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EXACT ASYMPTOTIC BEHAVIOR OF THE POSITIVE SOLUTIONS FOR SOME SINGULAR DIRICHLET PROBLEMS ON THE HALF LINE

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Abstract. In this article, we give an exact behavior at infinity of the unique solution to the following singular boundary value problem

$$-\frac{1}{A}(Au')' = q(t)g(u), \quad t \in (0,\infty),$$
$$u > 0, \quad \lim_{t\to 0} A(t)u'(t) = 0, \quad \lim_{t\to\infty} u(t) = 0.$$

Here A is a nonnegative continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that

$$\lim_{t\to\infty}\frac{tA'(t)}{A(t)} = \alpha > 1, \quad g \in C^1((0,\infty),(0,\infty))$$

is non-increasing on $(0,\infty)$ with $\lim_{t\to 0} g'(t)\int_0^t \frac{ds}{g(s)} = -C_g \leq 0$ and the function q is a nonnegative continuous, satisfying

$$0 < a_1 = \liminf_{t\to\infty}\frac{q(t)}{h(t)} \leq \limsup_{t\to\infty}\frac{q(t)}{h(t)} = a_2 < \infty,$$

where $h(t) = ct^{-\lambda}\exp(\int_1^t \frac{y(s)}{s}ds)$, $\lambda \geq 2$, $c > 0$ and y is continuous on $[1,\infty)$ such that $\lim_{t\to\infty} y(t) = 0$.

1. Introduction

In this article, we give the exact asymptotic behavior at infinity of the unique positive solution to the singular problem

$$\begin{gathered}\frac{1}{A}(Au')' = -q(t)g(u)\,, \quad t \in (0,\infty),\\ u > 0\,, \quad \text{in } (0,\infty)\\ \lim_{t\to 0^+} A(t)u'(t) = 0, \quad \lim_{t\to\infty} u(t) = 0,\end{gathered} \tag{1.1}$$

where the functions A, q and g satisfy the following assumptions.

(H1) A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that

$$\lim_{t\to\infty}\frac{t\,A'(t)}{A(t)} = \alpha > 1.$$

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(H2) q is a nonnegative continuous function on $(0,\infty)$ satisfying
$$0 < a_1 = \liminf_{t\to\infty} \frac{t^\lambda q(t)}{L(t)} \le \limsup_{t\to\infty} \frac{t^\lambda q(t)}{L(t)} = a_2 < \infty,$$
where $\lambda \ge 2$ and $L \in \mathcal{K}$ (see (1.3) below), such that $\int_1^\infty s^{1-\lambda} L(s)\, ds < \infty$.

(H3) The function $g : (0,\infty) \to (0,\infty)$ is nonincreasing, continuously differentiable such that
$$\lim_{t\to 0^+} g'(t) \int_0^t \frac{1}{g(s)}\, ds = -C_g \quad \text{with } C_g \ge 0.$$

(H4) $\alpha + 1 - \lambda + (\lambda - 2)C_g > 0$.

Using that g is non-increasing, for $t > 0$, we obtain
$$0 < g(t) \int_0^t \frac{1}{g(s)}\, ds \le t.$$
This implies $\lim_{t\to 0} g(t) \int_0^t \frac{1}{g(s)}\, ds = 0$. Now, since for $t > 0$,
$$\int_0^t g'(s) \int_0^s \frac{1}{g(r)}\, dr\, ds = g(t) \int_0^t \frac{1}{g(s)} ds - t,$$
we obtain
$$\lim_{t\to 0} \frac{g(t)}{t} \int_0^t \frac{1}{g(s)}\, ds = 1 - C_g. \tag{1.2}$$
This implies that $0 \le C_g \le 1$. The functions $t^{-1}\log(1+t)$, $\log(\log(e + \frac{1}{t}))$, $t^{-\nu}\log(1+\frac{1}{t})$, $\exp\{(\log(1+\frac{1}{t}))^\nu\}$, $\nu \in (0,1)$ satisfy the assumption (H3), as well as the function
$$\begin{gathered} t^2 e^{1/t}, \quad \text{if } 0 < t < \frac{1}{2}, \\ \frac{1}{4} e^2, \quad \text{if } t \ge \frac{1}{2}. \end{gathered}$$

Singular nonlinear boundary value problems appear in a variety of applications and often only positive solutions are important. When $A(t) = 1$, problems of type (1.1) with various boundary conditions arise in the study of boundary layer equations for the class of pseudoplastic fluids and have been studied for both bounded and unbounded intervals of $\mathbb{R}$ (see [6, 11, 16, 22, 23, 29]) and the references therein. When $A(t) = t^{n-1}$ $(n \ge 1)$, the operator $u \to \frac{1}{A}(Au')'$ appears as the radial part of the laplace operator Δ (see [10, 30]). Other results of existence and uniqueness of positive solutions were obtained by Agarwal and O'Regan in [1] on the interval $(0,1)$ and in the case where A is continuous on $[0,1]$, positive and differentiable on $(0,1)$ and satisfying an integrability condition. In general the exact asymptotic behavior of the unique positive solution of (1.1) is extremely complex when the coefficients are in general continuous functions, even though upper and lower bounds for this solution are often given (see [1, 4, 10, 15]). Recent research (see [2, 8, 16]) show that these problems should be studied in the case of Karamata regularly varying functions. This approach was initiated by Avakumovic [3] and followed by Maric and Tomic (see [20, 21]). Our aim in this paper is to give a contribution to the qualitative analysis of problem (1.1) by giving the exact asymptotic behavior at infinity of the unique positive solution under the previous assumptions on A, q and g. We note that the existence and uniqueness of such a solution are established by Mâagli and Masmoudi in [17]. For related results, we refer to Barile and Salvatore

[5], Cencelj, Repovš, and Virk [7], Ghergu and Rădulescu [13, 14, 15, 16], Rădulescu and Repovš [24, 25], Repovš [26, 27].

