $y$ be a solution of problem* (1.1) *with $f\in L^1(0,\infty)$. Then*

$$\begin{aligned}
&\lim_{t\to\infty}\frac{y(t)}{t^\alpha} \\
&= \lim_{t\to\infty}\frac{(D^\alpha_{0+}y)(t)}{\Gamma(\alpha+1)} \\
&= \frac{1}{\Gamma(\alpha+1)}\Big(a_2 + \int_0^\infty f\Big(s,(D^\beta_{0+}y)(s),\int_0^s k(s,\tau,(D^\gamma_{0+}y)(\tau))d\tau\Big)ds\Big).
\end{aligned}$$

*Proof.* Applying $I^1_{0+}$ to both sides of the equation in (1.1) yields

$$(D^\alpha_{0+}y)(t) = a_2 + \int_0^t f\Big(s,(D^\beta_{0+}y)(s),\int_0^s k(s,\tau,(D^\gamma_{0+}y)(\tau))d\tau\Big)ds, \tag{3.2}$$

Applying $I^{\alpha}_{a^+}$ to the 3.2 with taking into account Lemmas 2.7, 2.8 and 2.10, we obtain

$$\begin{aligned} y(t) &= \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \Big(I^{\alpha+1}_{0^+} f\Big(s, (D^{\beta}_{0^+}y)(s), \int_0^s k(s,\tau,(D^{\gamma}_{0^+}y)(\tau))d\tau\Big)\Big)(t), \end{aligned} \tag{3.3}$$

for all $t > 0$. Taking the limit of the ratio $\frac{y(t)}{t^{\alpha}}$ as $t \to \infty$ gives the desired result with the help of Lemma 3.4. □

The next two lemmas provide estimates for some integrals which will appear later in our arguments.

**Lemma 3.6.** *Let $b_2, b_3$ and $b_4$ be positive constants and $z(t)$ be a continuous and nonnegative function on $[0,\infty)$. Assume that*

$$z(t) \le b_2 + b_3 t + b_4 t \int_0^t (h_1(s)g_1(z(s)) + h_2(s)g_2(z(s)))ds, \quad t \ge 0, \tag{3.4}$$

*where $h_1, h_2 \in \mathcal{H}_1$ and $g_1, g_2 \in \mathcal{G}$ with $g_1 \propto g_2$. Then*

$$z(t) \le \begin{cases} G_2^{-1}(d_2), & 0 \le t < 1 \\ tG_2^{-1}(d_3), & t \ge 1, \end{cases}$$

*where*

$$d_2 = G_2(d_1) + \int_0^1 h_2(s)ds, \quad d_1 = G_1^{-1}(G_1(d_0) + \frac{\int_0^1 h_1(s)ds}{\Gamma(\alpha+1)}),$$

$$d_0 = b_2 + b_3, \quad d_3 = G_2(e_1) + b_4 \int_1^{\infty} s h_2(s)ds,$$

$$e_1 = G_1^{-1}\Big(G_1(d_4) + b_4 \int_1^{\infty} s h_1(s)ds\Big),$$

$$d_4 = d_0 + b_4 g_1(G_2^{-1}(d_2)) \int_0^1 h_1(s)ds + b_4 g_2(G_2^{-1}(d_2)) \int_0^1 h_2(s)ds,$$

*and $G_i^{-1}$ is the inverse function of $G_i(t) = \int_{t_0}^t \frac{d\tau}{g_i(\tau)}$, $i = 1, 2$, $t \ge t_0 > 0$.*

*Proof.* For $0 \le t < 1$, from (3.4) we obtain

$$z(t) \le b_2 + b_3 + b_4 \int_0^t h_1(s)g_1(z(s))ds + b_4 \int_0^t h_2(s)g_2(z(s))ds.$$

From Lemma 2.11 it follows $z(t) \le G_2^{-1}(d_2)$. For $t \ge 1$, from (3.4) we have

$$\begin{aligned} \frac{z(t)}{t} &\le b_2 + b_3 + b_4 \int_0^t h_1(s)g_1(z(s))ds + b_4 \int_0^t h_2(s)g_2(z(s))ds \\ &\le d_4 + b_4 \int_1^t s h_1(s) g_1(\frac{z(s)}{s})ds + b_4 \int_1^t s h_2(s) g_2(\frac{z(s)}{s})ds\,. \end{aligned} \tag{3.5}$$

