Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 117, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR FOURTH-ORDER EQUATIONS DEPENDING ON TWO PARAMETERS

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Communicated by Vicentiu Radulescu

Abstract. By using variational methods and critical point theory, we establish the existence of infinitely many classical solutions for a fourth-order differential equation. This equation has nonlinear boundary conditions and depends on two real parameters.

1. Introduction

The aim of this article is to study the fourth-order problem

$$\begin{gathered} u^{(iv)}(x) = \lambda f(x, u(x)) \quad \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)), \end{gathered} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Caratéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and λ, μ are positive parameters. Problem (1.1) describes the static equilibrium of a flexible elastic beam of length 1 when, along its length, a load f is added to cause deformation. Precisely, conditions $u(0) = u'(0) = 0$ mean that the left end of the beam is fixed and conditions $u''(1) = 0$, $u'''(1) = \mu g(u(1))$ mean that the right end of the beam is attached to a bearing device, given by the function g.

Existence and multiplicity of solutions for fourth-order boundary value problems has been discussed by several authors in the last decades; see for example [1, 3, 4, 8, 9, 11, 12, 20, 21, 22] and references therein.

Yang et al. [22], used Ricceri's variational principle [19] to establish the existence of at least two classical solutions generated from g for problem (1.1), with $\mu = 1$.

The authors in [8], using a multiplicity result by Cabada and Iannizzotto [7], ensured the existence of at least two nontrivial classical solutions for the problem

$$\begin{gathered} u^{(4)}(x) + \lambda f(x, u(x)) = 0, \quad 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = \lambda g(u(1)), \end{gathered}$$

2010 *Mathematics Subject Classification.* 34B15, 58E05.
Key words and phrases. Fourth-order equations; variational methods; infinitely many solutions.

Submitted December 21, 2016. Published May 3, 2017.

where the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and $\lambda \geq 0$ is a real parameter.

More recently, Bonanno et al. [3], by means of an abstract critical points result of Bonanno [2], studied the existence of at least one non-zero classical solution for problem (1.1).

Our goal in this article is to obtain sufficient conditions to guarantee that problem (1.1) has infinitely many classical solutions. To this end, we require that the primitive F of f satisfy a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at the origin (for finding arbitrarily small solutions), while G, the primitive of g, have an appropriate growth (see Theorems 3.1 and 3.6). Our analysis is mainly based on a general critical point theorem (see Lemma 2.1 below) contained in [5]; see also [18].

We just point out that Song [20, Theorem 3.1], using the same variational setting but different technical arguments, ensured the existence of infinitely many classical solutions for the problem

$$\begin{gathered} u^{(4)} = \lambda f(x,u) + \mu h(x,u), \quad 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{gathered}$$

where λ, μ are two positive parameters, f, h are two L^1-Carathéodory functions, and $g \in C(\mathbb{R})$ is a real function. A special case of our main result reads as follows.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad 0 < B^\star := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} \leq +\infty.$$

Then, for each $\lambda > \frac{2^7 \pi^4}{27 B^\star}$, for every nonpositive continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$g_\infty := \limsup_{\xi \to +\infty} \frac{-\int_0^\xi g(t)dt}{\xi^2} < +\infty,$$

and for each $\mu \in \left]0, \frac{1}{2g_\infty}\right[$, the problem

$$\begin{gathered} u^{(iv)}(x) = \lambda f(u(x)) \quad \textit{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)), \end{gathered} \tag{1.2}$$

admits infinitely many classical solutions.

The plan of the article is as follows. In Section 2 we introduce our notation and a suitable abstract setting (see Lemma 2.1). In Section 3 we present our main result (see Theorems 3.1 and 3.6) and some significative consequences (see Theorem 3.8 as well as Corollaries 3.4, 3.5 and 3.9). A concrete example of an application is exhibited in Example 3.3.

In the conclusion, we cite a recent monograph by Kristály, Rădulescu and Varga [10] as a general reference on variational methods adopted here.