To state our results, we denote by $\mathcal{K}$ the set of Karamata functions L defined on $[1,\infty)$ by

$$L(t) := c\exp\Big(\int_1^t \frac{y(s)}{s}ds\Big), \tag{1.3}$$

where $c>0$ and $y\in C([1,\infty))$ such that $\lim_{t\to\infty} y(t)=0$.

Remark 1.1. It is clear that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1([1,\infty))$ such that

$$\lim_{t\to\infty}\frac{t\,L'(t)}{L(t)}=0. \tag{1.4}$$

Throughout this paper, we denote by ψ_g the unique solution of the equation

$$\int_0^{\psi_g(t)}\frac{ds}{g(s)}=t\,,\quad \text{for } t\in[0,\infty), \tag{1.5}$$

and we mention that

$$\lim_{t\to 0} tg'(\psi_g(t))=-C_g. \tag{1.6}$$

Theorem 1.2. *Assume* (H1)–(H4). *Then problem* (1.1) *has a unique solution* $u\in C^2((0,\infty))\cap C([0,\infty))$ *satisfying*

(i) *If* $\lambda>2$,

$$\Big(\frac{\xi_1}{\lambda-2}\Big)^{1-C_g}\le \liminf_{t\to\infty}\frac{u(x)}{\psi_g(t^{2-\lambda}L(t))}\le \limsup_{t\to\infty}\frac{u(x)}{\psi_g(t^{2-\lambda}L(t))}\le\Big(\frac{\xi_2}{\lambda-2}\Big)^{1-C_g},$$

where $\xi_i=\frac{a_i}{\alpha+1-\lambda+(\lambda-2)C_g}$ *for* $i\in\{1\,2\}$.

(ii) *If* $\lambda=2$,

$$\xi_1{}^{1-C_g}\le \liminf_{t\to\infty}\frac{u(t)}{\psi_g(\int_t^\infty \frac{L(s)}{s}\,ds)}\le \limsup_{t\to\infty}\frac{u(t)}{\psi_g(\int_t^\infty \frac{L(s)}{s}\,ds)}\le \xi_2{}^{1-C_g}$$

An immediate consequence of Theorem 1.2 is the following result.

Corollary 1.3. *Let* u *be the unique solution of* (1.1). *Then, we have the following exact asymptotic behavior:*

(a) When $C_g=1$, *we have*

(i) $\lim_{t\to\infty}\frac{u(t)}{\psi_g(t^{2-\lambda}L(t))}=1$ *if* $\lambda>2$;

(ii) $\lim_{t\to\infty}\frac{u(t)}{\psi_g(\int_t^\infty\frac{L(s)}{s}\,ds)}=1$ *if* $\lambda=2$.

(b) When $C_g<1$ *and* $a_1=a_2=a_0$, *we have*

(i) $\lim_{t\to\infty}\frac{u(t)}{\psi_g(t^{2-\lambda}L(t))}=\big(\frac{a_0}{(\lambda-2)(\alpha+1-\lambda+(\lambda-2)C_g)}\big)^{1-C_g}$ *if* $\lambda>2$;

(ii) $\lim_{t\to\infty}\frac{u(t)}{\psi_g(\int_t^\infty\frac{L(s)}{s}\,ds)}=\big(\frac{a_0}{\alpha-1}\big)^{1-C_g}$ *if* $\lambda=2$.

Remark 1.4. In the hypothesis (H3), we do not need the monotonicity of the function g on $(0,\infty)$, but only the fact that g is non-increasing in a neighborhood of zero.

Example 1.5. Let g be the function

$$g(t)=\begin{cases} t^2e^{1/t}\,, & \text{if } 0<t<\frac{1}{2},\\ \frac{1}{4}e^2\,, & \text{if } t\geq \frac{1}{2}. \end{cases}$$

and let q be a nonnegative function in $(0,\infty)$ such that

$$\lim_{t\to\infty}\frac{q(t)}{h(t)}=b_0\in(0,\infty),$$

where $h(t)=t^{-\lambda}L(t)$, $\lambda\geq 2$ and $L\in\mathcal{K}$ such that $\int_1^\infty s^{1-\lambda}\,L(s)\,ds<\infty$. Then, we have $C_g=1$ and $\psi_g(\xi)=\frac{-1}{log(\xi)}$ for $\xi\in(0,e^{-2})$. Let u be the unique solution of (1.1), then we have the following exact behavior:

(i) $\lim_{t\to\infty}u(t)\log\big(\frac{1}{t^{2-\lambda}L(t)}\big)=1$ if $\lambda>2$;
(ii) $\lim_{t\to\infty}u(t)\log\big(\frac{1}{\int_t^\infty\frac{L(s)}{s}\,ds}\big)=1$ if $\lambda=2$.

To establish our second result, we consider the special case where $g(t)=t^{-\gamma}$ with $\gamma\geq 0$, and $\lambda=\alpha+1+\gamma(\alpha-1)$. Note that in this case $C_g=\frac{\gamma}{\gamma+1}$ and $(\alpha+1-\lambda)+(\lambda-2)C_g=0$. We assume that A and q satisfy the following hypotheses:

(H5) A is a continuous function on $(0,\infty)$ such that $A(t)=t^\alpha B(t)$ with $\alpha>1$ and $\frac{t^\nu B'(t)}{B(t)}$ is bounded for t large and $\nu\in(0,1)$.
(H6) q is a nonnegative continuous function in $(0,\infty)$ and satisfies

$$0<a_1=\liminf_{t\to\infty}\frac{q(t)}{t^{\gamma-1-\alpha(\gamma+1)}L(t)}\leq\limsup_{t\to\infty}\frac{q(t)}{t^{\gamma-1-\alpha(\gamma+1)}L(t)}=a_2<\infty,$$

where $L\in\mathcal{K}$ with $\int_1^\infty\frac{L(s)}{s}\,ds=\infty$.

