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ENTIRE FUNCTIONS RELATED TO STATIONARY SOLUTIONS OF THE KAWAHARA EQUATION

ANDRÉ LUIZ C. DOS SANTOS, PATRÍCIA N. DA SILVA,
CARLOS FREDERICO VASCONCELLOS

Abstract. In this study, we characterize the lengths of intervals for which the linear Kawahara equation has a non-trivial solution, whose energy is stationary. This gives rise to a family of complex functions. Characterizing the lengths amounts to deciding which members of this family are entire functions. Our approach is essentially based on determining the existence of certain Möbius transformation.

1. Introduction

In the Kawahara equation

$$u_t + u_x + \kappa u_{xxx} + \eta u_{xxxxx} + uu_x = 0, \tag{1.1}$$

the conservative dispersive effect is represented by the term $(\kappa u_{xxx} + \eta u_{xxxxx})$. This equation is a model for plasma wave, capilarity-gravity water waves and other dispersive phenomena when the cubic KdV-type equation is weak. Kawahara [11] pointed out that it happens when the coefficient of the third order derivative in the KdV equation becomes very small or even zero. It is then necessary to take into account the higher order effect of dispersion in order to balance the nonlinear effect. Kakutani and Ono [10] showed that for a critical value of angle between the magneto-acoustic wave in a cold collision-free plasma and the external magnetic field, the third order derivative term in the KdV equation vanishes and may be replaced by the fifth order derivative term. Following this idea, Kawahara [11] studied a generalized nonlinear dispersive equation which has a form of the KdV equation with an additional fifth order derivative term. This equation has also been obtainded by Hasimoto [9] for the shallow wave near critical values of surface tension. More precisely, in this work Hasimoto found these critical values when the Bond number is near one third.

While analyzing the evolution of solutions of the water wave-problem, Schneider and Wayne [17] also showed that the coefficient of the third order dispersive term in nondimensionalized statements of the KdV equation vanishes when the Bond number is equal to one third. The Bond number is proportional to the strength

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of the surface tension and in the KdV equation it is related to the leading order dispersive effects in the water-waves problem. With its disappearance, the resulting equation is just Burger's equation whose solutions typically form shocks in finite time. Thus, if we wish to model interesting behavior in the water-wave problem it is necessary to include higher order terms. That is, it is necessary to consider the Kawahara equation. In any case, the inclusion of the fifth order derivative term takes into account the comparative magnitude of the coefficients of the third and fifth power terms in the linearized dispersion relation.

Berloff and Howard [3] presented the Kawahara equation as the purely dispersive form of the nonlinear partial differential equation

$$u_t + u^r u_x + a u_{xx} + b u_{xxx} + c u_{xxxx} + d u_{xxxxx} = 0.$$

The above equation describes the evolution of long waves in various problems in fluid dynamics. The Kawahara equation corresponds to the choice $a = c = 0$ and $r = 1$ and describes water waves with surface tension. Bridges and Derks [6] presented the Kawahara equation – or fifth-order KdV-type equation – as a particular case of the general form

$$u_t + \kappa u_{xxx} + \eta u_{xxxxx} = \frac{\partial}{\partial x} f(u, u_x, u_{xx}) \tag{1.2}$$

where $u(x,t)$ is a scalar real valued function, κ and $\eta \neq 0$ are real parameters and $f(u, u_x, u_{xx})$ is some smooth function. The form (1.1) occurs most often in applications and corresponds to the choice of f in (1.2) with the form $f(u, u_x, u_{xx}) = -u^2/2$.

As noted by Kawahara [11], we may assume without loss of generality that $\eta < 0$ in (1.1). In fact, if we introduce the following simple transformations

$$u \to -u, \quad x \to -x, \quad t \to t,$$

we can obtain an equation of the form of (1.1) in which κ and η are replaced, respectively, by $-\kappa$ and $-\eta$.

Hagarus, Lombardi and Scheel [8] pointed out that the Kawahara equation

$$u_t = u_{xxxxx} - \varepsilon u_{xxx} + u u_x, \tag{1.3}$$

in which ε is a real parameter, models water waves in the long-wave regime for moderate values of surface tension (Weber numbers close to 1/3). For such Weber numbers, the usual description of long water waves via the Korteweg-de Vries (KdV) equation fails since the cubic term in the linear dispersion relation vanishes and fifth order dispersion becomes relevant at leading order, $\omega(k) = k^5 + \varepsilon k^3$. Positive (resp. negative) values of the parameter ε in (1.3) correspond to Weber numbers larger (resp. smaller) than 1/3.

