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KERNEL FUNCTION AND INTEGRAL REPRESENTATIONS ON KLEIN SURFACES

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Abstract. Some representation theorems for the solutions of the Dirichlet problem and the Neumann problem on Klein surfaces are proved by using an analogue of the harmonic kernel function on symmetric Riemann surfaces.

1. Introduction

In this article we study boundary-value problems for harmonic functions on Klein surfaces, through their double covers by symmetric Riemann surfaces in the sense of Klein, that is, Riemann surfaces endowed with fixed point free antianalytic involutions. The harmonic kernel function is related to the classical domain functions, such as the Green function and the Neumann function on a Klein surface, introduced earlier, see [4, 10]. Thus, it is possible to solve both the boundary value problems of potential theory on a Klein surface, once the harmonic kernel function on a symmetric Riemann surface is known. We develop a symmetrization technique based on the correspondence between Klein surfaces and symmetric Riemann surfaces. The idea of using the double cover has been successfully used to study objects on a Klein surface by Alling and Greenleaf [2], Andreian Cazacu [3], Bârză and Ghişa [4, 5].

2. Preliminaries

Klein surfaces are the most general two-dimensional real manifolds that support harmonic functions. In this paper, the methods introduced in [2, 4, 10] are used to extend results about boundary value problems for harmonic functions on Riemann surfaces to Klein surfaces. The extension required new concepts and techniques such as the well known relationship between Riemann surfaces and Klein surfaces, see [2, 11]. Namely, given a compact Klein surface (X, A), there exists a double cover $f : O_2 \to X$ of X by a Riemann surface O_2, such that O_2 has an antianalytic involution k (called symmetry in the sense of Klein) with $f \circ k = f$. Moreover, X is dianalytically equivalent with O_2/H, where H is the group generated by k, with respect to the usual composition of functions. Also, if O_2 is a Riemann surface on which a fixed point free antianalytic involution k exists, then O_2/H carries a

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unique dianalytic structure, which makes the canonical projection $\pi : O_2 \to O_2/H$ a dianalytic function. Throughout this paper, we identify X with the orbit space O_2/H.

By Klein's definition, the pair (O_2, k) is a symmetric Riemann surface.

Any compact Klein surface can be conceived as a region X of the complex plane bounded by a finite number of analytic Jordan curves, see [11].

Let $F : X \to \mathbf{R}$ be a function on X. Its lifting f to O_2 is given by

$$F(\widetilde{z}) = f(z) = f(k(z)), \quad z \in O_2,\ \widetilde{z} = \pi(z). \tag{2.1}$$

A function f on O_2 with the property (2.1) is called a symmetric function.

Also, if $g : O_2 \to \mathbf{R}$ is a function on O_2, then the function $f = g + g \circ k$ is a symmetric function on O_2. Thus, relation (2.1) defines a function F on X.

We consider the symmetric metric on O_2, defined by $d\sigma = \frac{1}{2}(|dz| + |dw|)$, where $w = k(z)$, $z \in O_2$. Then

$$d\Sigma(\widetilde{z}) = d\sigma(z) = d\sigma(k(z)),\ z \in O_2,$$

is a metric on X. The metric $d\Sigma$ is invariant with respect to the group of conformal or anticonformal transition functions of X.

Let $\widetilde{\gamma}$ be a piecewise smooth Jordan curve on X. Then $\widetilde{\gamma}$ has exactly two liftings γ and $k \circ \gamma$ on O_2, see [7], and by definition

$$\int_{\widetilde{\gamma}} F d\Sigma = \int_{\gamma} f d\sigma = \int_{k\circ\gamma} f d\sigma.$$

For more details about measure and integration on Klein surfaces, see [2].

Let u be a C^1-function defined in a neighborhood of the σ-rectifiable Jordan curve γ, parameterized in terms of the arc σ-length. Therefore, $\gamma : z = z(s) = x(s)+iy(s)$, $s \in [0, l]$, where l is the σ-length of γ. Then the normal derivative of u on γ with respect to $d\sigma$, denoted by $\frac{\partial u}{\partial n_\sigma}$, is the directional derivative of u in the direction of the unit normal vector $n_\sigma = (\frac{dy}{d\sigma}, -\frac{dx}{d\sigma})$.

