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COEXISTENCE OF SOME CHAOS SYNCHRONIZATION TYPES IN FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Referring to incommensurate and commensurate fractional systems, this article presents a new approach to investigate the coexistence of some synchronization types between non-identical systems characterized by different dimensions and different orders. In particular, the paper shows that complete synchronization (CS), anti-synchronization (AS) and inverse full state hybrid function projective synchronization (IFSHFPS) coexist when synchronizing a three-dimensional master system with a four-dimensional slave system. The approach is based on two new results involving stability theory of linear fractional systems and the fractional Lyapunov method. A number of examples are provided to highlight the applicability of the method.

1. Introduction

Over the last few years, substantial efforts have been devoted to the study of chaos synchronization in dynamical systems described by integer-order differential equations [5, 15, 29]. Different types of synchronization have been proposed in the literature for continuous–time systems [17, 18, 19, 23] as well as discrete-time [16, 24, 27, 28]. Recently, a lot of attention has been paid to dynamical systems described by fractional-order differential equations [2, 8, 35]. Research studies have shown that fractional-order systems, as generalizations of the more well–known integer-order systems, may also have complex dynamics such as chaos and bifurcation [3, 7, 32]. Some recent studies such as [9, 37] have also shown that chaotic fractional-order systems can be synchronized. However, since the subject is still relatively new, fewer synchronization types have been introduced for fractional-order systems compared to integer-order ones. It is important to note that most of the approaches available in the literature are related to the synchronization of identical fractional-order systems [38]. Very few methods for synchronizing non-identical fractional-order chaotic systems have been established, see [20].

When studying the synchronization of chaotic systems, an interesting phenomenon that may occur is the coexistence of several synchronization types. In fact, the coexistence of these types between different dimensional chaotic (hyperchaotic)

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systems remains entirely unexplored. Perhaps the most relevant studies dedicated to the coexistence of synchronization types between two chaotic systems include:

- [25]: the approach developed in this study proves rigorously the coexistence of some synchronization types between discrete–time chaotic (hyperchaotic) systems.
- [21]: this study proposes two synchronization schemes of coexistence for integer-order chaotic systems.
- [22]: a robust method is applied to study the coexistence of two generalized types of synchronization in fractional chaotic systems with different dimensions. The coexistence of synchronization types can be used to enhance the security in communications and chaotic encryption schemes.

This article investigates the coexistence of various synchronization types between fractional chaotic (hyperchaotic) systems with different dimensions. In particular, the paper shows that complete synchronization (CS) [11], anti-synchronization (AS) [14] and inverse full state hybrid function projective synchronization (IFSHFPS) [26] coexist between a three-dimensional fractional-order master system and a four-dimensional fractional-order slave system. By exploiting the stability theory of fractional linear systems, the coexistence of CS, AS and IFSHFPS between two incommensurate fractional-order systems with different dimensions is proved. Additionally, by using a fractional Lyapunov approach, the coexistence of CS, AS and IFSHFPS is illustrated when the slave system is of the commensurate fractional-order type. Numerical examples are used to confirm the capability of the proposed approach in successfully achieving the coexistence of these synchronization types in the commensurate and incommensurate cases.

The paper is organized as follows: Section 2 lists some preliminaries relating to fractional calculus and the stability of fractional systems. In Section 3, the coexistence of CS, AS and IFSHFPS in fractional-order systems is formulated. The main results of the study are presented in Section 4, followed by some numerical examples in Section 5 that confirm the formulated problem. A summary of the conclusions is, then, given in the last section.

2. Preliminaries

Definition 2.1 ([33]). *The Riemann-Liouville fractional integral operator of order $p > 0$ of the function $f(t)$ is defined as*

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau, \quad t > 0, \tag{2.1}$$

where Γ denotes Gamma function.

Definition 2.2 ([4]). *The Caputo fractional derivative of $f(t)$ is defined as*

$$D_t^p f(t) = J^{m-p}\Big(\frac{d^m}{dt^m} f(t)\Big) = \frac{1}{\Gamma(m-p)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{p-m+1}} d\tau, \tag{2.2}$$

for $m-1 < p \le m$, $m \in \mathbf{N}$, $t > 0$.

Lemma 2.3 ([13]). *The Laplace transform of the Caputo fractional derivative rule reads*

$$\mathbf{L}(D_t^p f(t)) = s^p \mathbf{F}(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \quad (p > 0,\ n-1 < p \le n). \tag{2.3}$$

Particularly, when $0 < p \leq 1$, we have

$$\mathbf{L}(D_t^p f(t)) = s^p \mathbf{F}(s) - s^{p-1} f(0). \tag{2.4}$$

Lemma 2.4 ([31]). *The Laplace transform of the Riemann-Liouville fractional integral rule satisfies*

$$\mathbf{L}(J^q f(t)) = s^{-q} \mathbf{F}(s), \quad (q > 0). \tag{2.5}$$

Lemma 2.5 ([12]). *The fractional-order linear system*

$$D_t^{p_i} x_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t), \quad i = 1, 2, \dots, n, \tag{2.6}$$

is asymptotically stable if all roots λ of the characteristic equation

$$\det(\operatorname{diag}(\lambda^{Mp_1}, \lambda^{Mp_2}, \dots, \lambda^{Mp_n}) - A) = 0, \tag{2.7}$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$, where $A = (a_{ij})$ and M is the least common multiple of the denominators of p_i's.