Theorem 1.6. *Assume* (H5), (H6) *are satisfied. Then the Dirichlet problem*

$$\begin{gathered} -\frac{1}{A}(A\,u')'=q(t)u^{-\gamma}\,,\quad t\in(0,\infty),\\ \lim_{t\to0^+}A(t)u'(t)=0,\quad \lim_{t\to\infty}u(t)=0, \end{gathered}\tag{1.7}$$

has a unique solution $u\in C([0,\infty))\cap C^2((0,\infty))$, *satisfying*

$$\begin{aligned} \Big(\frac{(\gamma+1)a_1}{\alpha-1}\Big)^{\frac{1}{1+\gamma}} &\leq\liminf_{t\to\infty}\frac{u(t)}{t^{1-\alpha}\big(\int_1^t\frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}}\\ &\leq\limsup_{t\to\infty}\frac{u(t)}{t^{1-\alpha}\big(\int_1^t\frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}}\\ &\leq\big(\frac{(\gamma+1)a_2}{\alpha-1}\big)^{\frac{1}{1+\gamma}}, \end{aligned}$$

In particular if $a_1=a_2$, *then*

$$\lim_{t\to\infty}\frac{u(t)}{t^{1-\alpha}\big(\int_1^t\frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}}=\Big(\frac{(\gamma+1)a_1}{\alpha-1}\Big)^{\frac{1}{1+\gamma}}.$$

2. On the Karamata class

To make the paper self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. The following result is due to [19, 28].

Lemma 2.1. *(i) Let $L \in \mathcal{K}$ and $\varepsilon > 0$, then*

$$\lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

(ii) Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.

Applying Karamata's theorem (see [19, 28]), we get the following result.

Lemma 2.2. *Let $\gamma \in \mathbb{R}$, L be a function in $\mathcal{K}$ defined on $[1,\infty)$. We have*

(i) *If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s)ds$ converges. Moreover*

$$\int_t^\infty s^\gamma L(s)ds \sim_{t\to\infty} -\frac{t^{1+\gamma}L(t)}{\gamma+1}.$$

(ii) *If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)ds$ diverges. Moreover*

$$\int_1^t s^\gamma L(s)ds \sim_{t\to\infty} \frac{t^{1+\gamma}L(t)}{\gamma+1}.$$

Lemma 2.3 ([8, 18])**.** *Let $L \in \mathcal{K}$ be defined on $[1,\infty)$. Then*

$$\lim_{t\to\infty} \frac{L(t)}{\int_1^t \frac{L(s)}{s}ds} = 0. \tag{2.1}$$

If further $\int_1^\infty \frac{L(s)}{s}ds$ converges, then

$$\lim_{t\to\infty} \frac{L(t)}{\int_t^\infty \frac{L(s)}{s}ds} = 0. \tag{2.2}$$

Remark 2.4. Let $L \in \mathcal{K}$, then using Remark 1.1 and (2.1), we deduce that

$$t \to \int_1^t \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

If further $\int_1^\infty \frac{L(s)}{s}\,ds$ converges, then $t \to \int_t^\infty \frac{L(s)}{s}\,ds \in \mathcal{K}$.

Definition 2.5. A positive measurable function k is called normalized regularly varying at infinity with index $\rho \in \mathbb{R}$ and we write $k \in NRVI_\rho$ if $k(s) = s^\rho L(s)$ for $s \in [1,\infty)$ with $L \in \mathcal{K}$.

Using the definition of the class $\mathcal{K}$ and the above Lemmas we obtain the following lemma.

Lemma 2.6 ([2])**.**
(i) *If $k \in NRVI_\rho$, then $\lim_{t\to\infty} \frac{k(\xi t)}{k(t)} = \xi^\rho$, uniformly for $\xi \in [c_1, c_2] \subset (0,\infty)$.*
(ii) *A positive measurable function k belongs to the class $NRVI_\rho$ if and only if $\lim_{t\to\infty} \frac{tk'(t)}{k(t)} = \rho$.*
(iii) *Let $L \in \mathcal{K}$ and assume that $\int_1^\infty s^{1-\lambda}\,L(s)\,ds < \infty$. Then the function $\theta(t) = \int_t^\infty s^{1-\lambda}\,L(s)\,ds$ belongs to $NRVI_{(2-\lambda)}$.*
(iv) *The function $\psi_g \circ \theta \in NRVI_{(2-\lambda)(1-C_g)}$.*
(v) *Let m_1, m_2 be positive functions on $(0,\infty)$ such that $\lim_{t\to\infty} m_1(t) = \lim_{t\to\infty} m_2(t) = 0$ and $\lim_{t\to\infty} \frac{m_1(t)}{m_2(t)} = 1$. Then $\lim_{t\to\infty} \frac{\psi_g(m_1(t))}{\psi_g(m_2(t))} = 1$.*

3. Proofs of Theorems 1.2 and 1.6

In the sequel, we denote by

$$v_0(t) = \int_t^\infty \frac{1}{A(s)}\,ds \quad \text{for } t \in (0,\infty).$$

Since the function A satisfies (H1), then using Definition 2.5 and assertion (ii) of Lemma 2.6, we deduce that there exists $L_0 \in \mathcal{K}$ such that $A(t) = t^\alpha L_0(t)$, for $t > 1$. Hence, using Lemma 2.1, we deduce that $1/A$ is integrable near infinity. So the function v_0 is well defined, and by Lemma 2.2 we have

$$v_0(t) = \int_t^\infty \frac{1}{A(s)}\,ds \sim \frac{t^{1-\alpha}}{(\alpha-1)\,L_0(t)} \quad \text{as } t \to \infty. \tag{3.1}$$

In the sequel, we denote also by $L_A u := \frac{1}{A}(Au')' = u'' + \frac{A'}{A}u'$ and we remark that $L_A v_0 = 0$.

