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LIOUVILLE-TYPE THEOREMS FOR AN ELLIPTIC SYSTEM INVOLVING FRACTIONAL LAPLACIAN OPERATORS WITH MIXED ORDER

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Abstract. We study the nonexistence of nontrivial solutions for the nonlinear elliptic system

$$\begin{gathered} G_{\alpha,\beta,\theta}(u^p, u^q) = v^r \\ G_{\lambda,\mu,\theta}(v^s, v^t) = u^m \\ u, v \geq 0, \end{gathered}$$

where $0 < \alpha, \beta, \lambda, \mu \leq 2$, $\theta \geq 0$, $m > q \geq p \geq 1$, $r > t \geq s \geq 1$, and $G_{\alpha,\beta,\theta}$ is the fractional operator of mixed orders α, β, defined by

$$G_{\alpha,\beta,\theta}(u, v) = (-\Delta_x)^{\alpha/2}u + |x|^{2\theta}(-\Delta_y)^{\beta/2}v, \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

Here, $(-\Delta_x)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian operator of order $\alpha/2$ with respect to the variable $x \in \mathbb{R}^{N_1}$, and $(-\Delta_y)^{\beta/2}$, $0 < \beta < 2$, is the fractional Laplacian operator of order $\beta/2$ with respect to the variable $y \in \mathbb{R}^{N_2}$. Via a weak formulation approach, sufficient conditions are provided in terms of space dimension and system parameters.

1. Introduction

Liouville theorem [18] states that any bounded complex function which is harmonic (or holomorphic) on the entire space is constant. The first proof of this theorem is credited to Cauchy [1]. In the recent literature, this result was extended to the case of non-negative solutions of semilinear elliptic equations in the whole space $\mathbb{R}^N$ or in half-spaces, by Gidas and Spruck [9]. In the case of the whole space $\mathbb{R}^N$, they established that if $1 \leq p < \frac{N+2}{N-2}$, then the unique non-negative solution of

$$-\Delta u = Cu^p \quad \text{in } \mathbb{R}^N,$$

where C is a stricly positive constant, is the trivial solution. Using the moving planes method, a simple proof was presented by Chen and Li [2] in the range $0 < p < \frac{N+2}{N-2}$. This result is optimal in the sense that for any $p \geq \frac{N+2}{N-2}$, we have infinitely many positive solutions.

Several Liouville-type results were proved for various classes of degenerate equations. In [24], Serrin and Zou generalized the standard Liouville theorem for

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p-harmonic functions on the whole space and on exterior domains. In [14, 15], Liouville-type properties for some degenerate elliptic operators such as X-elliptic operators, Kohn-Laplacian and Ornstein-Uhlenbeck operators were presented. In [5], Dolcetta and Cutri considered an elliptic inequality involving the Grushin operator. More precisely, they studied the problem

$$u \geq 0, \quad G_\theta u \geq u^p \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{1.1}$$

where $\theta > 1$ and G_θ is the Grushin operator defined by

$$G_\theta u = (-\Delta_x)u + |x|^{2\theta}(-\Delta_y)u, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.2}$$

They proved that if $1 < p < \frac{Q}{Q-2}$, then the only solution of (1.1) is the trivial solution. Here, Q is the homogeneous dimension of the space, given by $Q = N_1 + (\theta+1)N_2$. In [26], Takase and Sleeman considered the system of semilinear parabolic equations

$$\begin{gathered} u_t = \Delta_1 u + v^p \\ v_t = \Delta_2 v + u^q \\ (x,t) \in \mathbb{R}^N \times [0,T), \quad u, v \geq 0, \end{gathered} \tag{1.3}$$

with $p, q \geq 1$, $pq > 1$, under the initial boundary conditions

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where

$$\Delta_i = \sum_{j=1}^{N_i} \frac{\partial^2}{\partial x_j^2}, \quad i = 1,2, \quad x_j \in R_i, \quad N_i = \dim(R_i) \leq N,$$

R_i is a subspace of $\mathbb{R}^N$, and the algebraic sum $R_1 + R_2 = \mathbb{R}^N$. In the case of $R_1 \neq R_2$, they proved that any solution to (1.3)-(1.4) blows up in finite time if

$$\max\big\{\alpha_1 - \frac{N_1}{2} - \frac{n_2}{2q}, \alpha_2 - \frac{N_2}{2} - \frac{n_1}{2p}\big\} > 0,$$

where $\alpha_1 = \frac{p+1}{pq-1}$, $\alpha_2 = \frac{q+1}{pq-1}$, and $n_i = N_i - \dim(R_1 \cap R_2)$, $i = 1,2$. For other results in this directions, we refer to [3, 17, 20, 21, 27].