2. Preliminaries

We shall prove our results applying the following smooth version of [5, Theorem 2.1], which is a more precise version of Ricceri's variational principle [18, Theorem 2.5]. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [16, 17] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\big(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\big) - \Psi(u)}{r-\Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

- (a) *If $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*
  - (a1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*
  - (a2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*
$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$
- (b) *If $\delta < +\infty$, then for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*
  - (b1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  - (b2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

We also refer the interested reader to [6, 13, 14, 15] and the references therein, in which Ricceri's variational principle and its variants have been successfully used to obtain the existence of multiple solutions for different boundary value problems.

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Caratéodory function and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

- (a) the mapping $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
- (b) the mapping $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;
- (c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that
$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$
for almost every $x \in [0,1]$.

Corresponding to f, g we introduce the functions F, G as follows

$$F(x,\xi) := \int_0^\xi f(x,t)dt, \quad G(\xi) := -\int_0^\xi g(t)dt,$$

for all $x \in [0,1]$ and $\xi \in \mathbb{R}$.

We consider the space

$$X := \{u \in H^2([0,1]) : u(0) = u'(0) = 0\},$$

where $H^2([0,1])$ is the Sobolev space of all function $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$. X is a Hilbert space with the inner product

$$\langle u, v\rangle := \int_0^1 u''(t)v''(t)dt$$

and the corresponding norm

$$\|u\| := \Big(\int_0^1 (u''(t))^2 dt\Big)^{1/2}.$$

It is easy to see that the norm $\|\cdot\|$ on X is equivalent to the usual norm

$$\int_0^1 \left(|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2\right) dt.$$

It is well known that the embedding $X \hookrightarrow C^1([0,1])$ is compact and

$$\|u\|_{C^1([0,1])} := \max\{\|u\|_\infty, \|u'\|_\infty\} \le \|u\| \tag{2.1}$$

for all $u \in X$ (see [22]).

We say that $u \in X$ is a *weak solution* of problem (1.1) if

$$\int_0^1 u''(x)v''(x)dt - \lambda \int_0^1 f(x,u(x))v(x)dx + \mu g(u(1))v(1) = 0$$

for all $v \in X$. By a *classical solution* of problem (1.1) we mean a function $u \in C^1([0,1])$ such that $u^{(iv)}(x) \in C([0,1])$ and the boundary conditions and the equation are satisfied in $[0,1]$. In [22, Lemma 2.1] it has been shown that the weak solutions are classical solutions of problem (1.1).

3. Main results

Before introducing the main result, we define some notation. We put

$$A := \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|t|\le\xi} F(x,t)dx}{\xi^2}, \quad B := \limsup_{\xi\to+\infty} \frac{\int_{3/4}^1 F(x,\xi)dx}{\xi^2}.$$

Theorem 3.1. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Caratéodory function. Assume that*

- (A1) *$F(x,\xi) \ge 0$ for all $(x,\xi) \in [0,3/4]\times\mathbb{R}$;*
- (A2) *$A < \frac{27}{64\pi^4}B$.*

Then, for every $\lambda \in \Lambda :=]\frac{32\pi^4}{27B}, \frac{1}{2A}[$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential G satisfying the conditions $\inf_{\xi>0} G(\xi) = 0$ and

$$G_\infty := \limsup_{\xi\to+\infty} \frac{\max_{|t|\le\xi} G(t)}{\xi^2} < +\infty, \tag{3.1}$$

if we put

$$\mu_{G,\lambda} := \frac{1}{2G_\infty}(1 - 2A\lambda),$$

where $\mu_{G,\lambda} = +\infty$ when $G_\infty = 0$, problem (1.1) has an unbounded sequence of classical solutions for every $\mu \in]0, \mu_{G,\lambda}[$ in X.