Proof of Theorem 1.2. Let $\varepsilon \in (0, a_1/2)$. Put

$$\xi_i = \frac{a_i}{(\alpha+1-\lambda)+(\lambda-2)C_g} \quad \text{for } i \in \{1,2\},$$

$\tau_1 = \xi_1 - \varepsilon\frac{\xi_1}{a_1}$ and $\tau_2 = \xi_2 + \varepsilon\frac{\xi_2}{a_2}$. Clearly, we have $\frac{\xi_1}{2} < \tau_1 < \tau_2 < \frac{3}{2}\xi_2$. Let $\theta(t) = \int_t^\infty s^{1-\lambda}\,L(s)\,ds$ and put

$$\omega_i(t) = \psi_g\Big(\tau_i \int_t^\infty s^{1-\lambda}\,L(s)\,ds\Big) = \psi_g(\tau_i\,\theta(t)), \quad \text{for } t > 0.$$

By a simple calculus, for $i \in \{1,2\}$ we obtain

$$\begin{aligned}
&L_A\omega_i(t) + q(t)g(\omega_i(t)) \\
&= g(\omega_i(t))t^{-\lambda}\,L(t))\Big[\tau_i(\tau_i t^{2-\lambda}L(t)g'(\omega_i(t)) + (\lambda-2)C_g) \\
&\quad - \tau_i\Big(\frac{t\,A'(t)}{A(t)} - \alpha + \frac{t\,L'(t)}{L(t)}\Big) - \tau_i((\alpha+1-\lambda+(\lambda-2)C_g) + a_i \\
&\quad + \Big(\frac{q(t)}{t^{-\lambda}\,L(t)} - a_i\Big)\Big].
\end{aligned}$$

So, for the fixed $\varepsilon > 0$, there exists $M_\varepsilon > 1$ such that for $t > M_\varepsilon$ and $i \in \{1,2\}$, we have

$$\tau_i\Big|\frac{t\,A'(t)}{A(t)} - \alpha + \frac{t\,L'(t)}{L(t)}\Big| \le \frac{3}{2}\xi_2\Big(\Big|\frac{t\,A'(t)}{A(t)} - \alpha\Big| + \Big|\frac{t\,L'(t)}{L(t)}\Big|\Big) \le \frac{\varepsilon}{4},$$

$$a_1 - \frac{\varepsilon}{2} \le \frac{a(t)}{t^{-\mu}\,L(t)} \le a_2 + \frac{\varepsilon}{2}$$

$$|\tau_i(\tau_i\,t^{2-\lambda}\,L(t)g'(\omega_i(t)) + (\lambda-2)C_g)| \le \frac{3}{2}\xi_2|\tau_i\,t^{2-\lambda}\,L(t)g'(\omega_i(t)) + (\lambda-2)C_g| \le \frac{\varepsilon}{4}.$$

Indeed, the last inequality follows from (1.6) and the fact that from Lemmas 2.2 and 2.3, we have

$$\lim_{t\to\infty} \frac{t^{2-\lambda}\,L(t)}{\int_t^\infty s^{1-\lambda}\,L(s)\,ds} = 2-\lambda,$$

for all $\lambda \ge 2$. This implies that for each $t > M_\varepsilon$, we have

$$L_A\omega_1(t) + q(t)g(\omega_1(t)) \ge g(\omega_1(t))t^{-\lambda}L(t)[-\varepsilon + a_1 - \tau_1((\alpha+1-\lambda)+(\lambda-2)C_g)] = 0$$

and

$$L_A\omega_2(t)+q(t)g(\omega_2(t)) \le g(\omega_2(t))t^{-\lambda} L(t)[\varepsilon+a_2-\tau_2((\alpha+1-\lambda)+(\lambda-2)C_g)]=0.$$

Let $u\in C^2((0,\infty))\cap C([0,\infty))$ be the unique solution of (1.1) (see [17]). Then, there exists $B>0$ such that

$$\omega_1(M_\varepsilon)-B\,v_0(M_\varepsilon)\le u(M_\varepsilon)\le \omega_2(M_\varepsilon)+B\,v_0(M_\varepsilon)\,. \tag{3.2}$$

We claim that

$$\omega_1(t)-B\,v_0(t)\le u(t)\le \omega_2(t)+B\,v_0(t)\quad \text{for all } t>M_\varepsilon. \tag{3.3}$$

Assume for instance that the right inequality of (3.3) is not true. Then the function $h(t)=u(t)-\omega_2(t)-B\,v_0(t)$ for $t>M_\varepsilon$ is not negative. Consequently, there exists $t_1>M_\varepsilon$ such that $h(t_1)=\max_{M_\varepsilon\le t<\infty}h(t)>0$. Since h is continuous on $[M_\varepsilon,\infty)$, $h(M_\varepsilon)\le 0$ and $\lim_{t\to\infty}h(t)=0$, then $h'(t_1)=0$ and $h(t)>0$ for $t\in(t_1-\delta,t_1+\delta)$ for some $\delta>0$, sufficiently small. Namely $u(t)>\omega_2(t)+B\,v_0(t)$ for $t\in(t_1-\delta,t_1+\delta)$. Since g is non-increasing on $(0,\infty)$, then

$$\frac{1}{A(t)}(A(t)h'(t))'=-q(t)g(u(t))-\frac{1}{A(t)}(A(t)\omega_2'(t))'\ge q(t)(g(\omega_2(t))-g(u(t)))\ge 0,$$

for $t\in(t_1-\delta,t_1+\delta)$. Which implies $h'(t)\le h'(t_1)=0$ for $t\in(t_1-\delta,t_1)$ and $h'(t)\ge h'(t_1)=0$ for $t\in(t_1,t_1+\delta)$. This implies that h has a local minimum at t_1. Which contradicts the fact that h a global maximum at t_1 on $[M_\varepsilon,\infty)$. This proves that

$$u(t)\le\omega_2(t)+B\,v_0(t)\quad\text{for all } t>M_\varepsilon.$$

Similarly, we show that

$$\omega_1(t)-B\,v_0(t)\le u(t)\quad\text{for all } t>M_\varepsilon.$$

This proves (3.3).