Lemma 2.6 ([6]). *The trivial solution of the fractional order system*

$$D_t^p X(t) = F(X(t)), \tag{2.8}$$

where $X(t) = (x_i(t))_{1 \leq i \leq n}$, p is a rational number between 0 and 1, and $F : \mathbf{R}^n \to \mathbf{R}^n$ is asymptotically stable if there exists a positive definite Lyapunov function $V(X(t))$ such that $D_t^p V(X(t)) < 0$ for all $t > 0$.

Lemma 2.7 ([1]). *For all $X(t) \in \mathbf{R}^n$, all $p \in]0, 1]$ and all $t > 0$,*

$$\frac{1}{2} D_t^p (X^T(t) X(t)) \leq X^T(t) D_t^p (X(t)). \tag{2.9}$$

3. Problem formulation

We consider the master system given by

$$\begin{aligned} D_t^{p_1} x_1(t) &= f_1(X(t)), \\ D_t^{p_2} x_2(t) &= f_2(X(t)), \\ D_t^{p_3} x_3(t) &= f_3(X(t)), \end{aligned} \tag{3.1}$$

where $X(t) = (x_1(t),\, x_2(t),\, x_3(t))^T$ is the state vector of the master system (3.1), $f_i : \mathbf{R}^n \to \mathbf{R}$, $0 < p_i < 1$, and $D_t^{p_i}$ is the Caputo fractional derivative of order p_i for $i = 1, 2, 3$. Also, consider the slave system defined as

$$\begin{aligned} D_t^{q_1} y_1(t) &= \sum_{j=1}^{4} b_{1j} y_j(t) + g_1(Y(t)) + u_1, \\ D_t^{q_2} y_2(t) &= \sum_{j=1}^{4} b_{2j} y_j(t) + g_2(Y(t)) + u_2, \\ D_t^{q_3} y_3(t) &= \sum_{j=1}^{4} b_{3j} y_j(t) + g_3(Y(t)) + u_3, \\ D_t^{q_4} y_4(t) &= \sum_{j=1}^{4} b_{4j} y_j(t) + g_4(Y(t)) + u_4, \end{aligned} \tag{3.2}$$

where $Y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ is the slave system's state vector, $(b_{ij}) \in \mathbf{R}^{4\times 4}$, $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, 3, 4$ are nonlinear functions, $0 < q_i < 1$, $D_t^{q_i}$ is the Caputo fractional derivative of order q_i, and u_i, $i = 1, 2, 3, 4$ are controllers to be designed. Based on the master–slave synchronizing system described by (3.1) and (3.2), the following definition for the coexistence of different synchronization types can be stated.

Definition 3.1. Complete synchronization (CS), anti–synchronization (AS) and inverse full state hybrid function projective synchronization (IFSHFPS) co–exist in the synchronization of the master system (3.1) and the slave system (3.2) if there exist controllers u_i $(1 \le i \le 4)$ and given differentiable functions $\alpha_i(t)$ $(1 \le i \le 4)$ such that the synchronization errors:

$$\begin{aligned} e_1(t) &= y_1(t) - x_1(t), \\ e_2(t) &= y_2(t) + x_2(t), \\ e_3(t) &= \sum_{j=1}^{4} \alpha_j(t) y_j(t) - x_3(t), \end{aligned} \tag{3.3}$$

satisfy

$$\lim_{t\to\infty} e_i(t) = 0, \tag{3.4}$$

for $i = 1, 2, 3$.

Before presenting the main result of this study, let us start by rewriting the synchronization error problem (3.3). The system can be differentiated to yield

$$\begin{aligned} D_t^{q_1} e_1(t) &= D_t^{q_1} y_1(t) - D_t^{q_1} x_1(t), \\ D_t^{q_2} e_2(t) &= D_t^{q_2} y_2(t) + D_t^{q_2} x_2(t), \\ \dot{e}_3(t) &= \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) + \sum_{j=1}^{4} \alpha_j(t) \dot{y}_j(t) - \dot{x}_3(t). \end{aligned} \tag{3.5}$$