Proof. Our aim is to apply Lemma 2.1(a) to problem (1.1). To this end, fix $\bar\lambda \in \Lambda$ and g satisfying our assumptions. Since $\bar\lambda < \frac{1}{2A}$, we have

$$\mu_{G,\bar\lambda} := \frac{1}{2G_\infty}\left(1 - 2A\bar\lambda\right) > 0.$$

Now fix $\bar\mu \in]0, \mu_{G,\bar\lambda}[$. For each $u \in X$, let the functionals $\Phi, \Psi_{\bar\lambda,\bar\mu} : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2,$$
$$\Psi_{\bar\lambda,\bar\mu}(u) := \int_0^1 F(x, u(x))dx + \frac{\bar\mu}{\bar\lambda} G(u(1))$$

and put

$$I_{\bar\lambda,\bar\mu}(u) := \Phi(u) - \bar\lambda\Psi(u), \quad u \in X.$$

By standard arguments, it follows that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive. Moreover, $\Phi, \Psi_{\bar\lambda,\bar\mu} \in C^1(X, \mathbb{R})$ and for any $u, v \in X$, we have

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx,$$
$$\Psi'_{\bar\lambda,\bar\mu}(u)(v) = \int_0^1 f(x, u(x))v(x)dx - \frac{\bar\mu}{\bar\lambda} g(u(1))v(1).$$

In [22] the authors proved that $\Psi'_{\bar\lambda,\bar\mu}$ is compact. Hence $\Psi_{\bar\lambda,\bar\mu}$ is sequentially weakly (upper) continuous (see [23, Corollary 41.9]).

First of all, we show that $\bar\lambda < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \max_{|t|\le\xi_n} F(x, t)dx}{\xi_n^2} = A.$$

Put $r_n := \xi_n^2/2$ for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2.1) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi_{\bar\lambda,\bar\mu}(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi_{\bar\lambda,\bar\mu}(v)\right) - \Psi_{\bar\lambda,\bar\mu}(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi_{\bar\lambda,\bar\mu}(v)}{r_n} \\
&\le 2\Big[\frac{\int_0^1 \max_{|t|\le\xi_n} F(x, t)dx}{\xi_n^2} + \frac{\bar\mu}{\bar\lambda}\frac{\max_{|t|\le\xi_n} G(t)}{\xi_n^2}\Big].
\end{aligned}$$

Therefore, from assumption (A2) and condition (3.1), we obtain

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le 2\Big(A + \frac{\bar\mu}{\bar\lambda} G_\infty\Big) < +\infty.$$

It follows from $\bar\mu \in]0, \mu_{G,\bar\lambda}[$ that $\gamma < 2A + \frac{1-2\bar\lambda A}{\bar\lambda}$. Hence $\bar\lambda < 1/\gamma$.

Let $\bar\lambda$ be fixed. We claim that the functional $I_{\bar\lambda,\bar\mu}$ is unbounded from below. Since $\frac{1}{\bar\lambda} < \frac{27}{32\pi^4} B$, there exist a sequence $\{d_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\frac{1}{\bar\lambda} < \tau < \frac{27}{32\pi^4}\frac{\int_{3/4}^1 F(x,d_n)dx}{d_n^2} \tag{3.2}$$

for all $n \in \mathbb{N}$ large enough. For $n \in \mathbb{N}$ we define

$$w_n(x) := \begin{cases} 0, & x \in [0,3/8], \\ d_n\cos^2(4\pi x/3), & x \in]3/8,3/4[, \\ d_n, & x \in [3/4,1]. \end{cases} \tag{3.3}$$