Notice that the hypotheses of Lemma 2.11 are satisfied with $b_1 = \infty$ (because $\int_{t_0}^{\infty} \frac{d\tau}{g_i(\tau)} = \infty$, $i = 1, 2$). Then, for $t \in [1, \infty)$ (3.5) leads to

$$\frac{z(t)}{t} \le G_2^{-1}(d_3).$$

This completes the proof. □

**Lemma 3.7.** *Let $b_2, b_3$ and $b_4$ be positive constants and let $z(t)$ be a continuous and nonnegative function on $[0,\infty)$. Assume further that*

$$z(t) \le b_2 + b_3 t + b_4 t \int_0^t \Big(h_1(s)g_1(z(s)) + h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\Big)\Big)ds, \tag{3.6}$$

*for $t \ge 0$, where $h_1, h_3$ are of type $\mathcal{H}_1$, $h_2$ is of type $\mathcal{H}_0$ and $g_i$ is of type $\mathcal{G}$, $i = 1,2,3$ with $g_1 \propto g_2 \propto g_3$. Then*

$$z(t) \le \begin{cases} G_3^{-1}(M), & 0 \le t < 1 \\ tG_3^{-1}(M_1), & t \ge 1, \end{cases}$$

*where*

$$\begin{gathered}
M = G_3(d_2) + \int_0^1 h_3(s)ds, \quad d_2 = G_2^{-1}(G_2(d_1) + b_4\int_0^1 h_2(s)ds), \\
d_1 = G_1^{-1}\Big(G_1(d_0) + b_4\int_0^1 h_1(s)ds\Big), \quad d_0 = b_2 + b_3, \\
M_1 = G_3(e_2) + \int_1^\infty sh_3(s)ds, \quad e_2 = G_2^{-1}\Big(G_2(e_1) + b_4\int_1^\infty h_2(s)ds\Big), \\
e_1 = G_1^{-1}(G_1(M_2) + b_4\int_1^\infty sh_1(s)ds), \\
\begin{aligned}
M_2 = d_0 &+ b_4 g_1(G_3^{-1}(M))\int_0^1 h_1(s)ds \\
&+ b_4 g_2\Big(g_3(G_3^{-1}(M))\int_0^1 h_3(\tau)d\tau\Big)\int_0^1 h_2(s)ds\,.
\end{aligned}
\end{gathered}$$

*Proof.* For $0 \le t < 1$, from (3.6) we obtain

$$z(t) \le b_2 + b_3 + b_4 \int_0^t \Big(h_1(s)g_1(z(s)) + h_2(s)g_2(\int_0^s h_3(\tau)g_3(z(\tau))d\tau)\Big)ds.$$

From Lemma 2.12 it follows that

$$z(t) \le G_3^{-1}(M) \quad \text{for all } 0 \le t < 1\,.$$

For $t \ge 1$, from (3.6) we have

$$\begin{aligned}
\frac{z(t)}{t} &\le d_0 + b_4 \int_0^1 \Big(h_1(s)g_1(G_3^{-1}(M)) + h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3(G_3^{-1}(M))d\tau\Big)\Big)ds \\
&\quad + b_4 \int_1^t \Big(h_1(s)g_1(z(s)) + h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\Big)\Big)ds \\
&\le M_2 + b_4 \int_1^t \Big(h_1(s)g_1(z(s)) + h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\Big)\Big)ds\,.
\end{aligned}$$

Let $u = u_1 + u_2 + u_3$, where

$$\begin{gathered}
u_1(t) = M_2 + b_4\int_0^t h_1(s)g_1(z(s))ds, \\
u_2(t) = b_4\int_0^t h_2(s)g_2(u_3(s))ds, \quad u_3(t) = \int_0^t h_3(s)g_3(z(s))ds, \quad t > 0.
\end{gathered}$$