This can, then, be divided in two subsystems as follows

$$(D_t^{q_1} e_1(t), D_t^{q_2} e_2(t))^T = (B - C)(e_1(t), e_2(t))^T + (u_1, u_2)^T + (R_1, R_2)^T, \tag{3.6}$$

and

$$\dot{e}_3(t) = \alpha_3(t)\dot{y}_3(t) + R_3, \tag{3.7}$$

where $B = (b_{ij})_{1\le i;j\le 2}$, $C = (c_{ij})_{1\le i;j\le 2}$ is a control matrix to be selected and

$$\begin{aligned} R_1 &= (c_{11} - b_{11})e_1(t) + (c_{12} - b_{12})e_2(t) + \sum_{j=1}^{4} b_{1j} y_j(t) \\ &\quad + g_1(Y(t)) - D_t^{q_1} x_1(t), \\ R_2 &= (c_{21} - b_{21})e_1(t) + (c_{22} - b_{22})e_2(t) + \sum_{j=1}^{4} b_{2j} y_j(t) \\ &\quad + g_2(Y(t)) - D_t^{q_2} x_2(t), \\ R_3 &= \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) + \sum_{\substack{j=1 \\ j\ne 3}}^{4} \alpha_j(t)\dot{y}_j(t) - \dot{x}_3(t). \end{aligned} \tag{3.8}$$

4. Coexistence of synchronization types

In this section, we show that three different synchronization types can coexist between the proposed systems (3.1) and (3.2) subject to some conditions. In order to achieve synchronization between the master and slave systems, we assume that $\alpha_3(t) \neq 0$ for all $t \geq 0$. Hence, we may now formulate the following theorem.

Theorem 4.1. *CS, AS and IFSHFPS coexist between the master system* (3.1) *and the slave system* (3.2) *under the following conditions:*

(i) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$,
$$u_3 = -\sum_{i=1}^{4} b_{3j} y_j(t) - g_3(Y(t)) + J^{q_3}\big[\frac{1}{\alpha_3(t)}((b_{33}-c)e_3(t) - R_3)\big],$$
and $u_4 = 0$.

(ii) *All roots of*
$$\det(\operatorname{diag}(\lambda^{Mq_1}, \lambda^{Mq_2}) + C - B) = 0$$
satisfy
$$|\arg(\lambda)| > \frac{\pi}{2M},$$
where M is the least common multiple of the denominators of q_1 and q_2.

(iii) *The control constant c is chosen such that $b_{33} - c < 0$.*

Proof. To prove Theorem 4.1, we need to show that equations (3.4) are satisfied. First of all, using (i), the error subsystem (3.6) can be written in the form
$$D_t^q \hat{e}(t) = (B - C)\hat{e}(t), \tag{4.1}$$
where
$$D_t^q \hat{e}(t) = (D_t^{q_2} e_1(t), D_t^{q_1} e_2(t))^T.$$
If the feedback gain matrix C is chosen according to (ii), then based on Lemma 2.5, we conclude that
$$\lim_{t\to+\infty} e_1(t) = \lim_{t\to+\infty} e_2(t) = 0. \tag{4.2}$$

As for the third error, we use the controller u_3 to obtain the following description for state $y_3(t)$
$$D_t^{q_3} y_3(t) = J^{q_3}\big[\frac{1}{\alpha_3(t)}((b_{33}-c)e_3(t) - R_3)\big]. \tag{4.3}$$

Applying the Laplace transform to (4.3) and letting
$$\mathbf{F}(s) = \mathbf{L}(y_3(t)), \tag{4.4}$$
we obtain,
$$s^{q_3}\mathbf{F}(s) - s^{q_3-1} y_3(0) = s^{q_3-1}\mathbf{L}(\frac{1}{\alpha_3(t)}((b_{33}-c)e_3(t) - R_3)). \tag{4.5}$$
Multiplying both sides of (4.5) by s^{1-q_3} and applying the inverse Laplace transform yields the equation
$$\dot{y}_3(t) = \frac{1}{\alpha_3(t)}((b_{33}-c)e_3(t) - R_3). \tag{4.6}$$
From (4.6) and (3.7), the dynamics of $e_3(t)$ can be given by
$$\dot{e}_3(t) = (b_{33}-c)e_3(t). \tag{4.7}$$

If c is selected according to (ii), we obtain

$$\lim_{t\to\infty} e_3(t) = 0. \tag{4.8}$$

Finally, from (4.2) and (4.8), we conclude that the master system (3.1) and the slave system (3.2) are globally synchronized. $\square$

The conditions derived in Theorem 4.1 can be simplified in the case where $q_1 = q_2 = q$. The following proposition shows the simplification.