Given Ω a region of X bounded by a finite number of σ-rectifiable Jordan curves, then $\pi^{-1}(\Omega) = D$ is a symmetric subset of O_2, since k is an antianalytic involution, without fixed points and $\pi \circ k = \pi$. For details about Green's identities for the symmetric region D in terms of $d\sigma$, see [4].

Let F be a continuous real-valued function on $\partial\Omega$. The Dirichlet problem on X for the region Ω, consists in finding a harmonic function U in Ω with prescribed values F on $\partial\Omega$. We define $f = F \circ \pi$ on ∂D. Then $f = f \circ k$ on ∂D, thus f is a symmetric, continuous real-valued functions on ∂D. The Dirichlet problem on X,

$$\begin{gathered} \Delta U = 0 \quad \text{in } \Omega \\ U = F \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

is equivalent with the Dirichlet problem on O_2

$$\begin{gathered} \Delta u = 0 \quad \text{in } D \\ u = f \quad \text{on } \partial D, \end{gathered} \tag{2.3}$$

see [4, 10].

Let G be a continuous real-valued function on $\partial\Omega$. The Neumann problem on X for the region Ω, consists in finding a harmonic function U in Ω with prescribed normal derivatives values G on $\partial\Omega$. We define $g = G \circ \pi$ on ∂D. Then $g = g \circ k$ on ∂D, thus g is a symmetric, continuous real-valued functions on ∂D. By the symmetry of g, we obtain the compatibility condition $\int_{\partial D} g d\sigma = 0$, for the existence of a solution to the Neumann problem on O_2 for the symmetric region D, see [8]. The Neumann problem on X

$$\begin{gathered} \Delta U = 0 \quad \text{in } \Omega \\ \frac{\partial U}{\partial n_\Sigma} = G \quad \text{on } \partial\Omega \end{gathered} \tag{2.4}$$

is equivalent with the Neumann problem on O_2

$$\begin{gathered} \Delta u = 0 \quad \text{in } D \\ \frac{\partial u}{\partial n_\sigma} = g \quad \text{on } \partial D, \end{gathered} \tag{2.5}$$

see [4].

The Dirichlet problem on O_2 for the region D and the boundary function f has a unique solution, provided that ∂D has only regular points, see [1]. If the Neumann problem (2.5) has a solution, then it is unique up to an additive constant, see [8].

The symmetric conditions on the boundary imply symmetric solutions for the problems (2.3) and (2.5), for details see [4] and the original source [11].

Proposition 2.1. *The solution u of problem* (2.3) *is a symmetric function in D.*

Proposition 2.2. *A solution u of problem* (2.5) *is a symmetric function in D.*

3. Symmetric harmonic kernel function

Let D be a symmetric region in the complex plane, bounded by a finite number of σ-rectifiable Jordan curves. In this section we introduce closed systems $(\varphi_i)_{i\in I}$ of harmonic functions in D, which are orthonormal with respect to the Dirichlet integral

$$D\{\varphi_i, \varphi_j\} = \iint_D \Big(\frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial y} + \frac{\partial \varphi_i}{\partial y}\frac{\partial \varphi_j}{\partial x}\Big)\, dx\, dy.$$

We recall some notions and results about orthogonal harmonic functions, see [6].

Let $\Lambda^2(D)$ be the set of harmonic functions $\varphi(z)$ in D with a finite Dirichlet integral

$$D\{\varphi\} = D\{\varphi, \varphi\} < \infty \tag{3.1}$$

such that

$$D\{N_D(z;\zeta), \varphi\} = -2\pi\varphi(\zeta), \tag{3.2}$$

where $N_D(z;\zeta)$ is the Neumann's function of D with its singularity at the fixed point ζ, $\zeta \in D$.