Dispersive problems have been object of intensive research (see, for instance, the classical paper of Benjamin, Bona and Mahoni [2], Biagioni and Linares [4], Bona and Chen [5], Menzala *et al.* [13], Rosier [15], and references therein). Recently global stabilization of the generalized KdV system have been obtained by Rosier and Zhang [16] and Linares and Pazoto [12] studied the stabilization of the generalized KdV system with critical exponents. For the stabilization of global solutions of the Kawahara under the effect of a localized damping mechanism, see Vasconcellos and Silva [19, 20, 21].

We consider the linear Kawahara equation

$$u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} = 0 \quad \text{with } (x,t) \in (0, L_0) \times (0, \infty), \tag{1.4}$$

where the coefficients β, κ and η are real numbers such that $\eta < 0$, $\kappa \neq 0$, $\beta \in \{0,1\}$. Sometimes, while discussing the existence of solutions of certain partial differential equations, it is necessary to establish when a certain quotient of entire functions still turns out to be an entire function (see, for instance, Rosier [15], Vasconcellos and Silva [19]). The problem of factoring an entire function is solved by the famous Weierstrass factorization theorem and its corollaries. However, applying this result may not be very practical or even viable in some cases.

Now we proceed to a general description of such kind of factoring problems. We have a polynomial $p : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_{\mathbf{a}} : \mathbb{C} \times (0,\infty) \to \mathbb{C},$$

$\mathbf{a} \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, whose restriction $N_{\mathbf{a}}(\cdot, L)$ is entire for each $L > 0$. We consider a family of functions $f_{\mathbf{a}}(\cdot, L)$ defined by

$$N_{\mathbf{a}}(\xi, L) = f_{\mathbf{a}}(\xi, L)\, p(\xi) \tag{1.5}$$

in its maximal domain. For a given polynomial $p(\cdot)$, the problem of characterizing the set of values $L_0 > 0$, for which it is possible to find a non null $\mathbf{a}_0 \in \mathbb{C}^4$ such that the function $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire, is a challenging problem and of particular interest of the academic community.

Vasconcellos and Silva [19, Lemma 2.1] discussed the existence of non-zero solutions for (1.4) whose energy is constant over time. Their results show that the existence of such solutions is equivalent to determining the lengths of interval $(0, L_0)$ for which it is possible to verify that the condition

$$(\exists\, \lambda \in \mathbb{C},\ u_0 \in (H_0^3(0,L_0) \cap H^5(0,L_0), \mathbb{C})\ \Rightarrow\ \lambda u_0 + \beta u_0' + \kappa u_0''' + \eta u_0''''' = 0) \tag{1.6}$$

is valid. Such condition in turn reduces to the problem of characterizing the set $\mathcal{X}$ of $L_0 > 0$ values, for which exist r and $\mathbf{a}_0$ providing that function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$ and $\mathbf{a} = \mathbf{a}_0$. In this case, using (1.5), $f_{\mathbf{a}}(\cdot, L)$ is defined by

$$\begin{gathered} N_{\mathbf{a}}(\xi, L) = a_1 i\xi - a_2 i\xi e^{-i\xi L} + a_3 - a_4 e^{-i\xi L} \\ p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5 \end{gathered} \tag{1.7}$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $r \in \mathbb{R}$. It follows from (1.6) that λ is a pure imaginary number. Thus, we only have to consider polynomials $p(\cdot)$ with $r \in \mathbb{R}$.

For each $r \in \mathbb{R}$ and $\mathbf{a}_0 \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, let $\mathcal{X}_{\mathbf{a}_0 r}$ be the set of $L_0 > 0$ values, for which the function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$ and $\mathbf{a} = \mathbf{a}_0$. The set $\mathcal{X}$ is the union of $\mathcal{X}_{\mathbf{a}_0 r}$ for $r \in \mathbb{R}$ and $\mathbf{a}_0 \in \mathbb{C}^4 \sim \{\mathbf{0}\}$.

Here, we place emphasis on the following statements:

- (S1) $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire;
- (S2) all the zeros, taking the respective multiplicities into account, of the polynomial p are zeros of $N_{\mathbf{a}_0}(\cdot, L)$;
- (S3) the maximal domain of $f_{\mathbf{a}_0}(\cdot, L_0)$ is $\mathbb{C}$;

which are, clearly, equivalent and will be widely used throughout this article. A closer look shows that determining the solution to the problem guarantees the existence of a Möbius transformation in some circumstances. Further, for the function $f_{\mathbf{a}}(\cdot, L)$, defined by (1.5) and (1.7), to be entire, given the equivalence between statements (S1) and (S2); informally, we must have

$$\frac{a_1 i\xi_0 + a_3}{a_2 i\xi_0 + a_4} = e^{-iL\xi_0} \tag{1.8}$$

for each root ξ_0 of the polynomial p. We note that for $\mathbf{a}$ such that $a_1a_3 - a_2a_4 \neq 0$, the left side of (1.8) suggests that a Möbius transformation is defined. Note that we already have an indication that for a polynomial p with at least two roots differing by an integer multiple of $2\pi/L$, we obtain $L \notin \mathcal{X}$. With this, a method for solving the problem is revealed: we must verify for which structures of the roots of the polynomial p is it possible to define a Möbius transformation M that satisfies $M(\xi_0) = e^{-iL\xi_0}$ for each zero ξ_0 of polynomial p (See Lemma 2.6).