Differentiating $u$, by the monotonicity of $g_i, i = 1,2,3$, we obtain

$$u'(t) \le b_4 t h_1(t)g_1(u(t)) + b_4 h_2(t)g_2(u(t)) + t h_3(t)g_3(u(t)), \tag{3.7}$$

for all $t \geq 1$. Integrating both sides of (3.7) over $[1, t]$ gives

$$\begin{aligned} u(t) \leq u(1) + b_4 \int_1^t s h_1(s) g_1(u(s)) ds + b_4 \int_1^t h_2(s) g_2(u(s)) ds \\ + \int_1^t s h_3(s) g_3(u(s)) ds \,. \end{aligned} \tag{3.8}$$

Now, since $\int_{t_0}^{\infty} \frac{d\tau}{g_i(\tau)} = \infty$, $i = 1, 2, 3$ for any $t_0 > 0$, the hypotheses of Lemma 2.11 are satisfied with $b_1 = \infty$. Therefore, by Lemma 2.11, the inequality (3.8) leads to

$$u(t) \leq G_3^{-1}(M_1), \quad \text{for all } t \geq 1.$$

The proof is now complete. □

Although the estimates in Lemmas 3.6 and 3.7 are not the best, they ensure that all the involved integrals are bounded, which is the most useful fact we need in the next subsections.

3.1. **Case of a non-fractional source.** In this subsection, we consider problem (1.1) with $\beta = \gamma = 0$ and $0 < \alpha < 1$; that is,

$$\begin{gathered} (D_{0^+}^{\alpha+1} y)(t) = f\Big(t, y(t), \int_0^t k(t, s, y(s)) ds\Big), \; t > 0, \\ (I_{0^+}^{1-\alpha} y)(0^+) = a_1, \quad (D_{0^+}^{\alpha} y)(0^+) = a_2, \quad a_1, a_2 \in \mathbb{R}. \end{gathered} \tag{3.9}$$

First, we need the following condition:

(A3) There are functions $h_1, h_3 \in \mathcal{H}_1, h_2 \in \mathcal{H}_0$ and $g_i \in \mathcal{G}$, $i = 1, 2, 3$ with $g_1 \propto g_2 \propto g_3$ such that

$$|f(t, u, v)| \leq h_1(t) g_1\big(\frac{|u|}{t^{\alpha-1}}\big) + h_2(t) g_2(|v|), \quad (t, u, v) \in D, \tag{3.10}$$

$$|k(t, s, y)| \leq h_3(s) g_3\big(\frac{|y|}{s^{\alpha-1}}\big), \quad (t, s, y) \in E\,. \tag{3.11}$$

Now, we prove the main result in this subsection.

**Theorem 3.8.** *Suppose that $f$ and $k$ satisfy* (A1)–(A3). *Then, any solution of problem* (3.9) *is asymptotic to $ct^{\alpha}$ as $t \to \infty$, for some $c \in \mathbb{R}$.*

*Proof.* Applying $I_{0^+}^{\alpha+1}$ to both sides of the equation in (3.9) gives

$$y(t) = \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^{\alpha}}{\Gamma(\alpha+1)} + \Big(I_{0^+}^{\alpha+1} f\Big(s, y(s), \int_0^s k(s, \tau, y(\tau)) d\tau\Big)\Big)(t).$$

Then, for all $t > 0$,

$$\begin{aligned} \frac{|y(t)|}{t^{\alpha-1}} \leq \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2| t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)} \int_0^t \Big[h_1(s) g_1\big(\frac{|y(s)|}{s^{\alpha-1}}\big) \\ + h_2(s) g_2\Big(\int_0^s h_3(\tau) g_3\big(\frac{|y(\tau)|}{\tau^{\alpha-1}}\big) d\tau\Big)\Big] ds. \end{aligned} \tag{3.12}$$

Let us denote the right hand side of (3.12) by $z(t)$ for all $t > 0$, then

$$\frac{|y(t)|}{t^{\alpha-1}} \leq z(t), \quad \text{for all } t > 0, \tag{3.13}$$

and consequently,

$$\begin{aligned} z(t) \le \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|}{\Gamma(\alpha+1)} t + \frac{t}{\Gamma(\alpha+1)} \int_0^t [h_1(s) g_1(z(s)) \\ + h_2(s) g_2\Big(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau\Big)] ds \quad \text{for all } t > 0. \end{aligned} \tag{3.14}$$