Remark 3.1. If φ has continuous boundary values, φ will be normalized by the condition

$$\int_{\partial D} \varphi d\sigma = 0. \tag{3.3}$$

Proposition 3.2. *There exists a closed system $(\varphi_i)_{i\in I}$ for $\Lambda^2(D)$, which is orthonormal with respect to the Dirichlet integral, i.e.*

$$D\{\varphi_i, \varphi_j\} = \delta_{ij}, \quad \delta_{ii} = 1,\ \delta_{ij} = 0,\ i \neq j.$$

Let ζ be a point inside D. The harmonic kernel function $K_D(z;\zeta)$ of the closed orthonormal system $(\varphi_i)_{i\in I}$, for the region D, with respect to the point ζ, is the function defined by

$$K_D(z;\zeta) = \sum_{i=1}^{\infty} \varphi_i(z)\varphi_i(\zeta), \quad z \in \overline{D}.$$

Remark 3.3. The harmonic kernel function depends on the domain D.

An extensive study of the harmonic kernel function can be found in Bergman [6]. It is known, see [6], that the harmonic kernel function $K_D(z;\zeta)$, the Green function $G_D(z;\zeta)$ and the Neumann function $N_D(z;\zeta)$ satisfy the relation

$$K_G(z;\zeta) = \frac{1}{2\pi}[N_D(z;\zeta) - G_D(z;\zeta)], \quad z \in \overline{D}. \tag{3.4}$$

First, we derive formulas that solve problem (2.3). We prove that if u is harmonic inside a region D and continuous on ∂D, then we can determine the values of u inside of D by integrating on ∂D the product of u times the normal derivative of the harmonic kernel function for the region D, which is a fixed function that depends only on D.

Proposition 3.4. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. If u is harmonic in D and continuous on $\overline{D}$, then for all ζ in D,*

$$u(\zeta) = -\int_{\partial D} u(z)\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} d\sigma. \tag{3.5}$$

Proof. From [4, 9], the solution of the Dirichlet problem (2.3) is

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial G_D(z;\zeta)}{\partial n_\sigma} d\sigma, \quad \zeta \in D. \tag{3.6}$$

Using (3.4), we obtain

$$\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} = \frac{1}{2\pi}\frac{\partial N_D(z;\zeta)}{\partial n_\sigma} - \frac{1}{2\pi}\frac{\partial g_D(z;\zeta)}{\partial n_\sigma} = -\frac{1}{L} - \frac{1}{2\pi}\frac{\partial G_D(z;\zeta)}{\partial n_\sigma},$$

for $z \in \partial D$, where L is the length of ∂D, see [9]. Combining this with (3.6), we find that

$$u(\zeta) = -\int_{\partial D} u(z)\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} d\sigma - \frac{1}{L}\int_{\partial D} u(z) d\sigma.$$

Since the boundary values satisfy the normalization condition $\int_{\partial D} u(z)d\sigma = 0$, we obtain relation (3.5). □

Next, we derive formulas that solve problem (2.5). From Green's formula for the Laplacian, in terms of $d\sigma$, the prescribed values of the normal derivatives must satisfy the compatibility condition

$$\int_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} d\sigma = 0.$$

Proposition 3.5. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. If u is harmonic in D, then, up to an additive constant,*

$$u(\zeta) = \int_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} K_D(z;\zeta) d\sigma, \quad \zeta \in D. \tag{3.7}$$

Proof. Using Green's second identity, it follows that, up to an additive constant, a solution of the Neumann problem is given by

$$u(\zeta) = \frac{1}{2\pi} \int_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} N_D(z;\zeta) d\sigma, \quad \zeta \in D. \tag{3.8}$$

The constant is chosen such that $u(z)$ is in $\Lambda^2(D)$. By (3.4), for $\zeta \in \partial D$, we obtain

$$K_D(z;\zeta) = \frac{1}{2\pi} N_D(z;\zeta).$$

Substituting this in (3.8), we obtain (3.7). □

Let $K_D^{(k)}(z;\widetilde{\zeta})$ be the function defined by

$$K_D^{(k)}(z;\widetilde{\zeta}) = \frac{1}{2}[K_D(z;\zeta) + K_D(z;k(\zeta))], \quad z \in \overline{D},$$

where $K_D(z;k(\zeta))$ is the harmonic kernel function of the closed orthonormal system $(\varphi_i)_{i\in I}$, for the region D, with respect to the point $k(\zeta)$. The function $K_D^{(k)}(z;\widetilde{\zeta})$ is in $\Lambda^2(D)$.