Recently, a lot of attention has been paid to the study of Liouville-type properties for elliptic equations and inequalities involving fractional operators. In [19], via the moving plane method, Ma and Chen obtained a Liouville-type result for the system of equations

$$\begin{gathered} (-\Delta)^{\mu/2} u = v^q \\ (-\Delta)^{\mu/2} v = u^p \\ u, v \geq 0, \end{gathered}$$

where $\mu \in (0,2)$, $1 < p, q \leq \frac{N+\mu}{N-\mu}$, and $N \geq 2$. Here, $(-\Delta)^{\mu/2}$ is the fractional Laplacian operator of order $\mu/2$. Using the test function method [5], Dahmani et al. [4] extended the result in [19] to various classes of systems involving fractional Laplacian operators with different orders. Some liouville-type results were established recently by Quaas and Xia in [23] for a class of fractional elliptic equations and systems in the half space. For other related works, we refer to [6, 7, 8, 10, 13], and the references therein.

In this work, we establish Liouville-type results for the nonlinear elliptic system

$$\begin{gathered} G_{\alpha,\beta,\theta}(u^p, u^q) = v^r \\ G_{\lambda,\mu,\theta}(v^s, v^t) = u^m \\ u, v \geq 0, \end{gathered} \tag{1.5}$$

where $0 < \alpha, \beta, \lambda, \mu \leq 2$, $\theta \geq 0$, $m > q \geq p \geq 1$, $r > t \geq s \geq 1$, and $G_{\alpha,\beta,\theta}$ is the fractional operator of mixed orders α, β, defined by defined by

$$G_{\alpha,\beta,\theta}(u, v) = (-\Delta_x)^{\alpha/2} u + |x|^{2\theta} (-\Delta_y)^{\beta/2} v, \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where, $(-\Delta_x)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian operator of order $\alpha/2$ with respect to the variable $x \in \mathbb{R}^{N_1}$, and $(-\Delta_y)^{\beta/2}$, $0 < \beta < 2$, is the fractional Laplacian operator of order $\beta/2$ with respect to the variable $y \in \mathbb{R}^{N_2}$. Observe that the standard Grushin operator defined by (1.2) can be written in the form

$$G_\theta u = G_{2,2,\theta}(u, u).$$

Via a weak formulation approach, we provide sufficient conditions for the nonexistence of nontrivial solutions to system (1.5) in terms of space dimension and system parameters.

Before stating and proving the main results of this work, let us present some basic definitions and some lemmas that will be used later.

The nonlocal operator $(-\Delta)^s$, $0 < s < 1$, is defined for any function h in the Schwartz class through the Fourier transform

$$(-\Delta)^s h(x) = \mathcal{F}^{-1}\left(|\xi|^{2s} \mathcal{F}(h)(\xi)\right)(x),$$

where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse. It can be also defined via the Riesz potential

$$(-\Delta)^s h(x) = c_{N,s} \text{ PV} \int_{\mathbb{R}^N} \frac{h(x) - h(\overline{x})}{|x - \overline{x}|^{N+2s}} \, d\overline{x},$$

where $c_{N,s}$ is a normalisation constant and PV is the Cauchy principal value (see [16, 25]).