Proposition 4.2. *Subject to $q_1 = q_2 = q$, condition (ii) of Theorem 4.1 may be replaced by the following condition:*

(ii) *The control matrix C is selected such that $B - C$ is a negative definite matrix.*

Proof. If a Lyapunov function candidate is chosen as $V(\hat{e}(t)) = \frac{1}{2}\hat{e}^T(t)\hat{e}(t)$, then the time Caputo fractional derivative of order q of V along the trajectory of the system (4.1) may be stated as

$$D_t^q V(\hat{e}(t)) = D_t^q\big(\frac{1}{2}\hat{e}^T(t)\hat{e}(t)\big). \tag{4.9}$$

Using Lemma 2.7 along with (4.9), we obtain

$$D_t^q V(e(t)) \le \hat{e}^T(t) D_t^q \hat{e}(t) = \hat{e}^T(t)(B - C)\hat{e}(t).$$

If we select the matrix C such that $B - C$ is negative definite, we obtain

$$D_t^q V(\hat{e}(t)) < 0.$$

From Lemma 2.6, the zero solution of system (4.1) is globally asymptotically stable, i.e

$$\lim_{t\to+\infty} e_1(t) = \lim_{t\to+\infty} e_2(t) = 0. \tag{4.10}$$

$\square$

5. Numerical examples

In this section, we use numerical simulations to validate the theoretical synchronization results proposed in the previous section given some examples of nonlinear chaotic fractional systems.

Case I: $q_1 \neq q_2$. Consider as master the fractional version of the modified coupled dynamos system proposed in [36] and given by

$$\begin{aligned} D^{p_1}x_1 &= -\alpha x_1 + (x_3 + \beta)x_2, \\ D^{p_2}x_2 &= -\alpha x_2 + (x_3 - \beta)x_1, \\ D^{p_3}x_3 &= x_3 - x_1 x_2. \end{aligned} \tag{5.1}$$

System (5.1) can exhibit chaotic behaviors when $(p_1, p_2, p_3) = (0.9, 0.93, 0.96)$ and $(\alpha, \beta) = (2, 1)$. Attractors of the master system (5.1) are shown in Figure 1.

Let us also consider the slave system

$$\begin{aligned} D^{q_1}y_1 &= \alpha(y_2 - y_1) + y_4 + u_1, \\ D^{q_2}y_2 &= \gamma y_1 - y_2 - y_1 y_3 + u_2, \\ D^{q_3}y_3 &= -\beta y_3 + y_1 y_2 + u_3, \\ D^{q_4}y_4 &= \delta y_4 + y_2 y_3 + u_4, \end{aligned} \tag{5.2}$$

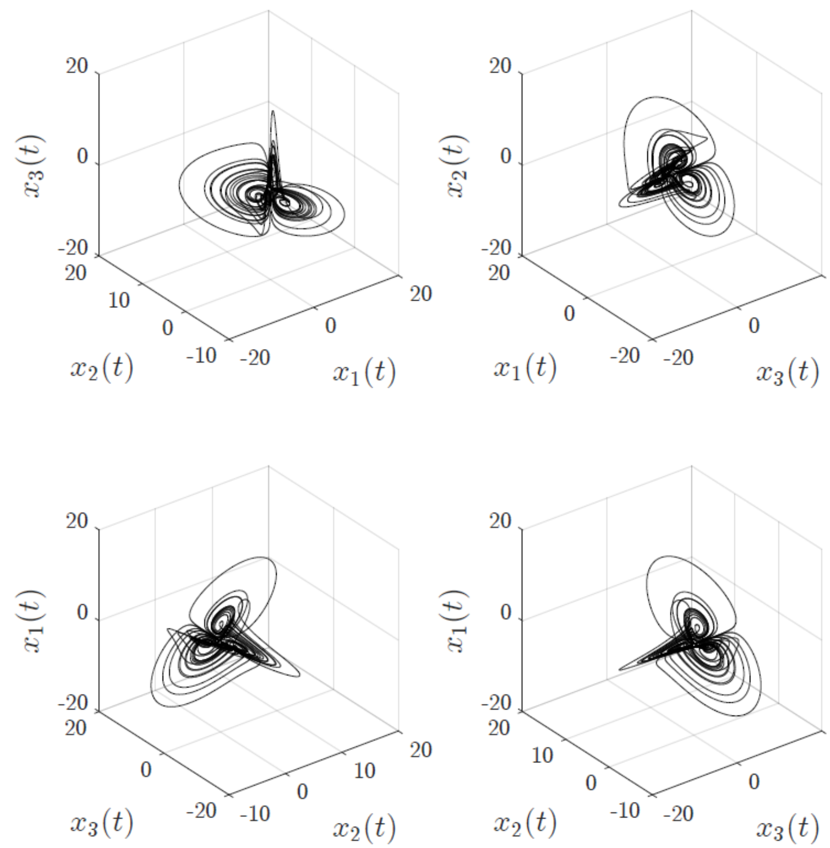

FIGURE 1. Chaotic attractors in 3-D of the master system (5.1) when $(p_1, p_2, p_3) = (0.9, 0.93, 0.96)$ and $(\alpha, \beta) = (2, 1)$.

where the vector controller is

$$U = (u_1, u_2, u_3, u_4)^T .$$

We observe that for $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$, $(q_1, q_2, q_3, q_4) = (0.94, 0.96, 0.97, 0.99)$ and $(\alpha, \beta, \gamma, \delta) = (10, \frac{8}{3}, 28, -1)$, system (5.2) exhibits a hyperchaotic behavior, see [34]. Attractors of the uncontrolled system (5.2) are shown in Figure 2.