Now, since $\psi_g\circ\theta\in NRVI_{(2-\lambda)(1-C_g)}$, there exists $\hat L\in\mathcal{K}$ such that $\psi_g\circ\theta=t^{(2-\lambda)(1-C_g)}\,\hat L(t)$ for $t\in[1,\infty)$. Moreover since $(\alpha-1)-(\lambda-2)(1-C_g)>0$, it follows by Lemma 2.1 that

$$\lim_{t\to\infty}\frac{t^{1-\alpha}}{t^{(2-\lambda)(1-C_g)}\hat L(t)}=0.$$

This implies that

$$\lim_{t\to\infty}\frac{t^{1-\alpha}}{\psi_g(\tau_i\int_t^\infty s^{1-\lambda}L(s)\,ds)}=\lim_{t\to\infty}\frac{t^{1-\alpha}}{\psi_g(\tau_i\theta(t))}=\lim_{t\to\infty}\frac{\psi_g(\theta(t))}{\psi_g(\tau_i\theta(t))}\frac{t^{1-\alpha}}{\psi_g(\theta(t))}=0$$

uniformly in $\tau_i\in[\frac{\xi_1}{2},\frac{3}{2}\xi_2]\subset(0,\infty)$. This together with (3.1) implies

$$\lim_{t\to\infty}\frac{v_0(t)}{\psi_g(\tau_1\theta(t))}=\lim_{t\to\infty}\frac{v_0(t)}{\psi_g(\tau_2\theta(t))}=0.$$

So, we obtain

$$\limsup_{t\to\infty}\frac{u(t)}{\omega_2(t)}\le 1\le\liminf_{t\to\infty}\frac{u(t)}{\omega_1(t)}.$$

Using this fact and assertions (i) and (iv) of Lemma 2.6, we deduce that

$$\liminf_{t\to\infty}\frac{u(t)}{\psi_g(\theta(t))}=\liminf_{t\to\infty}\frac{u(t)}{\omega_1(t)}\,\frac{\omega_1(t)}{\psi_g(\theta(t))}\ge\lim_{t\to\infty}\frac{\psi_g(\tau_1\theta(t))}{\psi_g(\theta(t))}={\tau_1}^{1-C_g}.$$

By letting ε approach zero, we obtain

$$\xi_1{}^{1-C_g} \le \liminf_{t\to\infty} \frac{u(t)}{\psi_g(\theta(t))}.$$

Similarly, we obtain

$$\limsup_{t\to\infty} \frac{u(t)}{\psi_g(\theta(t))} \le \xi_2{}^{1-C_g}.$$

This proves in particular the exact behavior at infinity in the case $\lambda = 2$. Now, for $\lambda > 2$, we have by Lemma 2.2 that $\theta(t) \sim_{t\to\infty} \frac{t^{2-\lambda}}{\lambda-2} L(t)$. Hence it follows by assertions (i), (iv) and (v) of Lemma 2.6 that for $\lambda > 2$, we have

$$\lim_{t\to\infty} \frac{\psi_g(\theta(t))}{\psi_g((t)^{2-\lambda}L(t))} = \lim_{t\to\infty} \frac{\psi_g(\theta(t))}{\psi_g((\lambda-2)\theta(t))} \frac{\psi_g((\lambda-2)\theta(t))}{\psi_g((t)^{2-\lambda}L(t))} = \frac{1}{(\lambda-2)^{1-C_g}}.$$

This achieves the proof of the Theorem. □

Proof of Theorem 1.6. We recall that $g(t) = t^{-\gamma}$, $\lambda = \alpha+1+(\alpha-1)\gamma$ and $C_g = \frac{\gamma}{1+\gamma}$. Let $\varepsilon \in (0, \frac{a_1}{2})$ and put $\tau_1 = (\gamma+1)(a_1 - \varepsilon)$ and $\tau_2 = (\gamma+1)(a_2+\varepsilon)$. Put $k(t) = \int_1^t \frac{L(s)}{s}\, ds$ and

$$\omega_i(t) = \Big((1+\gamma)\tau_i \int_t^\infty s^{1-\lambda}\, k(s)\, ds\Big)^{\frac{1}{1+\gamma}} \quad \text{for } i \in \{1,2\},$$

where L is the function given in hypothesis (H_6). Then, by a simple computation, we have

$$\begin{aligned}
&L_A\omega_i(t) + q(t)g(\omega_i(t))\\
&= g(\omega_i(t))t^{-\lambda}\, L(t)\Big[\tau_i\Big(\frac{k(t)}{L(t)}(\tau_i t^{2-\lambda}k(t)g'(\omega_i) + (\lambda-1-\alpha)) - \frac{\gamma}{\gamma+1}\Big)\\
&\quad - \tau_i\frac{k(t)}{L(t)}\Big(\frac{t\,A'(t)}{A(t)} - \alpha\Big) + \frac{\gamma}{\gamma+1}\tau_i - \tau_i + a_i + \Big(\frac{q(t)}{t^{-\lambda}\, L(t)} - a_i\Big)\Big]\\
&= g(\omega_i(t))t^{-\lambda}\, L(t)\Big[\tau_i\Big(\frac{k(t)}{L(t)}(\tau_i t^{2-\lambda}k(t)g'(\omega_i) + (\alpha-1)\gamma) - \frac{\gamma}{\gamma+1}\Big)\\
&\quad - \tau_i\, \frac{k(t)}{L(t)}\, \frac{t\,B'(t)}{B(t)} - \frac{\tau_i}{\gamma+1} + a_i + \Big(\frac{q(t)}{t^{-\lambda}\, L(t)} - a_i\Big)\Big].
\end{aligned}$$