Lemma 1.1 ([11])**.** *Suppose that $\delta \in (0, 2]$, $\beta + 1 \geq 0$, and $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$. Then the following point-wise inequality holds:*

$$(-\Delta)^{\delta/2} \psi^{\beta+2}(x) \leq (\beta + 2)\psi^{\beta+1}(x)(-\Delta)^{\delta/2}\psi(x).$$

Lemma 1.2 ([12])**.** *Let X, Y, A_1, B_1, A_2, B_2 be non-negative functions, and let α_i and θ_i, $i = 1, 2$, be positive reals such that $\alpha_1, \alpha_2 \geq 1$ and $\alpha_1\theta_1 > \max\{\alpha_2, \theta_2, \alpha_2\theta_2\}$. Suppose that*

$$\begin{gathered} X^{\alpha_1} \leq A_1 Y + B_1 Y^{\theta_2}, \\ Y^{\theta_1} \leq A_2 X + B_2 X^{\alpha_2}. \end{gathered}$$

Then there is some constant $C > 0$ such that

$$\begin{aligned} Y^{\alpha_1\theta_1} \leq C\Big[&(A_2^{\alpha_1} A_1)^{\frac{\alpha_1\theta_1}{\alpha_1\theta_1 - 1}} + (A_2^{\alpha_1} B_1)^{\frac{\alpha_1\theta_1}{\alpha_1\theta_1 - \theta_2}} \\ &+ (B_2^{\alpha_1} A_1^{\alpha_2})^{\frac{\alpha_1\theta_1}{\alpha_1\theta_1 - \alpha_2}} + (B_2^{\alpha_1} B_1^{\alpha_2})^{\frac{\alpha_1\theta_1}{\alpha_1\theta_1 - \alpha_2\theta_2}} \Big]. \end{aligned}$$

2. Main results

In this section, we state an prove the main results in this paper. We consider the elliptic system (1.5) under the assumptions

$$0<\alpha,\beta,\lambda,\mu\le 2,\quad \theta\ge 0,\ m>q\ge p\ge 1,\quad r>t\ge s\ge 1. \tag{2.1}$$

We adopt the following definition of solutions for (1.5).

Definition 2.1. We say that the pair (u,v) is a weak solution of (1.5) if, $u\ge 0$, $v\ge 0$, $(u,v)\in L^m_{\rm loc}(\mathbb{R}^{\mathbb{N}})\times L^r_{\rm loc}(\mathbb{R}^{\mathbb{N}})$, $N=N_1+N_2$, and

$$\begin{gathered}
\int_{\mathbb{R}^N} v^r\varphi\,dx\,dy=\int_{\mathbb{R}^N} u^p(-\Delta_x)^{\alpha/2}\varphi\,dx\,dy+\int_{\mathbb{R}^N}|x|^{2\theta}u^q(-\Delta_y)^{\beta/2}\varphi\,dx\,dy,\\
\int_{\mathbb{R}^N} u^m\varphi\,dx\,dy=\int_{\mathbb{R}^N} v^s(-\Delta_x)^{\lambda/2}\varphi\,dx\,dy+\int_{\mathbb{R}^N}|x|^{2\theta}v^t(-\Delta_y)^{\mu/2}\varphi\,dx\,dy,
\end{gathered}$$

for every $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi\ge 0$.

Let us introduce the following parameters:

$$\begin{gathered}
Q_1=\frac{m}{mr-ps}\,(\alpha s+\lambda r),\quad \overline{Q}_1=\frac{r}{mr-ps}\,(\lambda p+\alpha m),\\
Q_2=\frac{m}{mr-qs}\big(\lambda r-(2\theta-\beta(\theta+1))s\big),\\
\overline{Q}_2=\frac{r}{mr-tp}\big(\alpha m-(2\theta-\mu(\theta+1))p\big),\\
Q_3=\frac{m}{mr-pt}\big(\alpha t-(2\theta-\mu(\theta+1))r\big),\\
\overline{Q}_3=\frac{r}{mr-sq}\big(\lambda q-(2\theta-\beta(\theta+1))m\big),\\
Q_4=\frac{m}{mr-qt}\Big((\mu(\theta+1)-2\theta)r+(\beta(\theta+1)-2\theta)t\Big),\\
\overline{Q}_4=\frac{r}{mr-qt}\Big((\beta(\theta+1)-2\theta)m+(\mu(\theta+1)-2\theta)q\Big).
\end{gathered}$$

Our main result in this article is the following Liouville-type theorem.

Theorem 2.2. *Let (u,v) be a weak solution of system (1.5). Under assumptions (2.1), if*

$$Q<\max\{\Lambda_1,\Lambda_2\}, \tag{2.2}$$

where

$$Q=N_1+N_2(\theta+1),\quad \Lambda_1=\min\{Q_1,Q_2,Q_3,Q_4\},\quad \Lambda_2=\min\{\overline{Q}_1,\overline{Q}_2,\overline{Q}_3,\overline{Q}_4\},$$

then the solution (u,v) is trivial.