According to our approach presented in the previous sections, the error system between the master (5.1) and slave (5.2) is defined as

$$\begin{gathered} e_1 = y_1 - x_1, \\ e_2 = y_2 + x_2, \\ e_3 = \alpha_1(t)y_1 + \alpha_2(t)y_2 + \alpha_3(t)y_3 + \alpha_4(t)y_4 - x_3, \end{gathered} \tag{5.3}$$

where $\alpha_1(t) = 1$, $\alpha_2(t) = \frac{1}{t^2+1}$, $\alpha_3(t) = \exp(-t)$ and $\alpha_4(t) = \sin t$. Using the notations defined in Section 3 above, we can write

$$B = \begin{pmatrix} -10 & 10 \\ 28 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 10 \\ 28 & 0 \end{pmatrix},$$

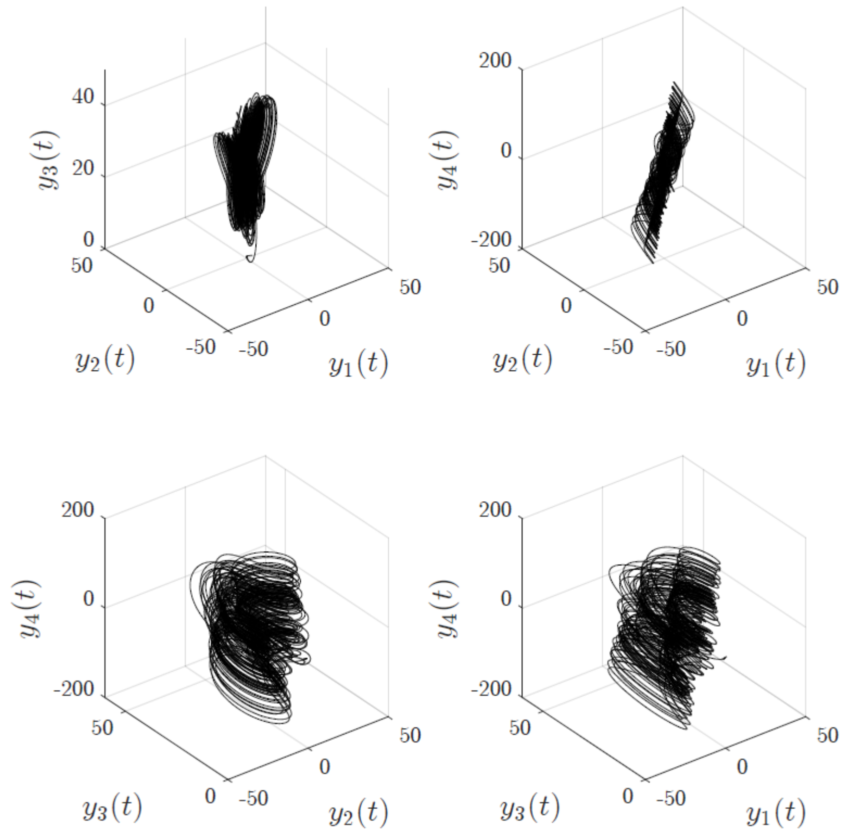

FIGURE 2. Hyperchaotic attractors in 3-D of the slave system (5.2) when $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$, $(q_1, q_2, q_3, q_4) = (0.94, 0.96, 0.97, 0.99)$ and $(\alpha, \beta, \gamma, \delta) = (10, \frac{8}{3}, 28, -1)$.

$b_{33} = -8/3$ and $c = 0$. According to Theorem 4.1, the controllers set of controllers (u_1, u_2, u_3, u_4) may be designed as

$$
\begin{aligned}
u_1 &= -10e_1 + \alpha(y_2 - y_1) - D_t^{0.94} x_1, \\
u_2 &= -e_2 + \gamma y_1 - y_2 - y_1 y_3 - D_t^{0.96} x_2, \\
u_3 &= \beta y_3 - y_1 y_2 + \frac{1}{3} J^{0.97}\Big(-\frac{8}{3} e_3(t) - \frac{1}{(t+1)^2} y_2 + \exp(-t) y_3 \\
&\quad - (\cos t) y_4 - y_1 - (\frac{1}{t+1}) y_2 - (\sin t) y_4 + \dot{x}_3 \Big), \\
u_4 &= 0.
\end{aligned}
\tag{5.4}
$$