It follows from Lemma 3.7 that

$$z(t) \le t G_3^{-1}(M_1), \quad \text{for all } t \ge 1,$$

and from (3.13) we have

$$\frac{|y(t)|}{t^\alpha} \le M_3 := G_3^{-1}(M_1), \quad \text{for all } t \ge 1. \tag{3.15}$$

Let

$$J := \int_0^t \Big| f\Big(s, y(s), \int_0^s k(s,\tau,y(\tau)) d\tau\Big)\Big| ds, \quad t > 0.$$

Using assumption (A3) and (3.13) we see that

$$\begin{aligned} J \le \int_0^1 \Big[h_1(s) g_1(z(s)) + h_2(s) g_2(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau)\Big] ds \\ + \int_1^t \Big[h_1(s) g_1(z(s)) + h_2(s) g_2(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau)\Big] ds, \quad t \ge 1. \end{aligned} \tag{3.16}$$

The second integral on the right-hand side of (3.16) can be estimated using (3.13) as follows

$$\begin{aligned} J_2 &\le \int_1^t s h_1(s) g_1(M_3) ds + \int_1^t h_2(s) g_2\Big(\int_0^1 h_3(\tau) g_3(z(\tau)) d\tau \\ &\quad + \int_1^s h_3(\tau) g_3(z(\tau)) d\tau\Big) ds \\ &\le g_1(M_3) \int_1^t s h_1(s) ds + g_2\Big(g_3(M_4) \int_0^1 h_3(\tau) d\tau + g_3(M_3) \int_1^t \tau h_3(\tau) d\tau\Big) \\ &\quad \times \int_1^t h_2(s) ds. \end{aligned}$$

for all $t \ge 1$. As $h_1, h_3 \in \mathcal{H}_1, h_2 \in \mathcal{H}_0$, we deduce that $J_2$ is uniformly bounded and so is $J$.

It means that the integral $\int_0^t f(s, y(s), \int_0^s k(s,\tau,y(\tau)) d\tau) ds$ is absolutely convergent and so

$$\lim_{t\to\infty} \int_0^t f\Big(s, y(s), \int_0^s k(s,\tau,y(\tau)) d\tau\Big) ds < \infty. \tag{3.17}$$

Integrating both sides of the equation in(3.9) over the interval $[0,t]$ yields

$$(D_{0+}^\alpha y)(t) = a_1 + \int_0^t f\Big(s, y(s), \int_0^s k(s,\tau,y(\tau)) d\tau\Big) ds.$$

Now, (3.17) ensures that there is a real number $\hat{c}$ such that

$$\lim_{t\to\infty} D_{0+}^\alpha y(t) = \hat{c}.$$

By Lemma 3.5,

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{(D_{0+}^\alpha y)(t)}{\Gamma(\alpha+1)} = c,$$

$c := \frac{\hat{c}}{\Gamma(\alpha+1)}$. This completes the proof. $\square$

3.2. **Case of a singular kernel.** Consider the problem

$$\begin{gathered} D_{0^+}^{\alpha+1} y(t) = f\Big(t, y(t), (I_{0^+}^{\beta} y)(t)\Big), \quad t > 0,\ 0 < \alpha < 1,\ 0 < \alpha + \beta < 1, \\ (I_{0^+}^{1-\alpha} y)(0^+) = a_1, \quad \big(D_{0^+}^{\alpha} y\big)(0^+) = a_2, \quad a_1, a_2 \in \mathbb{R}. \end{gathered} \tag{3.18}$$

To study the asymptotic behavior of solutions for the problem (3.18), we assume that the function $f$ satisfies the condition

(A4) There are functions $h_1, h_2 \in \mathcal{H}_1$ and $g_1, g_2 \in \mathcal{G}$ with $g_1 \propto g_2$ such that
$$|f(t,u,v)| \le h_1(t) g_1\big(\frac{|u|}{t^{\alpha-1}}\big) + h_2(t) g_2\big(\frac{|v|}{t^{\alpha+\beta-1}}\big), \quad (t,u,v) \in D.$$

**Theorem 3.9.** *Suppose that $f$ satisfies conditions* (A1), )A4). *Then, every solution of problem* (3.18) *is asymptotic to $ct^{\alpha}$ when $t \to \infty$, for some $c \in \mathbb{R}$.*