Proof. Suppose that (u,v) is a weak solution of (1.5) such that $(u,v)\not\equiv(0,0)$. Let ω be a real number such that

$$\omega>\max\Big\{\frac{m}{m-q},\frac{r}{r-t}\Big\}. \tag{2.3}$$

By the weak formulation of (1.5), for all $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi\ge 0$, we have

$$\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy=\int_{\mathbb{R}^N} u^p(-\Delta_x)^{\alpha/2}\varphi^\omega\,dx\,dy+\int_{\mathbb{R}^N}|x|^{2\theta}u^q(-\Delta_y)^{\beta/2}\varphi^\omega\,dx\,dy \tag{2.4}$$

and

$$\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy=\int_{\mathbb{R}^N} v^s(-\Delta_x)^{\lambda/2}\varphi^\omega\,dx\,dy+\int_{\mathbb{R}^N}|x|^{2\theta}v^t(-\Delta_y)^{\mu/2}\varphi^\omega\,dx\,dy. \tag{2.5}$$

Using Lemma 1.1 and Hölder's inequality with parameters $\frac{m}{p}$ and $\frac{m}{m-p}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} u^p(-\Delta_x)^{\alpha/2}\varphi^\omega\,dx\,dy\\
&\le \omega\int_{\mathbb{R}^N} u^p\varphi^{\omega-1}|(-\Delta_x)^{\alpha/2}\varphi|\,dx\,dy\\
&=\omega\int_{\mathbb{R}^N} u^p\varphi^{\frac{\omega p}{m}}\varphi^{(\omega-1-\frac{\omega p}{m})}|(-\Delta_x)^{\alpha/2}\varphi|\,dx\,dy\\
&\le \omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{p/m}\Big(\int_{\mathbb{R}^N}\varphi^{(\omega-1-\frac{\omega p}{m})\frac{m}{m-p}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{m}{m-p}}\,dx\,dy\Big)^{\frac{m-p}{m}}\\
&=\omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{p/m}\Big(\int_{\mathbb{R}^N}\varphi^{\omega-\frac{m}{m-p}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{m}{m-p}}\,dx\,dy\Big)^{\frac{m-p}{m}}.
\end{aligned}$$

Note that thanks to the choice (2.3) of the parameter ω, we have

$$\int_{\mathbb{R}^N}\varphi^{\omega-\frac{m}{m-p}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{m}{m-p}}\,dx\,dy<\infty.$$

Therefore, we have the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} u^p(-\Delta_x)^{\alpha/2}\varphi^\omega\,dx\,dy\\
&\le \omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{p/m}\Big(\int_{\mathbb{R}^N}\varphi^{\omega-\frac{m}{m-p}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{m}{m-p}}\,dx\,dy\Big)^{\frac{m-p}{m}}.
\end{aligned}\tag{2.6}$$

Again, using Lemma 1.1 and Hölder's inequality with parameters $\frac{m}{q}$ and $\frac{m}{m-q}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|x|^{2\theta}u^q(-\Delta_y)^{\beta/2}\varphi^\omega\,dx\,dy\\
&\le \omega\int_{\mathbb{R}^N} u^q|x|^{2\theta}\varphi^{\omega-1}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy\\
&=\omega\int_{\mathbb{R}^N} u^q\varphi^{\frac{\omega q}{m}}|x|^{2\theta}\varphi^{(\omega-1-\frac{\omega q}{m})}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy\\
&\le \omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{q/m}\Big(\int_{\mathbb{R}^N}|x|^{\frac{2\theta m}{m-q}}\varphi^{\big(\omega-1-\frac{\omega q}{m}\big)\frac{m}{m-q}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{m}{m-q}}\,dx\,dy\Big)^{\frac{m-q}{m}}\\
&=\omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{q/m}\Big(\int_{\mathbb{R}^N}|x|^{\frac{2\theta m}{m-q}}\varphi^{\omega-\frac{m}{m-q}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{m}{m-q}}\,dx\,dy\Big)^{\frac{m-q}{m}}.
\end{aligned}$$