The roots of equation

$$
\det(\operatorname{diag}(\lambda^{0.94M}, \lambda^{0.96M}) + C - B) = 0
$$

are

$$
\lambda_1 = 10^{\frac{1}{0.94M}} (\cos \frac{\pi}{0.94M} + \mathbf{i} \sin \frac{\pi}{0.94M}),
$$

$$\lambda_2 = \cos\frac{\pi}{0.96M} + \mathbf{i}\sin\frac{\pi}{0.96M},$$

where M is the least common multiple of the denominators of 0.94 and 0.96. It is easy to show that $|\arg(\lambda_i)| > \frac{\pi}{2M}$ for $i = 1, 2$. Hence, the conditions of Theorem 4.1 are satisfied and, consequently, systems (5.1) and (5.2) are globally synchronized. The error system can be summarized by the two subsystems

$$\begin{aligned} D^{0.94}e_1 &= -10e_1, \\ D^{0.96}e_2 &= -e_2, \end{aligned} \tag{5.5}$$

and

$$\dot{e}_3 = -\frac{8}{3}e_3. \tag{5.6}$$

Fractional Euler integration and fourth order Runge-Kutta integration methods have been used to solve systems (5.5) and (5.6), respectively. Time evolution of the errors e_1, e_2 and e_3 are shown in Figures 3 and 4, respectively.

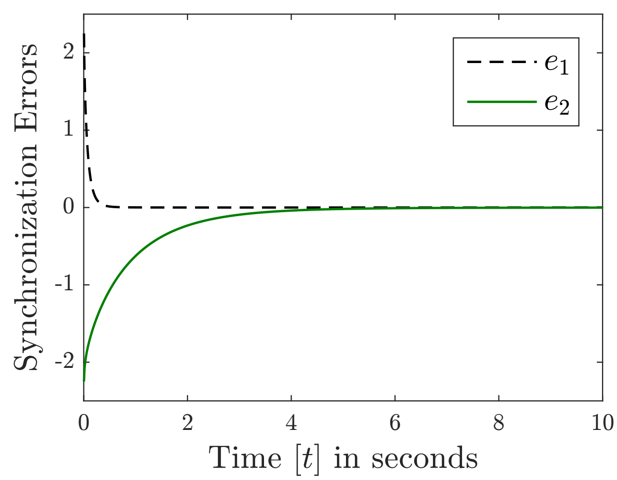

FIGURE 3. Time series of the synchronized error signals e_1 and e_2 between the master system (5.1) and the slave system (5.2).

Case II: $q_1 = q_2 = q$**.** Now, let us consider as master the fractional-order Liu system with the hyperchaotic fractional-order Lorenz system as its slave. The master system is

$$\begin{aligned} D^{p_1}x_1 &= a(x_2 - x_1), \\ D^{p_2}x_2 &= bx_1 - x_1x_3, \\ D^{p_3}x_3 &= -cx_3 + 4x_1^2. \end{aligned} \tag{5.7}$$

This system exhibits chaotic behavior when $(p_1, p_2, p_3) = (0.93, 0.94, 0.95)$ and $(a, b, c) = (10, 40, 2.5)$ [10]. The attractors for (5.7) are shown in Figure 5.

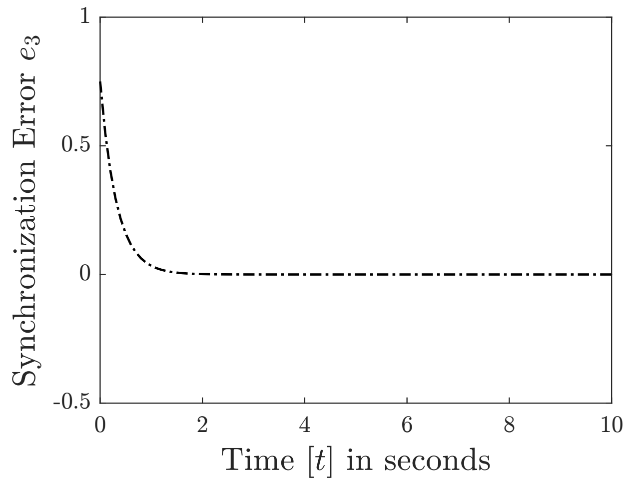

FIGURE 4. Time evolution of the error e_3 between the master system (5.1) and the slave system (5.2).

The slave system is

$$\begin{aligned} D^{q_1}y_1 &= 0.56y_1 - y_2 + u_1, \\ D^{q_2}y_2 &= y_1 - 0.1y_2y_3^2 + u_2, \\ D^{q_3}y_3 &= 4y_2 - y_3 - 6y_4 + u_3, \\ D^{q_4}y_4 &= 0.5y_3 + 0.8y_4 + u_4, \end{aligned} \tag{5.8}$$

where u_1, u_2, u_3, u_4 are the synchronization controllers. This system, as illustrated in [30], exhibits a hyperchaotic behavior when $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$ and $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.95, 0.95)$. The attractors of (5.8) are shown in Figure 6.