Since $g(t) = t^{-\gamma}$ and $\lambda = \alpha+1+(\alpha-1)\gamma$, integrating by parts, we obtain

$$\begin{aligned}
&\tau_i t^{2-\lambda}k(t)g'(\omega_i(t)) + (\alpha-1)\gamma\\
&= -\gamma\tau_i\, t^{2-\lambda}k(t)(\omega_i(t))^{-(1+\gamma)} + (\alpha-1)\gamma\\
&= \gamma\Big((\alpha-1) - \frac{t^{2-\alpha}k(t)}{(\gamma+1)\int_t^\infty s^{1-\lambda}k(s)\, ds}\Big)\\
&= \gamma\Big(\frac{(\alpha-1)(1+\gamma)\int_t^\infty s^{1-\lambda}k(s)ds - t^{2-\lambda}k(t)}{(1+\gamma)\int_t^\infty s^{1-\lambda}k(s)ds}\Big)\\
&= \frac{\gamma}{\gamma+1}\, \frac{\int_t^\infty s^{1-\lambda}L(s)\, ds}{\int_t^\infty s^{1-\lambda}k(s)ds}.
\end{aligned}$$

This gives

$$\frac{k(t)}{L(t)}(\tau_i t^{2-\lambda}k(t)g'(\omega_i(t)) + (\alpha-1)\gamma) - \frac{\gamma}{\gamma+1}$$

$$= \frac{\gamma}{\gamma+1}\Big[\frac{\int_t^\infty s^{1-\lambda}L(s)\,ds}{t^{2-\lambda}\,L(t)}\frac{t^{2-\lambda}\,k(t)}{\int_t^\infty s^{1-\lambda}k(s)ds} - 1\Big].$$

This together with Lemma 2.2 and the fact that k and L are in $\mathcal{K}$, implies

$$\lim_{t\to\infty}\frac{k(t)}{L(t)}(\tau_i t^{2-\lambda}k(t)g'(\omega_i(t)) + (\alpha-1)\gamma) - \frac{\gamma}{\gamma+1} = 0.$$

Now since $\frac{t^\nu\,B'(t)}{B(t)}$ is bounded for t large and by Lemma 2.1, we have $\frac{k}{L}\in\mathcal{K}$ and $\lim_{t\to\infty}\frac{t^{1-\nu}\,k(t)}{L(t)} = 0$, we deduce that

$$\lim_{t\to\infty}\frac{k(t)}{L(t)}\Big(\frac{t\,B'(t)}{B(t)}\Big) = \lim_{t\to\infty}\frac{t^{1-\nu}\,k(t)}{L(t)}\Big(\frac{t^\nu\,B'(t)}{B(t)}\Big) = 0.$$

So, for the fixed $\varepsilon>0$, there exists $M_\varepsilon>1$ such that for $t\geq M_\varepsilon$, we have

$$L_A\omega_2(t) + q(t)g(\omega_2(t)) \leq g(\omega_2(t))t^{-\lambda}L(t)\Big[\frac{\varepsilon}{3}+\frac{\varepsilon}{3}-\frac{\tau_2}{\gamma+1}+a_2+\frac{\varepsilon}{3}\Big] = 0,$$

$$L_A\omega_1(t) + q(t)g(\omega_1(t)) \geq g(\omega_1(t))t^{-\mu}\,L(t)[-\frac{\varepsilon}{3}-\frac{\varepsilon}{3}-\frac{\tau_1}{\gamma+1}+a_1-\frac{\varepsilon}{3}\Big] = 0.$$

Let $u\in C([0,\infty))\cap C^2((0,\infty))$ be the unique solution of (1.7). As in the proof of Theorem 1.2, we choose $C>0$ such that

$$\omega_1(t) - Cv_0(t) \leq u(t) \leq \omega_2(t) + Cv_0(t) \quad \text{for } t\geq M_\varepsilon.$$

Moreover, thanks to (H6), we have $\lim_{t\to\infty}k(t)=\infty$. So, using Lemma 2.2, we obtain

$$\lim_{t\to\infty}\frac{1}{t^{\alpha-1}\Big((1+\gamma)\tau_1\int_t^\infty s^{1-\lambda}\,k(s)\,ds\Big)^{\frac{1}{1+\gamma}}} = \lim_{t\to\infty}\frac{1}{t^{\alpha-1}\big(t^{(2-\lambda)}\frac{\tau_1\,k(t)}{\alpha-1}\big)^{\frac{1}{1+\gamma}}}$$

$$= \lim_{t\to\infty}\Big(\frac{\alpha-1}{\tau_1\,k(t)}\Big)^{\frac{1}{1+\gamma}} = 0.$$

This and (3.1) gives $\lim_{t\to\infty}\frac{v_0(t)}{\omega_1(t)}=0$. Similarly, we obtain $\lim_{t\to\infty}\frac{v_0(t)}{\omega_2(t)}=0$. So we have

$$\limsup_{t\to\infty}\frac{u(t)}{\omega_2(t)} \leq 1 \leq \liminf_{t\to\infty}\frac{u(t)}{\omega_1(t)}.$$

This implies that

$$\liminf_{t\to\infty}\frac{u(t)}{\Big((1+\gamma)\int_t^\infty s^{1-\lambda}\,k(s)\,ds\Big)^{\frac{1}{1+\gamma}}} \geq \tau_1^{\frac{1}{1+\gamma}}.$$

Now, as ε tends to zero, we obtain

$$\liminf_{t\to\infty}\frac{u(t)}{\Big((1+\gamma)\int_t^\infty s^{1-\lambda}\,k(s)\,ds\Big)^{\frac{1}{1+\gamma}}} \geq ((\gamma+1)a_1)^{\frac{1}{1+\gamma}}.$$

Similarly, we obtain

$$\limsup_{t\to\infty}\frac{u(t)}{\Big((1+\gamma)\int_t^\infty s^{1-\lambda}\,k(s)\,ds\Big)^{\frac{1}{1+\gamma}}} \leq ((\gamma+1)a_2)^{\frac{1}{1+\gamma}}.$$