From the choice (2.3) of the parameter ω, we have

$$\int_{\mathbb{R}^N}|x|^{\frac{2\theta m}{m-q}}\varphi^{\omega-\frac{m}{m-q}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{m}{m-q}}\,dx\,dy<\infty.$$

Therefore, we have the estimate

$$\int_{\mathbb{R}^N}|x|^{2\theta}u^q(-\Delta_y)^{\beta/2}\varphi^\omega\,dx\,dy$$

$$\leq \omega\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{q/m}\Big(\int_{\mathbb{R}^N} |x|^{\frac{2\theta m}{m-q}}\varphi^{\omega-\frac{m}{m-q}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{m}{m-q}}\,dx\,dy\Big)^{\frac{m-q}{m}}.$$

Combining this with (2.4) and (2.6), we obtain

$$\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy \leq A_\varphi\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{p/m} + B_\varphi\Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{q/m}, \quad (2.7)$$

where

$$A_\varphi = \omega\Big(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{m}{m-p}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{m}{m-p}}\,dx\,dy\Big)^{\frac{m-p}{m}},$$

$$B_\varphi = \omega\Big(\int_{\mathbb{R}^N} |x|^{\frac{2\theta m}{m-q}}\varphi^{\omega-\frac{m}{m-q}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{m}{m-q}}\,dx\,dy\Big)^{\frac{m-q}{m}}.$$

Similarly, using Hölder's inequality with parameters $\frac{r}{s}$ and $\frac{r}{r-s}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} v^s(-\Delta_x)^{\lambda/2}\varphi^\omega\,dx\,dy\\ &\leq \omega\Big(\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy\Big)^{s/r}\Big(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{r}{r-s}}|(-\Delta_x)^{\lambda/2}\varphi|^{\frac{r}{r-s}}\,dx\,dy\Big)^{\frac{r-s}{r}}.\end{aligned} \quad (2.8)$$

Again, Hölder's inequality with parameters $\frac{r}{t}$ and $\frac{r}{r-t}$ yields

$$\begin{aligned}&\int_{\mathbb{R}^N} |x|^{2\theta}v^t(-\Delta_y)^{\mu/2}\varphi^\omega\,dx\,dy\\ &\leq \omega\Big(\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy\Big)^{t/r}\Big(\int_{\mathbb{R}^N} |x|^{\frac{2\theta r}{r-t}}\varphi^{\omega-\frac{r}{r-t}}|(-\Delta_y)^{\mu/2}\varphi|^{\frac{r}{r-t}}\,dx\,dy\Big)^{\frac{r-t}{r}}.\end{aligned} \quad (2.9)$$

Combining (2.5) with the estimates (2.8) and (2.9), we obtain

$$\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy \leq C_\varphi\Big(\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy\Big)^{s/r} + D_\varphi\Big(\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy\Big)^{t/r}, \quad (2.10)$$

where

$$C_\varphi = \omega\Big(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{r}{r-s}}|(-\Delta_x)^{\lambda/2}\varphi|^{\frac{r}{r-s}}\,dx\,dy\Big)^{\frac{r-s}{r}},$$

$$D_\varphi = \omega\Big(\int_{\mathbb{R}^N} |x|^{\frac{2\theta r}{r-t}}\varphi^{\omega-\frac{r}{r-t}}|(-\Delta_y)^{\mu/2}\varphi|^{\frac{r}{r-t}}\,dx\,dy\Big)^{\frac{r-t}{r}}.$$

Let

$$X = \Big(\int_{\mathbb{R}^N} u^m\varphi^\omega\,dx\,dy\Big)^{p/m}, \quad Y = \Big(\int_{\mathbb{R}^N} v^r\varphi^\omega\,dx\,dy\Big)^{s/r}.$$

Combining the estimates (2.7) and (2.10), we obtain the system of inequalities

$$\begin{gathered}X^{m/p} \leq C_\varphi Y + D_\varphi Y^{\frac{t}{s}},\\ Y^{r/s} \leq A_\varphi X + B_\varphi X^{\frac{q}{p}}.\end{gathered}$$