*Proof.* From condition (A4), after applying $I_{0^+}^{\alpha+1}$ to both sides of the equation in (3.18), we have

$$\begin{aligned} t^{1-\alpha} |y(t)| \le \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2| t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)} \int_0^t \Big[ h_1(s) g_1\big(\frac{|y(s)|}{s^{\alpha-1}}\big) \\ + h_2(s) g_2\Big(\frac{|(I_{0^+}^{\beta} y)(s)|}{s^{\alpha+\beta-1}}\Big)\Big] ds, \quad t > 0. \end{aligned} \tag{3.19}$$

Since

$$(I_{0^+}^{\beta} y)(t) = \frac{a_1 t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{a_2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + I_{0^+}^{\alpha+\beta+1} f\big(\tau, y(\tau), (I_{0^+}^{\beta} y)(\tau)\big)(s)(t),$$

for all $t > 0$, we arrive at

$$\begin{aligned} &|(I_{0^+}^{\beta} y)(t)| \\ &\le \frac{|a_1| t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{|a_2| t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + I_{0^+}^{\alpha+\beta+1} \big|f(\tau, y(\tau), (I_{0^+}^{\beta} y)(\tau))\big|(s)(t) \\ &\le \frac{|a_1| t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big(|a_2| + \int_0^t |f(s, y(s), (I_{0^+}^{\beta} y)(s))| ds\Big), \end{aligned}$$

or equivalently with the help of (A4),

$$\begin{aligned} t^{1-\alpha-\beta} |(I_{0^+}^{\beta} y)(t)| \le \frac{|a_1|}{\Gamma(\alpha+\beta)} + \frac{t}{\Gamma(\alpha+\beta+1)} \Big(|a_2| + \int_0^t \Big[h_1(s) g_1\big(\frac{|y(s)|}{s^{\alpha-1}}\big) \\ + h_2(s) g_2\Big(\frac{|(I_{0^+}^{\beta} y)(s)|}{s^{\alpha+\beta-1}}\Big)\Big] ds\Big) \quad \forall t > 0. \end{aligned} \tag{3.20}$$

Now, let

$$z(t) = A_1 + A_2 t + A_3 t \int_0^t \Big(h_1(s) g_1\big(\frac{|y(s)|}{s^{\alpha-1}}\big) + h_2(s) g_2\big(\frac{|(I_{0^+}^{\beta} y)(s)|}{s^{\alpha+\beta-1}}\big)\Big) ds, \tag{3.21}$$

for all $t > 0$, where

$$\begin{gathered} A_1 = \max\big\{\frac{|a_1|}{\Gamma(\alpha)}, \frac{|a_1|}{\Gamma(\alpha+\beta)}\big\}, \quad A_2 = \max\big\{\frac{|a_2|}{\Gamma(\alpha+1)}, \frac{|a_2|}{\Gamma(\alpha+\beta+1)}\big\}, \\ A_3 = \max\big\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha+\beta+1)}\big\}. \end{gathered}$$

It is not difficult to see from the relations (3.19)–(3.21), that

$$t^{1-\alpha}|y(t)|,\ t^{1-\alpha-\beta}|(I_{0^+}^{\beta}y)(t)| \le z(t), \quad t>0,$$

and consequently, for $t>0$,

$$z(t) \le A_1 + A_2 t + A_3 t\int_0^t h_1(s)g_1(z(s))ds + A_3 t\int_0^t h_2(s)g_2(z(s))ds, \quad t>0.$$

It follows from Lemma 3.6 that

$$z(t) \le tG_2^{-1}(d_3), \quad \text{for all } t \ge 1,$$

where $G_2^{-1}$ and $d_3$ are given in Lemma 3.6. Now, the proof can be completed as the proof of Theorem 3.8. $\square$

3.3. **Case of fractional source terms.** In this subsection we study the asymptotic behavior of solutions for problem (1.1) under the following condition:

(A5) There are functions $h_1, h_3 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_0$ and $g_i \in \mathcal{G}$, $i=1,2,3$, with $g_1 \propto g_2 \propto g_3$ such that

$$|f(t,u,v)| \le h_1(t)g_1\big(\frac{|u|}{t^{\alpha-\beta-1}}\big) + h_2(t)g_2(|v|), \quad (t,u,v) \in D,$$
$$|k(t,s,y)| \le h_3(s)g_3\big(\frac{|y|}{t^{\alpha-\gamma-1}}\big), \quad (t,s,y) \in E.$$

The main result of this subsection is as follows.