Now, since $(\gamma+1)\int_t^\infty s^{1-\lambda}\,k(s)\,ds \sim_{t\to\infty} \frac{t^{2-\lambda}\,k(t)}{\alpha-1} = \frac{t^{(1-\alpha)(\gamma+1)}}{\alpha}\int_1^t \frac{L(s)}{s}\,ds$, we deduce that

$$
\begin{aligned}
\Big(\frac{(\gamma+1)a_1}{\alpha-1}\Big)^{\frac{1}{1+\gamma}} &\le \liminf_{t\to\infty} \frac{u(t)}{t^{1-\alpha}\big(\int_1^t \frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}} \\
&\le \limsup_{t\to\infty} \frac{u(t)}{t^{1-\alpha}\big(\int_1^t \frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}} \\
&\le (\frac{(\gamma+1)a_2}{\alpha})^{\frac{1}{1+\gamma}}.
\end{aligned}
$$

In particular, if $a_1 = a_2$, we obtain

$$
\lim_{t\to\infty} \frac{u(t)}{t^{1-\alpha}\big(\int_1^t \frac{L(s)}{s}\,ds\big)^{\frac{1}{1+\gamma}}} = \Big(\frac{(\gamma+1)a_1}{\alpha-1}\Big)^{\frac{1}{1+\gamma}}.
$$

□

4. Applications

Application 1. We consider the Dirichlet problem

$$
\begin{gathered}
-\frac{1}{A}(Au')' + \frac{\beta}{u}(u')^2 = q(t)g(u)\,, \quad t \in (0,\infty), \\
u > 0\,, \quad \text{in } (0,\infty)\,, \\
\lim_{t\to 0^+} A(t)u'(t) = 0 \qquad \lim_{t\to\infty} u(t) = 0,
\end{gathered} \tag{4.1}
$$

where $\beta < 1$ and $\lim_{t\to\infty} \frac{q(t)}{t^{-\lambda}L(t)} = a_0 > 0$ with $\lambda \ge 2$ and $L \in \mathcal{K}$ with $\int_1^\infty s^{1-\lambda}\,L(s)\,ds < \infty$.

We assume that A satisfies (H1) and g satisfies the following hypotheses:

- (A1) The function $t \to t^{-\beta}\,g(t)$ is non-increasing from $(0,\infty)$ into $(0,\infty)$.
- (A2) $\lim_{t\to 0} g'(t)\int_0^t \frac{1}{g(s)}\,ds = -C_g$ with $\max(0, \frac{\beta}{\beta-1}) \le C_g \le 1$.
- (A3) $(\alpha-1) - (\lambda-2)(1-\beta)(1-C_g) > 0$

Note that for $\gamma > 0$ and $-\gamma < \beta < 1$, the function $g(t) = t^{-\gamma}$ satisfies (A_1) and (A2). Put $u = v^{\frac{1}{1-\beta}}$. Then v satisfies

$$
\begin{gathered}
-\frac{1}{A}(Av')' = (1-\beta)q(t)g(v^{\frac{1}{1-\beta}})v^{\frac{-\beta}{1-\beta}}\,, \quad t \in (0,\infty), \\
v > 0\,, \quad \text{in } (0,\infty)\,, \\
\lim_{t\to 0^+} A(t)v'(t) = 0, \quad \lim_{t\to\infty} v(t) = 0,
\end{gathered} \tag{4.2}
$$

The function $f(r) = (1-\beta)g(r^{\frac{1}{1-\beta}})r^{-\frac{\beta}{1-\beta}}$ is non-increasing on $(0,\infty)$ and a simple computation shows that $\psi_g = (\psi_f)^{\frac{1}{1-\beta}}$ and

$$
\lim_{r\to 0} f'(r)\int_0^r \frac{1}{f(s)}\,ds = (1-\beta)(1-C_g) - 1 =: -C_f\,, \quad \text{with } 0 \le C_f \le 1.
$$

Applying Corollary 1.3 to problem (4.2), we deduce that there exists a unique solution v to (4.2) such that

(a) When $C_f = 1$, we have

- (i) $\lim_{t\to\infty} \frac{v(t)}{\psi_f(t^{2-\lambda}L(t))} = 1$ if $\lambda > 2$;

(ii) $\lim_{t\to\infty} \frac{v(t)}{\psi_f\left(\int_t^\infty \frac{L(s)}{s}\,ds\right)} = 1$ if $\lambda = 2$;

(b) When $C_f < 1$, we have:

(i) $\lim_{t\to\infty} \frac{v(t)}{\psi_f(t^{2-\lambda}L(t))} = [\frac{a_0}{\alpha+1-\lambda+(\lambda-2)C_f}]^{1-C_f}$ if $2 < \lambda < 2 + \frac{\alpha-1}{(1-\beta)(1-c_g)}$;

(ii) $\lim_{t\to\infty} \frac{v(t)}{\psi_f(\int_t^\infty \frac{L(s)}{s}\,ds)} = [\frac{a_0}{\alpha-1}]^{1-C_f}$ if $\lambda = 2$.

This implies that problem (4.1) has a solution $u \in C([0,\infty)) \cap C^2((0,\infty))$ satisfying the following exact behavior

(a) When $C_g = 1$, we have:

(i) if $\lambda > 2$, then

$$\lim_{t\to\infty} \frac{u(t)}{\psi_g(t^{2-\lambda}L(t))} = 1;$$

(ii) if $\lambda = 2$, then

$$\lim_{t\to\infty} \frac{u(t)}{\psi_g(\int_t^\infty \frac{L(s)}{s}\,ds)} = 1;$$

(b) If $\max(0, \frac{\beta}{\beta-1}) \le C_g < 1$, then:

(i) if $2 < \lambda < 2 + \frac{\alpha-1}{(1-\beta)(1-C_g)}$, then

$$\lim_{t\to\infty} \frac{u(t)}{\psi_g(t^{2-\lambda}L(t))} = [\frac{a_0}{\alpha-1-(\lambda-2)(1-\beta)(1-C_g)}]^{1-C_g}$$

(ii) if $\lambda = 2$, then

$$\lim_{|x|\to\infty} \frac{u(t)}{\psi_g(\int_t^\infty \frac{L(s)}{s}\,ds)} = [\frac{a_0}{\alpha-1}]^{1-C_g}.$$

Application 2. In this subsection, we assume that the function A satisfy the following hypothesis

(A4) A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\frac{1}{A}$ is integrable near 0 and $\lim_{t\to\infty} \frac{t\,A'(t)}{A(t)} = \sigma \in \mathbb{R} - \{1\}$.