Using Lemma 1.2, we obtain

$$\begin{aligned}Y^{\frac{mr}{ps}} \leq C\Big(&\Big(A_\varphi^{m/p}C_\varphi\Big)^{\frac{mr}{mr-ps}} + \Big(A_\varphi^{m/p}D_\varphi\Big)^{\frac{mr}{mr-pt}}\\ &+ \Big(B_\varphi^{m/p}C_\varphi^{\frac{q}{p}}\Big)^{\frac{mr}{mr-qs}} + \Big(B_\varphi^{m/p}D_\varphi^{\frac{q}{p}}\Big)^{\frac{mr}{mr-qt}}\Big).\end{aligned} \quad (2.11)$$

Similarly, we obtain

$$\begin{aligned} X^{\frac{mr}{ps}} \le C\Big(&\Big(C_\varphi^{r/s}A_\varphi\Big)^{\frac{mr}{mr-ps}} + \Big(C_\varphi^{r/s}B_\varphi\Big)^{\frac{mr}{mr-qs}} \\ &+ \Big(D_\varphi^{r/s}A_\varphi^{\frac{t}{s}}\Big)^{\frac{mr}{mr-pt}} + \Big(D_\varphi^{r/s}B_\varphi^{\frac{t}{s}}\Big)^{\frac{mr}{mr-qt}}\Big). \end{aligned} \tag{2.12}$$

Now, as a test function, we take

$$\varphi(x,y) = \varphi_0\Big(\frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}}\Big), \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where φ_0 is the classical cutoff function, that is, $\varphi_0 \in C_0^\infty(0,\infty)$ is a smooth decreasing function such that

$$0 \le \varphi_0 \le 1, \quad |\varphi_0'(\eta)| \le C\eta^{-1},$$

$$\varphi_0(\eta) = \begin{cases} 1 & \text{if } 0 < \eta \le 1, \\ 0 & \text{if } \eta \ge 2. \end{cases}$$

We use the change of variables

$$x = Rz \quad \text{and} \quad y = R^{\theta+1}w.$$

In this case, we have

$$\eta := \frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}} = |z|^2 + |w|^2, \quad (z,w) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

Let Ω be the subset of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ defined by

$$\Omega = \{(z,w) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : 1 \le |z|^2 + |w|^2 \le 2\}.$$

We have the following estimates.

• Estimate of A_φ. Using the above change of variables, we obtain

$$A_\varphi = \omega R^{\frac{Q(m-p)-\alpha m}{m}} \Big(\int_\Omega [\varphi_0(\eta)]^{\omega - \frac{m}{m-p}} |(-\Delta_z)^{\alpha/2}\varphi_0(\eta)|^{\frac{m}{m-p}}\, dz\, dw\Big)^{\frac{m-p}{m}}.$$

Observe that

$$\int_\Omega [\varphi_0(\eta)]^{\omega - \frac{m}{m-p}} |(-\Delta_z)^{\alpha/2}\varphi_0(\eta)|^{\frac{m}{m-p}}\, dz\, dw$$

is a real number independent on R. Therefore, we have

$$A_\varphi = CR^{\frac{Q(m-p)-\alpha m}{m}}, \tag{2.13}$$

where C is a positive constant independent on R.

• Estimate of B_φ. Using the same change of variable as above, we obtain

$$\begin{aligned} B_\varphi &= \omega R^{\frac{(2\theta-\beta(\theta+1))m+Q(m-q)}{m}} \\ &\quad \times \Big(\int_\Omega |z|^{\frac{2\theta m}{m-q}} [\varphi_0(\eta)]^{\omega - \frac{m}{m-q}} |(-\Delta_w)^{\beta/2}\varphi_0(\eta)|^{\frac{m}{m-q}}\, dz\, dw\Big)^{\frac{m-q}{m}}. \end{aligned}$$