For any fixed $n \in \mathbb{N}$, it is easy to see that $w_n \in X$ and, in particular, one has

$$\Phi(w_n) = \frac{32}{27}\pi^4 d_n^2. \tag{3.4}$$

On the other hand, bearing (A1) and $\inf_{\xi>0} G(\xi) = 0$ in mind, from the definition of $\Psi_{\bar\lambda,\bar\mu}$ and (3.2), we infer that

$$\Psi_{\bar\lambda,\bar\mu}(w_n) \geq \int_{3/4}^1 F(x,d_n)dx + \frac{\bar\mu}{\bar\lambda}G(d_n) \geq \frac{32}{27}\pi^4\tau d_n^2. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$I_{\bar\lambda,\bar\mu}(w_n) \leq \frac{32}{27}\pi^4 d_n^2 - \frac{32}{27}\pi^4\bar\lambda\tau d_n^2 = \frac{32}{27}\pi^4(1-\bar\lambda\tau)d_n^2$$

for all $n \in \mathbb{N}$ large enough. Since $\bar\lambda\tau > 1$ and $d_n \to +\infty$ as $n \to +\infty$, we have

$$\lim_{n\to+\infty} I_{\bar\lambda,\bar\mu}(w_n) = -\infty.$$

Hence, our claim is proved. It follows that $I_{\bar\lambda,\bar\mu}$ has no global minimum. Therefore, by Lemma 2.1(a), there exists a sequence $\{u_n\}$ of critical points of $I_{\bar\lambda,\bar\mu}$ such that $\lim_{n\to+\infty}\|u_n\| = +\infty$, and the proof is complete. $\square$

Remark 3.2. Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in]0, \frac{1}{2G_\infty}[$, problem (1.1) admits a sequence of classical solutions which is unbounded in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu > 0$.

Here, we present a concrete application of Theorem 3.1.

Example 3.3. Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x,t) = \begin{cases} 0, & (x,t) \in [0,1]\times\{0\}, \\ x^2t\left(2 - 2\sin(\ln|t|) - \cos(\ln|t|)\right), & (x,t) \in [0,1]\times(\mathbb{R}-\{0\}). \end{cases}$$

A direct calculation yields

$$F(x,t) = \begin{cases} 0, & (x,t) \in [0,1]\times\{0\}, \\ x^2t^2\big(1 - \sin(\ln|t|)\big), & (x,t) \in [0,1]\times(\mathbb{R}-\{0\}). \end{cases}$$

It is easy to see that $A = 0$ and $B = 37/96$. Hence, denoting $u^+ := \max\{u,0\}$, by taking Remark 3.2 into account, we have that for every $(\lambda,\mu) \in]0, \frac{2^{10}\pi^4}{333}[\times]0,1[$ the problem

$$u^{(iv)}(x) = \lambda f(x,u(x)) \quad \text{in } [0,1],$$
$$u(0) = u'(0) = 0,$$

$$u''(1) = 0, \quad u'''(1) = \mu\left(u^+(1) + 1\right)$$

has a sequence of classical solutions which is unbounded in X.

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Caratéodory function. Assume that hypothesis* (A1) *holds, and*

$$A < \frac{1}{2}, \quad B > \frac{32\pi^4}{27}.$$

Then, for every continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential G satisfying the conditions $\inf_{\xi>0} G(\xi) = 0$ and (3.1)*, if we put*

$$\mu_G := \frac{1}{2G_\infty}\left(1 - 2A\right),$$

where $\mu_G = +\infty$ when $G_\infty = 0$, the problem

$$\begin{gathered} u^{(iv)}(x) = f(x, u(x)) \quad \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{gathered}$$

has an unbounded sequence of classical solutions for every $\mu \in]0, \mu_G[$ in X.

We remark that Theorem 1.1 follows immediately from Theorem 3.1. Now, we point out a special situation of our main result when the nonlinear term has separated variables. To be precise, let $h \in L^1([0,1])$ such that $h(x) \geq 0$ a.e. $x \in [0,1]$, $h \not\equiv 0$, and let $k : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the fourth-order problem

$$\begin{gathered} u^{(iv)}(x) = \lambda h(x) k(u(x)) \quad \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)). \end{gathered} \tag{3.6}$$

Put $K(\xi) := \int_0^\xi k(t)dt$ for all $\xi \in \mathbb{R}$, and set $\|h\|_1 := \int_0^1 h(x)dx$ and $h_0 := \int_{\frac{3}{4}}^1 h(x)dx$.