We are interested in the exact behavior at infinity of the unique positive solution of the problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = -p(t)u^{-\gamma}, \quad t \in (0,\infty), \\ u > 0, \quad \text{in } (0,\infty), \\ u(0) = 0, \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 0, \end{gathered} \tag{4.3}$$

where $\gamma > 0$ and $\rho(t) = \int_0^t \frac{ds}{A(s)}$. Let $u(t) = \rho(t)\,v(t)$ and $B(t) = A(t)\rho^2(t)$ for $t \in [0,\infty)$. Then u is a positive solution of (4.3) if and only if v is a positive solution of the problem

$$\begin{gathered} \frac{1}{B(t)}(B(t)v'(t))' = -\frac{p(t)}{(\rho(t))^{\gamma+1}}\,v^{-\gamma}, \quad t \in (0,\infty), \\ v > 0, \quad \text{in } (0,\infty), \\ \lim_{t\to 0^+} B(t)v'(t) = 0, \quad \lim_{t\to\infty} v(t) = 0. \end{gathered} \tag{4.4}$$

First, we claim that if A satisfies (A4), then

$$\lim_{t\to\infty}\frac{tB'(t)}{B(t)}=1+|\sigma-1|>1. \tag{4.5}$$

Since $\frac{t\,B'(t)}{B(t)}=\frac{t\,A'(t)}{A(t)}+\frac{2t}{A(t)\,\rho(t)}$ and by Definition 2.5 and assertion (ii) of Lemma 2.6, we have $A(t)=t^{\sigma}L_0(t)$ for $t\geq a>1$ with $L_0\in\mathcal{K}$, then we deduce from Lemma 2.2 that

- For $\sigma<1$, we have $\rho(\infty)=\infty$ and so

$$\rho(t)=\int_0^a\frac{ds}{A(s)}+\int_a^t\frac{1}{s^{\sigma}\,L_0(s)}\,ds\sim\frac{1}{1-\sigma}\frac{t^{1-\sigma}}{L_0(t)}\quad\text{as } t\to\infty.$$

So

$$\frac{2t}{A(t)\,\rho(t)}\sim(1-\sigma)\frac{L_0(t)}{t^{1-\sigma}}\frac{2t}{t^{\sigma}L_0(t)}=2(1-\sigma)\quad\text{as } t\to\infty.$$

Consequently in this case we have

$$\lim_{t\to\infty}\frac{t\,B'(t)}{B(t)}=\sigma+2(1-\sigma)=2-\sigma=1+|\sigma-1|.$$

- For $\sigma>1$, we have $\rho(\infty)=\int_0^{\infty}\frac{ds}{A(s)}\,ds<\infty$. So

$$\frac{2t}{A(t)\,\rho(t)}\sim\frac{2t}{t^{\sigma}L_0(t)\rho(\infty)}\to 0\quad\text{as } t\to\infty.$$

In this case we have

$$\lim_{t\to\infty}\frac{t\,B'(t)}{B(t)}=\sigma=1+|\sigma-1|.$$

This proves (4.5). Taking into account this fact, we assume that the function p satisfies the following hypotheses

(A5) p is a nonnegative continuous function $(0,\infty)$ satisfying

$$0<a_0=\lim_{t\to\infty}\frac{t^{\lambda}\,p(t)}{L(t)(\rho(t))^{\gamma+1}}<\infty\,,$$

where $\lambda\geq 2$ and $L\in\mathcal{K}$ such that $\int_1^{\infty}s^{1-\lambda}\,L(s)\,ds<\infty$.

(A6) $2+|\sigma-1|-\lambda+(\lambda-2)\frac{\gamma}{\gamma+1}>0$.

Assume that A and p satisfy (A4)–(A6) and let v be the unique positive solution of problem (4.4). Then v has the following exact behavior at infinity

(i) if $\lambda>2$, then

$$\lim_{t\to\infty}\frac{v(t)}{[(\gamma+1)t^{2-\lambda}L(t)]^{\frac{1}{1+\gamma}}}=\Big[\frac{a_0}{(\lambda-2)(2+|\sigma-1|-\lambda+(\lambda-2)\frac{\gamma}{\gamma+1})}\Big]^{\frac{1}{1+\gamma}}.$$

(ii) if $\lambda=2$, then

$$\lim_{t\to\infty}\frac{v(t)}{[(\gamma+1)\int_t^{\infty}\frac{L(s)}{s}ds]^{\frac{1}{1+\gamma}}}=\Big[\frac{a_0}{|\sigma-1|}\Big]^{\frac{1}{1+\gamma}}$$

Consequently, the unique positive solution u of problem (4.3) has the following exact behavior at infinity

(i) if $\lambda>2$, then

$$\lim_{t\to\infty}\frac{u(t)}{\rho(t)[(\gamma+1)t^{2-\lambda}\,L(t)]^{\frac{1}{1+\gamma}}}=\Big[\frac{a_0}{(\lambda-2)(2+|\sigma-1|-\lambda+(\lambda-2)\frac{\gamma}{\gamma+1})}\Big]^{\frac{1}{1+\gamma}}.$$

ii) if $\lambda = 2$, then

$$\lim_{t\to\infty} \frac{u(t)}{\rho(t)[(\gamma+1)\int_t^\infty \frac{L(s)}{s}ds]^{\frac{1}{1+\gamma}}} = \Big[\frac{a_0}{|\sigma-1|}\Big]^{\frac{1}{1+\gamma}}$$

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