Proposition 3.6. *If D is a symmetric region, then the function $K_D^{(k)}(z;\widetilde{\zeta})$ is symmetric with respect to z on $\overline{D}$ i.e. for every $z \in \overline{D}$,*

$$K_D^{(k)}(z;\widetilde{\zeta}) = K_D^{(k)}(k(z);\widetilde{\zeta}).$$

Proof. We use (3.4) and the symmetric properties of the corresponding symmetric Green's function and symmetric Neumann's function, see [4, 10]. □

The function $K_D^{(k)}(z;\widetilde{\zeta})$ is called the symmetric harmonic kernel function of the closed orthonormal system $(\varphi_i)_{i\in I}$, for the region D, with respect to the point $\widetilde{\zeta} = \{\zeta, k(\zeta)\}$.

4. Integral representations on the double cover

The next theorem yields a formula for the symmetric solution of the problem (2.3).

Theorem 4.1. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let f be a symmetric, continuous function on ∂D. There exists a unique symmetric function u on $\overline{D}$, which is harmonic on D, continuous on $\overline{D}$, such that $u = f$ on ∂D. For all ζ in D,*

$$u(\zeta) = -\frac{1}{2} \int_{\partial D} f(z) \big[\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} + \frac{\partial K_D(z;k(\zeta))}{\partial n_\sigma} \big] d\sigma. \tag{4.1}$$

Proof. Since k is an involution of D, the function $\frac{u(\zeta)+u(k(\zeta))}{2}$ is a symmetric function on D. By Proposition 3.4,

$$u(\zeta) = -\int_{\partial D} u(z) \frac{\partial K_D(z;\zeta)}{\partial n_\sigma} d\sigma, \quad \zeta \in D. \tag{4.2}$$

Replacing ζ with $k(\zeta)$ in (4.2), we obtain

$$u(k(\zeta)) = -\int_{\partial D} u(z) \frac{\partial K_D(z;k(\zeta))}{\partial n_\sigma} d\sigma, \quad \zeta \in D. \tag{4.3}$$

Adding (4.2) to (4.3) and dividing by 2, it follows that

$$\frac{u(\zeta)+u(k(\zeta))}{2} = -\frac{1}{2}\int_{\partial D} u(z)\big[\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} + \frac{\partial K_D(z;k(\zeta))}{\partial n_\sigma}\big]d\sigma, \quad \zeta \in D.$$

By Proposition 2.1, u is a symmetric function on D, then the left-hand side of the last equality is $u(\zeta)$ and we conclude that for all ζ in D,

$$u(\zeta) = -\frac{1}{2}\int_{\partial D} u(z)\big[\frac{\partial K_D(z;\zeta)}{\partial n_\sigma} + \frac{\partial K_D(z;k(\zeta))}{\partial n_\sigma}\big]d\sigma.$$

The uniqueness of the solution of the Dirichlet problem for harmonic functions implies relation (4.1). □

Next we obtain a formula for the symmetric solution of the problem (2.5).

Theorem 4.2. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let g be a symmetric, continuous function on ∂D. If u is harmonic in D and g is its normal derivative on ∂D, then up to an additive constant,*

$$u(\zeta) = \frac{1}{2}\int_{\partial D} g(z)[K_D(z;\zeta) + K_D(z;k(\zeta))]d\sigma, \zeta \in D. \tag{4.4}$$

Proof. By analogy with the proof of the Theorem 4.1, we are using Proposition 3.5 instead of Proposition 3.4. □

5. Integral representations on a Klein surface

Let X be compact Klein surface and let Ω be a region bounded by a finite number of σ-rectifiable Jordan curves. Then there exists a symmetric Riemann surface (O_2, k) such that X is dianalytically equivalent with O_2/H, where H is the group generated by k, with respect to the usual composition of functions. Then, Ω is obtained from the symmetric region D by identifying the corresponding symmetric points.