Corollary 3.5. *Suppose that*

$$\liminf_{\xi \to +\infty} \frac{K(\xi)}{\xi^2} < \frac{27 h_0}{64\pi^4 \|h\|_1} \limsup_{\xi \to +\infty} \frac{K(\xi)}{\xi^2}.$$

Then, for each

$$\lambda \in \Big] \frac{32\pi^4}{(27h_0) \limsup_{\xi\to+\infty} \frac{K(\xi)}{\xi^2}}, \frac{1}{(2\|h\|_1) \liminf_{\xi\to+\infty} \frac{K(\xi)}{\xi^2}} \Big[,$$

and every continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential G satisfies the conditions $\inf_{\xi>0} G(\xi) = 0$ and (3.1)*, if we put*

$$\mu'_{G,\lambda} := \frac{1}{2G_\infty}\Big(1 - (2\lambda\|h\|_1) \liminf_{\xi\to+\infty} \frac{K(\xi)}{\xi^2}\Big),$$

where $\mu'_{G,\lambda} = +\infty$ when $G_\infty = 0$, problem (3.6) *has an unbounded sequence of classical solutions for every $\mu \in]0, \mu'_{G,\lambda}[$ in X.*

Put

$$A' := \liminf_{\xi\to 0^+} \frac{\int_0^1 \max_{|t|\leq\xi} F(x,t)dx}{\xi^2}, \quad B' := \limsup_{\xi\to 0^+} \frac{\int_{3/4}^1 F(x,\xi)dx}{\xi^2}.$$

Using Lemma 2.1(b) and arguing as in the proof of Theorem 3.1, we can obtain the following multiplicity result.

Theorem 3.6. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Caratéodory function. Assume that* (A1) *and*

(A3) $A' < \frac{27}{64\pi^4} B'$,

are satisfied. Then, for every $\lambda \in \Lambda :=]\frac{32\pi^4}{27B'}, \frac{1}{2A'}[$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential G satisfying the conditions $\inf_{\xi>0} G(\xi) = 0$ and

$$G_0 := \limsup_{\xi\to 0^+} \frac{\max_{|t|\leq\xi} G(t)}{\xi^2} < +\infty,$$

if we put

$$\widetilde{\mu}_{G,\lambda} := \frac{1}{2G_0}(1 - 2A'\lambda),$$

where $\widetilde{\mu}_{G,\lambda} = +\infty$ when $G_0 = 0$, for every $\mu \in [0, \widetilde{\mu}_{G,\lambda})$ problem (1.1) *has a sequence of classical solutions, which converges strongly to zero in X.*

Remark 3.7. Applying Theorem 3.6, results similar to Theorem 1.1 and Corollaries 3.4 and 3.5 can be obtained. We omit the discussions here.

Now, we put

$$A'' := \liminf_{\xi\to +\infty} \frac{\max_{|t|\leq\xi} G(t)}{\xi^2}, \quad B'' := \limsup_{\xi\to +\infty} \frac{G(\xi)}{\xi^2}.$$

By reversing the roles of λ and μ, we can obtain the following result.

Theorem 3.8. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A4) $A'' < \frac{27}{64\pi^4} B''$.

Then, for every $\mu \in \Gamma :=]\frac{32\pi^4}{27B''}, \frac{1}{2A''}[$ and for every L^1-Caratéodory function $f : [0,1]\times\mathbb{R} \to \mathbb{R}$, whose potential F is a nonnegative function satisfying the condition

$$F_\infty := \limsup_{\xi\to +\infty} \frac{\int_0^1 \max_{|t|\leq\xi} F(x,t)dx}{\xi^2} < +\infty, \tag{3.7}$$

there exists $\lambda_{F,\mu}$, where

$$\lambda_{F,\mu} := \frac{1}{2F_\infty}(1 - 2A''\mu),$$

such that for every $\lambda \in]0, \lambda_{F,\mu}[$, problem (1.1) *has an unbounded sequence of classical solutions in X.*