The error system between the master (5.7) and slave (5.8) is

$$\begin{aligned} &e_1 = y_1 - x_1, \\ &e_2 = y_2 + x_2, \\ &e_3 = \sin(t)y_1 + \cos(t)y_2 + \tfrac{1}{t^2+1}y_3 + y_4 - x_3. \end{aligned} \tag{5.9}$$

In this case, based on the notation presented in the Section 3, we write

$$B = \begin{pmatrix} 0.56 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$b_{33} = -1$ and $c = 0$, and using Theorem 4.1 and Proposition 4.2, the controllers can constructed as

$$\begin{gathered} u_1 = -0.44e_1 - 0.56y_1 + y_2 - D_t^{0.98}x_1, \\ u_2 = -e_2 + -y_1 + 0.1y_2y_3^2 - D_t^{0.98}x_2, \\ u_3 = -4y_2 + y_3 + 6y_4 + J^{0.97}(t^2+1)(-e_3 - \cos(t)y_1 + \sin(t)y_2) \\ + J^{0.97}(t^2+1)\Big(- \frac{2t}{(t^2+1)^2}y_3 - \sin(t)y_1 - \cos(t)y_2 - y_4 + \dot{x}_3\Big), \\ u_4 = 0. \end{gathered} \tag{5.10}$$

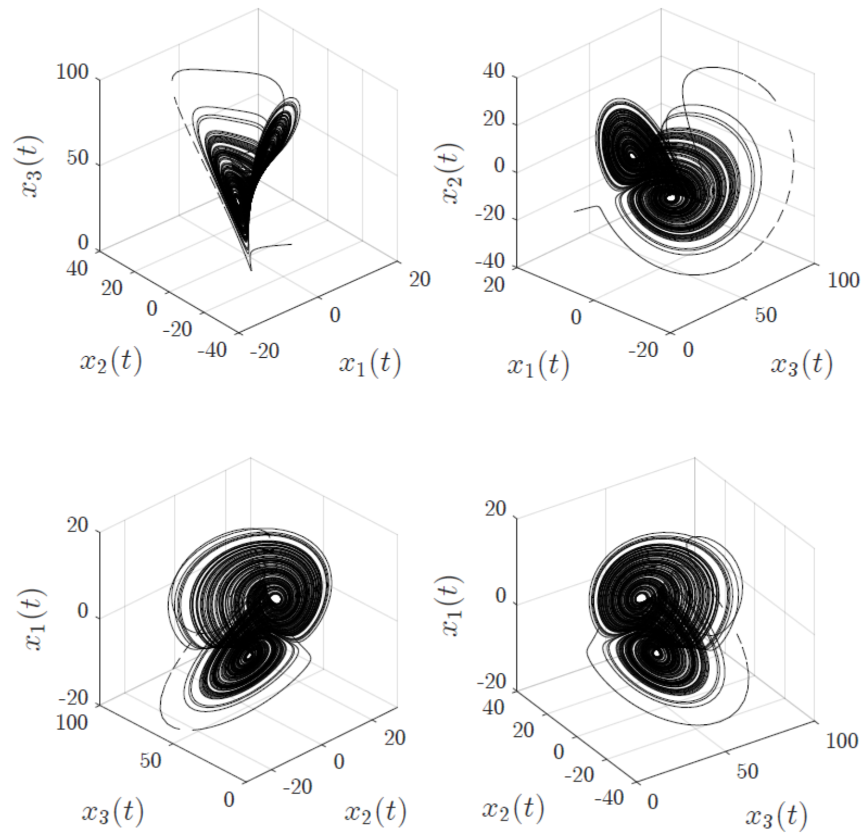

FIGURE 5. Chaotic attractors in 3-D of the master system (5.7) when $(p_1, p_2, p_3) = (0.93, 0.94, 0.95)$ and $(a, b, c) = (10, 40, 2.5)$.

We can show that $B - C$ is a negative definite matrix, which fulfills the condition of Proposition 4.2. Therefore, systems (5.10) and (5.11) are globally synchronized. The error systems is

$$\begin{gathered} D^{0.98}e_1 = -0.44e_1, \\ D^{0.98}e_2 = -e_2, \end{gathered} \tag{5.11}$$

and

$$\dot{e}_3 = -e_3. \tag{5.12}$$

Again, similar to the previous example, the Fractional Euler and fourth order Runge-Kutta integration methods have been used to solve systems (5.11) and (5.12), respectively. The time evolutions of e_1, e_2 and e_3 are shown in Figures 7 and 8.

Conclusions. This paper has proposed a new method to analyze the coexistence problem of some fractional synchronization types. In particular, the approach developed in this paper has proven the coexistence of complete synchronization (CS), anti–synchronization(AS) and inverse full state hybrid function projective synchronization (IFSHFPS) between a three-dimensional fractional-order master system and a four-dimensional fractional-order slave system. It has been shown that the