**Theorem 3.10.** *Suppose that $f$ and $k$ satisfy conditions* (A1), (A2), (A5)*. Then, every solution of the problem* (1.1) *is asymptotic to $ct^{\alpha}$ when $t \to \infty$, for some $c \in \mathbb{R}$.*

*Proof.* Here we have

$$\begin{aligned} y(t) &= \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \Big(I_{0^+}^{\alpha+1} f\Big(s, (D_{0^+}^{\beta}y)(s), \int_0^s k(s,\tau,(D_{0^+}^{\gamma}y)(\tau))d\tau\Big)\Big)(t), \end{aligned} \tag{3.22}$$

$$\begin{aligned} \frac{|y(t)|}{t^{\alpha-1}} &\le \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)}\int_0^t \Big[h_1(s)g_1\Big(\frac{|(D_{0^+}^{\beta}y)(s)|}{s^{\alpha-\beta-1}}\Big) \\ &\quad + h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3\big(\frac{|(D_{0^+}^{\gamma}y)(\tau)|}{\tau^{\alpha-\gamma-1}}\big)d\tau\Big)\Big]ds, \quad t>0. \end{aligned} \tag{3.23}$$

Applying $D_{0^+}^{\beta}$ and $D_{0^+}^{\gamma}$ to both sides of (3.22), and taking Lemma 2.7 and Lemma 2.9 into account, we have

$$\begin{aligned} (D_{0^+}^{\beta}y)(t) &= \frac{a_1 t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{a_2 t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} \\ &\quad + \Big(I_{0^+}^{\alpha+1-\beta} f\Big(s, (D_{0^+}^{\beta}y)(s), \int_0^s k(s,\tau,(D_{0^+}^{\gamma}y)(\tau))d\tau\Big)\Big)(t), \quad t>0, \\ (D_{0^+}^{\gamma}y)(t) &= \frac{a_1 t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} + \frac{a_2 t^{\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} \\ &\quad + \Big(I_{0^+}^{\alpha+1-\gamma} f\Big(s, (D_{0^+}^{\beta}y)(s), \int_0^s k(s,\tau,(D_{0^+}^{\gamma}y)(\tau))d\tau\Big)\Big)(t), \quad t>0, \end{aligned}$$

respectively. Therefore for all $t > 0$,

$$\begin{aligned} &t^{1-(\alpha-\beta)}|(D^{\beta}_{0+}y)(t)| \\ &\leq \frac{|a_1|}{\Gamma(\alpha-\beta)} + \frac{|a_2|t}{\Gamma(1+\alpha-\beta)} + \frac{t}{\Gamma(1+\alpha-\beta)}\int_0^t h_1(s)g_1\Big(\frac{|(D^{\beta}_{0+}y)(s)|}{s^{\alpha-\beta-1}}\Big)ds \\ &\quad + \frac{t}{\Gamma(1+\alpha-\beta)}\int_0^t h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3\Big(\frac{|(D^{\gamma}_{0+}y)(\tau)|}{\tau^{\alpha-\gamma-1}}\Big)d\tau\Big)ds, \quad t>0, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &t^{1-(\alpha-\gamma)}|(D^{\gamma}_{0+}y)(t)| \\ &\leq \frac{|a_1|}{\Gamma(\alpha-\gamma)} + \frac{|a_2|t}{\Gamma(1+\alpha-\gamma)} + \frac{t}{\Gamma(1+\alpha-\gamma)}\int_0^t h_1(s)g_1\Big(\frac{|(D^{\beta}_{0+}y)(s)|}{s^{\alpha-\beta-1}}\Big)ds \\ &\quad + \frac{t}{\Gamma(1+\alpha-\gamma)}\int_0^t h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3\Big(\frac{|(D^{\gamma}_{0+}y)(\tau)|}{\tau^{\alpha-\gamma-1}}\Big)d\tau\Big)ds, \quad t>0. \end{aligned} \tag{3.25}$$