Proof. Fix $\bar{\mu} \in \Gamma$ and f satisfying our assumptions. Since $\bar{\mu} < \frac{1}{2A''}$, we have $\lambda_{F,\bar{\mu}} > 0$. Now fix $\bar{\lambda} \in]0, \lambda_{F,\bar{\mu}}[$. Set

$$\widetilde{\Psi}_{\bar{\lambda},\bar{\mu}}(u) := \frac{\bar{\lambda}}{\bar{\mu}} \int_0^1 F(x, u(x))dx + G(u(1)),$$
$$\widetilde{I}_{\bar{\lambda},\bar{\mu}}(u) := \Phi(u) - \bar{\mu}\widetilde{\Psi}_{\bar{\lambda},\bar{\mu}}(u),$$

for all $u \in X$. Clearly, $\widetilde{I}_{\bar{\lambda},\bar{\mu}} = I_{\bar{\lambda},\bar{\mu}}$.

Let ξ_n be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n=+\infty$ and

$$\lim_{n\to+\infty}\frac{\max_{|t|\le\xi_n}G(t)}{\xi_n^2}=A''.$$

Let $r_n=\xi_n^2/2$ for all $n\in\mathbb{N}$. Arguing as in the proof of Theorem 3.1 and from the conditions (A4) and (3.7) we obtain

$$\gamma\le\liminf_{n\to+\infty}\varphi(r_n)\le 2\frac{\bar\lambda}{\bar\mu}F_\infty+2A''<+\infty.$$

Therefore, from $\bar\lambda\in]0,\lambda_{F,\bar\mu}[$ we obtain $\bar\mu<1/\gamma$.

Let $\bar\mu$ be fixed. We claim that the functional $\widetilde{I}_{\bar\lambda,\bar\mu}$ is unbounded from below. Since $1/\bar\mu<\frac{27}{32\pi^4}B''$, there exist a sequence $\{d_n\}$ and $\theta>0$ such that $\lim_{n\to+\infty}d_n=+\infty$ and

$$\frac{1}{\bar\mu}<\theta<\frac{27}{32\pi^4}\frac{G(d_n)}{d_n^2}\tag{3.8}$$

for all $n\in\mathbb{N}$ large enough. Now, for every $n\in\mathbb{N}$, let $w_n\in X$ the function as given in (3.3). Since F is nonnegative, from (3.8) we have

$$\widetilde{\Psi}_{\bar\lambda,\bar\mu}(w_n)\ge G(d_n)>\frac{32}{27}\pi^4\theta d_n^2.$$

It follows that

$$\widetilde{I}_{\bar\lambda,\bar\mu}(w_n)=\Phi(w_n)-\bar\mu\widetilde{\Psi}_{\bar\lambda,\bar\mu}(w_n)\le\frac{32}{27}\pi^4(1-\bar\mu\theta)d_n^2<0$$

for all $n\in\mathbb{N}$ large enough. Therefore, $\lim_{n\to+\infty}\widetilde{I}_{\bar\lambda,\bar\mu}(w_n)=-\infty$, and the proof is complete. $\square$

Corollary 3.9. *Assume that $g:\mathbb{R}\to\mathbb{R}$ be a nonpositive continuous function such that*

$$\liminf_{\xi\to+\infty}\frac{-\int_0^\xi g(t)dt}{\xi^2}=0,\quad \limsup_{\xi\to+\infty}\frac{-\int_0^\xi g(t)dt}{\xi^2}=+\infty.$$

Then, for each $\mu>0$ and for every nonnegative continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$f_\infty:=\limsup_{\xi\to+\infty}\frac{\int_0^\xi f(t)dt}{\xi^2}<+\infty,$$

and for each $\lambda\in\left]0,\frac{1}{2f_\infty}\right[$, problem (1.2) admits infinitely many classical solutions.

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