Since

$$\int_\Omega |z|^{\frac{2\theta m}{m-q}} [\varphi_0(\eta)]^{\omega - \frac{m}{m-q}} |(-\Delta_w)^{\beta/2}\varphi_0(\eta)|^{\frac{m}{m-q}}\, dz\, dw$$

is a real number independent on R, we have

$$B_\varphi = CR^{\frac{(2\theta-\beta(\theta+1))m+Q(m-q)}{m}}. \tag{2.14}$$

• Estimate of C_φ. We argue as previously, to obtain

$$C_\varphi = CR^{\frac{Q(r-s)-\lambda r}{r}}. \tag{2.15}$$

• Estimate of D_φ. We have the estimate

$$D_\varphi = CR^{\frac{(2\theta-\mu(\theta+1))r+Q(r-t)}{r}}. \tag{2.16}$$

Using the estimates (2.12), (2.13), (2.14), (2.15) and (2.16), we obtain

$$X^{\frac{mr}{ps}} \le C\left(R^{\tau_1}+R^{\tau_2}+R^{\tau_3}+R^{\tau_4}\right), \tag{2.17}$$

where

$$\begin{gathered}
\tau_1 = \Big(\frac{rm}{rm-ps}\Big)\Big(\frac{Q(mr-ps)-m(\lambda r+\alpha s)}{ms}\Big),\\
\tau_2 = \Big(\frac{rm}{rm-qs}\Big)\Big(\frac{Q(mr-sq)+m(s(2\theta-\beta(\theta+1))-\lambda r)}{ms}\Big),\\
\tau_3 = \Big(\frac{rm}{rm-pt}\Big)\Big(\frac{Q(mr-pt)+m(r(2\theta-\mu(\theta+1))-\alpha t)}{ms}\Big),\\
\tau_4 = \Big(\frac{rm}{rm-qt}\Big)\Big(\frac{Q(mr-qt)+m(r(2\theta-\mu(\theta+1))+(2\theta-\beta(\theta+1)))}{ms}\Big).
\end{gathered}$$

Similarly, using the estimates (2.11), (2.13), (2.14), (2.15) and (2.16), we obtain

$$Y^{\frac{mr}{ps}} \le C\left(R^{\kappa_1}+R^{\kappa_2}+R^{\kappa_3}+R^{\kappa_4}\right), \tag{2.18}$$

where

$$\begin{gathered}
\kappa_1 = \Big(\frac{rm}{rm-ps}\Big)\Big(\frac{Q(mr-ps)-r(\alpha m+\lambda p)}{rp}\Big),\\
\kappa_2 = \Big(\frac{rm}{rm-tp}\Big)\Big(\frac{Q(mr-pt)+r(p(2\theta-\mu(\theta+1))-\alpha m)}{rp}\Big),\\
\kappa_3 = \Big(\frac{rm}{rm-sq}\Big)\Big(\frac{Q(mr-sq)+r(m(2\theta-\beta(\theta+1))-\lambda q)}{rp}\Big),\\
\kappa_4 = \Big(\frac{rm}{rm-tq}\Big)\Big(\frac{Q(mr-qt)+r(m(2\theta-\beta(\theta+1))+(2\theta-\mu(\theta+1)))}{rp}\Big).
\end{gathered}$$

Now, using (2.2), we can see that

$$\max\{\tau_i : i =, 1, 2, 3, 4\} < 0$$

or

$$\max\{\kappa_i : i =, 1, 2, 3, 4\} < 0.$$

Case 1. If $\max\{\tau_i : i =, 1, 2, 3, 4\} < 0$. In this case, passing to the limit as $R \to \infty$ in (2.17), and using the monotone convergence theorem, we obtain

$$\lim_{R\to\infty}\Big(\int_{\mathbb{R}^N} u^m\Big[\varphi_0\Big(\frac{|x|^2}{R^2}+\frac{|y|^2}{R^{2(\theta+1)}}\Big)\Big]^\omega\,dx\,dy\Big)^{r/s} = \Big(\int_{\mathbb{R}^N} u^m\,dx\,dy\Big)^{r/s} = 0,$$

which yields $(u, v) \equiv (0, 0)$, that is a contradiction with the fact that (u, v) is a nontrivial solution.

Case 2. If $\max\{\kappa_i : i =, 1, 2, 3, 4\} < 0$. As in the previous case, passing to the limit as $R \to \infty$ in (2.18), and using the monotone convergence theorem, we obtain

$$\lim_{R\to\infty} \Big(\int_{\mathbb{R}^N} v^r \Big[\varphi_0\Big(\frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}}\Big)\Big]^{\omega} \,dx\,dy\Big)^{m/p} = \Big(\int_{\mathbb{R}^N} v^r \,dx\,dy\Big)^{m/p} = 0,$$

which yields $(u, v) \equiv (0, 0)$, that is a contradiction.