Let $\widetilde{\zeta}$ be a point inside Ω. The harmonic kernel function $K_\Omega(\widetilde{z};\widetilde{\zeta})$ of the closed orthonormal system $(\varphi_i)_{i\in I}$, for the region Ω, with respect to the point $\widetilde{\zeta} = \{\zeta, k(\zeta)\}$ is defined by

$$K_\Omega(\widetilde{z};\widetilde{\zeta}) = K_D^{(k)}(z;\widetilde{\zeta}) = K_D^{(k)}(k(z);\widetilde{\zeta}), \quad \widetilde{z} = \pi(z) \in \Omega.$$

Remark 5.1. From Proposition (3.6), it follows that $K_\Omega(\widetilde{z};\widetilde{\zeta})$ is well defined on Ω.

By Theorem 4.1, we obtain the following representation of the solution of problem (2.3) on a symmetric region D, in terms of the symmetric harmonic kernel function.

Theorem 5.2. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let f be a symmetric, continuous function on ∂D. There exists a unique symmetric function u on $\overline{D}$, which is harmonic on D, continuous on $\overline{D}$, such that $u = f$ on ∂D. For all ζ in D we have*

$$u(\zeta) = -\int_{\partial D} f(z)\frac{\partial K_D^{(k)}(z;\widetilde{\zeta})}{\partial n_\sigma}d\sigma. \tag{5.1}$$

By Theorem 4.2, we obtain the following representation of the solution of (2.5) on a symmetric region D, in terms of the symmetric harmonic kernel function.

Theorem 5.3. *Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let g be a symmetric, continuous function on ∂D. If u is harmonic in D and g is its normal derivative on ∂D, then up to an additive constant,*

$$u(\zeta) = \int_{\partial D} g(z) K_D^{(k)}(z; \widetilde{\zeta}) d\sigma, \zeta \in D. \tag{5.2}$$

The symmetric solutions on O_2 determine the solutions of the similar problems on the Klein surface X.

We obtain the solution of (2.2) on the region Ω, with respect to the harmonic kernel function, for the region Ω.

Theorem 5.4. *Let F be a continuous real-valued function on the border $\partial\Omega$. The solution of* (2.2) *with the boundary function F is the function U defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where π is the canonical projection of O_2 on X and u is the solution* (5.1) *of the problem* (2.3) *on the symmetric region D, with the boundary function $f = F \circ \pi$.*

Proof. By definition, $\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0$, for all $\widetilde{\zeta} \in \Omega$, where $\widetilde{\zeta} = \pi(\zeta)$, thus U is a harmonic function. The symmetry of the function f on ∂D, implies

$$U(\widetilde{\zeta}) = u(\zeta) = f(\zeta) = f(k(\zeta)) = F(\widetilde{\zeta}), \quad \text{for all } \widetilde{\zeta} \in \partial\Omega.$$

By the uniqueness of the solution, the function U defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\overline{\Omega}$, where $\widetilde{\zeta} = \pi(\zeta)$, is the solution of the problem (2.2) on Ω. □

The next theorem gives a solution of the problem (2.4) on the region Ω, with respect to the harmonic kernel function, for the region Ω.

Theorem 5.5. *Let G be a continuous real-valued function on the border $\partial\Omega$. Then, up to an additive constant, the solution of* (2.4) *with the normal derivative G on $\partial\Omega$ is the function U defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where π is the canonical projection of O_2 on X and u is the solution* (5.2) *of the problem* (2.5) *on the symmetric region D, with the normal derivative function $g = G \circ \pi$ on ∂D.*

Proof. By definition, $\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0$, for all $\widetilde{\zeta} \in \Omega$, where $\widetilde{\zeta} = \pi(\zeta)$, thus U is a harmonic function. The symmetry of the function g on ∂D, implies

$$\frac{\partial U(\widetilde{\zeta})}{\partial n_\sigma} = \frac{\partial u(\zeta)}{\partial n_\sigma} = g(\zeta) = g(k(\zeta)) = G(\widetilde{\zeta}), \quad \text{for all } \widetilde{\zeta} \in \partial\Omega.$$

Thus, up to an additive constant, the function U defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

is the solution of problem (2.4) on Ω. □

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