Now, let

$$\begin{gathered} b_2 = |a_1|\max\Big\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}, \frac{1}{\Gamma(\alpha-\gamma)}\Big\}, \quad b_3 = |a_2|b_4, \\ b_4 = \max\Big\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(1+\alpha-\beta)}, \frac{1}{\Gamma(1+\alpha-\gamma)}\Big\}, \\ \begin{aligned} z(t) = b_2 + b_3 t + b_4 t\int_0^t \Big[&h_1(s)g_1\Big(\frac{|(D^{\beta}_{0+}y)(s)|}{s^{\alpha-\beta-1}}\Big) \\ &+ h_2(s)g_2\Big(\int_0^s h_3(\tau)g_3(\frac{|(D^{\gamma}_{0+}y)(\tau)|}{\tau^{\alpha-\gamma-1}})d\tau\Big)\Big]ds, \quad t>0. \end{aligned} \end{gathered}$$

Then, for all $t > 0$ we obtain

$$\frac{|y(t)|}{t^{\alpha-1}}, \ \frac{|(D^{\beta}_{0+}y)(t)|}{t^{\alpha-\beta-1}}, \ \frac{|(D^{\gamma}_{0+}y)(t)|}{t^{\alpha-\gamma-1}} \leq z(t). \tag{3.26}$$

The remaining steps of the proof are similar to those of the proof of Theorem 3.8. □

## 4. Example

The next example provides some functions to which Theorem 3.8 applies.

**Example 4.1.** Consider the equation

$$(D^{\alpha+1}_{0^+}y)(t) = t^{\mu_1}e^{-t}\ y(t) + t^{\mu_2}e^{-t}\int_0^t s^{\mu_3}e^{-(s+t)}y(s)ds, \quad t>0, \tag{4.1}$$

where $0 < \alpha < 1$, $\mu_1 > -\alpha - 1$, $\mu_2 > -1$ and $\mu_3 > -\alpha - 1$. Notice that the right-hand side of the equation (4.1) can be rewritten as

$$t^{\mu_1+\alpha-1}e^{-t}\frac{y(t)}{t^{\alpha-1}} + t^{\mu_2}e^{-t}\int_0^t s^{\mu_3+\alpha-1}e^{-(s+t)}\frac{y(s)}{s^{\alpha-1}}ds, \quad t>0.$$

Let $h_1(t) = t^{\mu_1+\alpha-1}e^{-\rho_1 t}$, $h_2(t) = t^{\mu_2}e^{-\rho_2 t}$, $h_3(t) = t^{\mu_3+\alpha-1}e^{-\rho_3 t}$ for $t > 0$,

$$g_i(t) = t, \quad 0 < \rho_i \leq 1, \quad i = 1, 2, 3,\ t > 0.$$

Then conditions (A1)–(A3) are satisfied,

$$\int_1^\infty th_1(t)dt < \int_0^\infty th_1(t)dt = \int_0^\infty t^{\mu_1+\alpha}e^{-\rho_1 t}dt = \frac{\Gamma(\mu_1+\alpha+1)}{\rho_1^{\mu_1+\alpha+1}} < \infty,$$
$$\int_1^\infty h_2(t)dt < \int_0^\infty h_2(t)dt = \int_0^\infty t^{\mu_2}e^{-\rho_2 t}dt = \frac{\Gamma(\mu_2+1)}{\rho_2^{\mu_2+1}} < \infty,$$
$$\int_1^\infty th_3(t)dt < \int_0^\infty th_3(t)dt = \int_0^\infty t^{\mu_3+\alpha}e^{-\rho_3 t}dt = \frac{\Gamma(\mu_3+\alpha+1)}{\rho_3^{\mu_3+\alpha+1}} < \infty,$$
$$\int_{t_0}^\infty \frac{1}{g_i(t)}dt = \int_{t_0}^\infty \frac{1}{t}dt = \infty \quad \text{for any } t_0 > 0.$$

From Theorem 3.8 every solution of (4.1), subject to the initial conditions given in (4.1), is asymptotic to $d_1 t^\alpha$ as $t \to \infty$, for some $d_1 \in \mathbb{R}$.

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