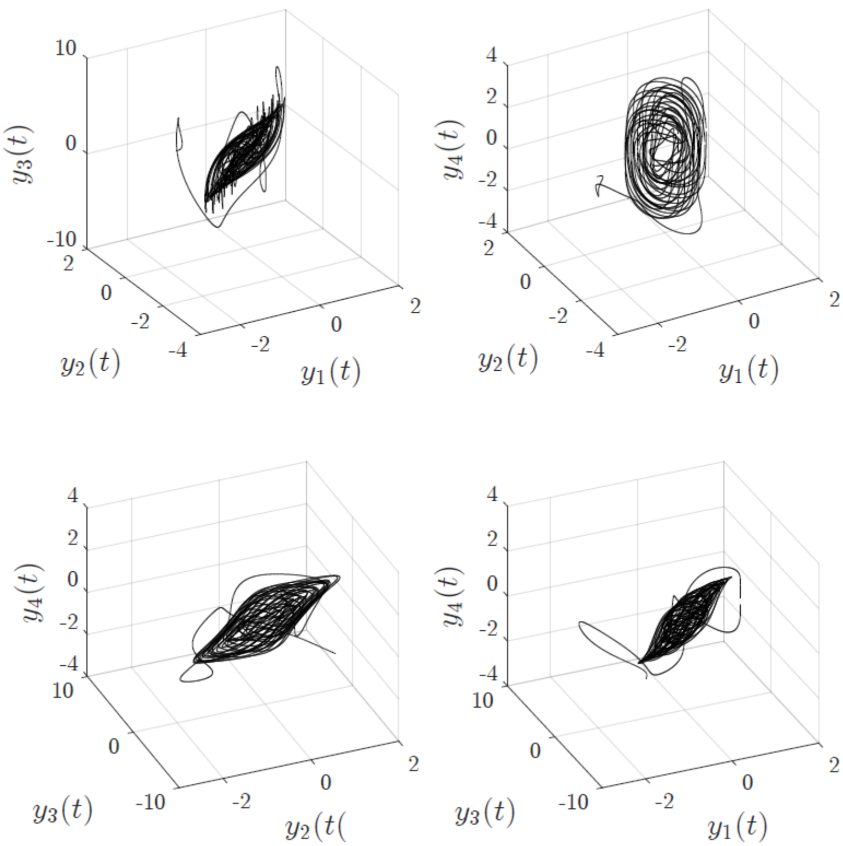

FIGURE 6. Hyperchaotic attractors in 3-D of the slave system (5.8) when when $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$ and $(q_1, q_2, q_3, q_4) = (0.98, 0.98, 0.95, 0.95)$.

approach presents the remarkable feature of being both rigorous and applicable to a wide class of commensurate and incommensurate systems with different dimensions and orders. The numerical examples reported in the paper have clearly highlighted the capability of the proposed approach in successfully achieving the coexistence of these synchronization types between chaotic and hyperchaotic systems of different dimensions for both commensurate and incommensurate fractional-order cases.

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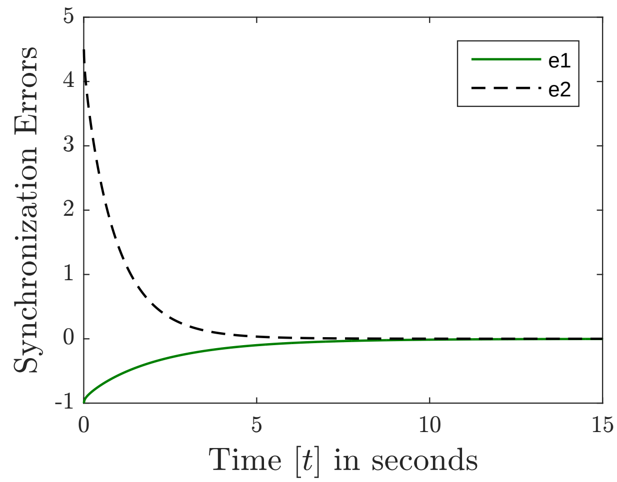

FIGURE 7. Time series of the synchronized error signals e_1 and e_2 between the master system (5.7) and the slave system (5.8).

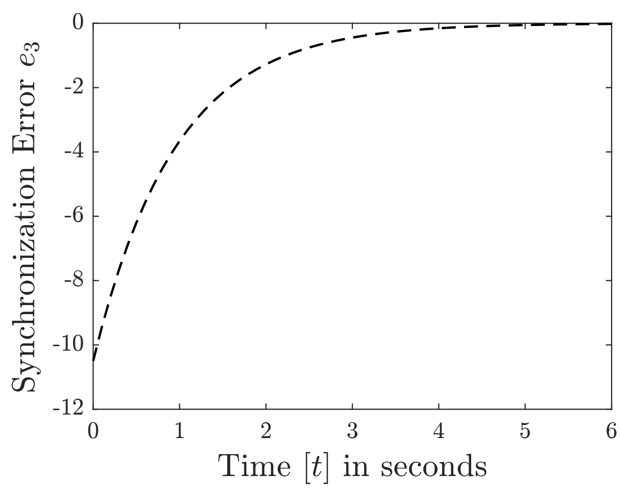

FIGURE 8. Time evolution of the error e_3 between the master system (5.7) and the slave system (5.8).

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