In both cases, we get a contradiction. As consequence, we infer that the only weak solution to system (1.5) is the trivial solution. $\square$

The following Liouville-type results follow from Theorem 2.2. Taking $\alpha = \lambda$, $\beta = \mu = 2$ and $p = s = q = t = 1$ in Theorem 2.2, we obtain the following Liouville-type property.

Corollary 2.3. *Let (u, v) be a weak solution of the elliptic system*

$$\begin{gathered}(-\Delta_x)^{\alpha/2}u + |x|^{2\theta}(-\Delta_y)u = v^r\\ (-\Delta_x)^{\alpha/2}v + |x|^{2\theta}(-\Delta_y)v = u^m, u, v \ge 0,\end{gathered}$$

where $0 < \alpha \le 2$, $\theta \ge 0$, $m > 1$ and $r > 1$. If

$$Q < \frac{\alpha}{mr - 1} \max\{m(r + 1), r(m + 1)\},$$

then (u, v) is trivial.

Taking $\alpha = 2$ in Corollary 2.3, we obtain the following Liouville-type property for an elliptic system involving the standard Grushin operator.

Corollary 2.4. *Let (u, v) be a weak solution of the elliptic system*

$$\begin{gathered}(-\Delta_x)u + |x|^{2\theta}(-\Delta_y)u = v^r\\ (-\Delta_x)v + |x|^{2\theta}(-\Delta_y)v = u^m\\ u, v \ge 0,\end{gathered}$$

where $\theta \ge 0$, $m > 1$ and $r > 1$. If

$$Q < \frac{2}{mr - 1} \max\{m(r + 1), r(m + 1)\},$$

then the solution (u, v) is trivial.

Taking $u = v$ and $m = r$ in Corollary 2.3, we obtain the following result.

Corollary 2.5. *Let u be a weak solution of the elliptic equation*

$$(-\Delta_x)^{\alpha/2}u + |x|^{2\theta}(-\Delta_y)u = u^r, \quad u \ge 0,$$

where $0 < \alpha \le 2$, $\theta \ge 0$. If

$$1 < r < \frac{Q}{Q - \alpha}, \tag{2.19}$$

then the solution u is trivial.

Remark 2.6. Taking $\alpha = 2$ in Corollary 2.5, condition (2.19) becomes

$$1 < r < \frac{Q}{Q - 2}.$$

Such condition was obtained by Dolcetta and Cutri in [5].

Taking $\alpha = \lambda = 2$, $\beta = \mu$ and $p = s = q = t = 1$ in Theorem 2.2, we obtain the following Liouville-type property.

Corollary 2.7. *Let* (u, v) *be a weak solution of the elliptic system*

$$\begin{gathered}(-\Delta_x)u + |x|^{2\theta}(-\Delta_y)^{\beta/2}u = v^r \\ (-\Delta_x)v + |x|^{2\theta}(-\Delta_y)^{\beta/2}v = u^m \\ u, v \geq 0,\end{gathered}$$

where $0 < \beta \leq 2$, $\theta \geq 0$, $m > 1$ *and* $r > 1$. *If*

$$Q < \frac{\beta(\theta+1) - 2\theta}{mr - 1} \max\{m(r+1), r(m+1)\},$$

then the solution (u, v) *is trivial.*

Remark 2.8. Taking $\beta = 2$ in Corollary 2.7, we obtain the Liouville-type property given by Corollary 2.4.

Taking $u = v$ and $m = r$ in Corollary 2.7, we obtain the following result.

Corollary 2.9. *Let* u *be a weak solution of the elliptic equation*

$$(-\Delta_x)u + |x|^{2\theta}(-\Delta_y)^{\beta/2}u = u^r, \quad u \geq 0,$$

where $0 < \beta \leq 2$, $\theta \geq 0$. *If*

$$1 < r < \frac{Q}{Q - \beta(\theta+1) + 2\theta},$$

then the solution u *is trivial.*

Remark 2.10. Taking $\beta = 2$ in Corollary 2.9, we obtain again the Dolcetta-Cutri condition [5]:

$$1 < r < \frac{Q}{Q-2}.$$

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