Taking (1.8), it is essential to define, for each non null $\mathbf{a} \in \mathbb{C}^4$, the discriminant of $\mathbf{a}$, specifically, the complex number $d(\mathbf{a}) = a_1a_4 - a_2a_3$. It is natural, however, to consider:

(i) $d(\mathbf{a}) = 0$ or
(ii) $d(\mathbf{a}) \neq 0$.

The main result shown in this article guarantees that the existence of pairs $(\mathbf{a}_0, L_0)$ that make $f_{\mathbf{a}}(\cdot, L)$ entire is intimately linked to whether or not the discriminant is zero. In fact, when the discriminant of $\mathbf{a}$ is zero, such pairs do not exist for any $r \in \mathbb{R}$. On the other hand, if the discriminant of $\mathbf{a}$ is non-zero, we identify situations where the pairs $(\mathbf{a}_0, L_0)$ can exist or not. Whereas case (i) has been completely solved here, in case (ii) there are situations where the problem remains to be solved, i.e., in some cases, we do not know whether or not it is possible to satisfy (1.8). As far as we know, Rosier [15] was the first to analyze these kinds of problems. In fact, he showed that the existence of non-trivial solutions for the Kortweg de Vries equation, whose energies do not decay over time, is equivalent to determining the set $\mathcal{U}$ of values $l_0 > 0$, for which there exists a non null $\mathbf{k}_0 \in \mathbb{C}^2$ and $s \in \mathbb{C}$, so that the function $g_{\mathbf{k}}(\cdot, L)$ with $\mathbf{k} = (k_1, k_2)$, defined by

$$M_{\mathbf{k}}(\xi, l) = g_{\mathbf{k}}(\xi, l)\, q(\xi), \tag{1.9}$$

is entire for $\mathbf{k} = \mathbf{k}_0$ and $l = l_0$. Here, in particular, $M_{\mathbf{k}}(\xi, l) = k_1 - k_2e^{-iL\xi}$ and $q(\xi) = \xi^3 - \xi + s$. Then Rosier [15] proves that

$$\mathcal{U} = \Big\{2\pi\sqrt{\frac{m^2 + mn + n^2}{3}} \;:\; n, m \in \mathbb{N}\Big\}.$$

Let us take case (i) from the same starting point as Rosier [15], i.e., the analysis of zeros of $N_{\mathbf{a}}(\cdot, L)$. Here, it makes no sense to argue about the existence of a Möbius transformation. Case (ii) is completely based on equation (1.8). Our strategy is quite efficient. It proved to be efficient in this situation, where using previously established results, such as the Weierstrass factorization theorem, is not possible.

Notice that, for each choice of the coefficients β, κ and η, condition (1.6) associates the Kawahara equation

$$u_t + \beta u_x + \kappa u_{xxx} + \eta u_{xxxxx} = 0$$

to a family of polynomials

$$p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5, \quad r \in \mathbb{R}.$$

Let $\mathcal{X}$ be the set of the lengths of interval $(0, L_0)$ for which exist non-zero solutions for (1.4) whose energy is constant over time. Consider for each $r \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{C}^4 \sim \{\mathbf{0}\}$, the set $\mathcal{X}_{\mathbf{a}r}$ of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ defined by (1.5) and (1.7) is entire for $L = L_0$. We can decompose $\mathcal{X}$ as the union of the sets $\mathcal{X}_{\mathbf{a}r}$ for $r \in \mathbb{R}$ and non null $\mathbf{a} \in \mathbb{C}^4$

We extend the results obtained by Vasconcellos and Silva [19, 20] for characterizing the set $\mathcal{X}$ for the Kawahara equation (1.4). They have partially analyzed the case $\kappa = 0$ in (1.4) and did not deal with the case $\kappa = 1$ in (1.4). In our proof, we argue by exhaustion characterizing the sets $\mathcal{X}_{\mathbf{a}r}$. In Part (I) of Theorem 1.1, we see that if $d(\mathbf{a}) = 0$, then $\mathcal{X}_{\mathbf{a}r} = \emptyset$ for all $r \in \mathbb{R}$. As a consequence of this result, it follows that for any Kawahara equation (1.4), the set $\mathcal{X}$ is given by the union of the set $\mathcal{X}_{\mathbf{a}r}$ for $r \in \mathbb{R}$ and $d(\mathbf{a}) \neq 0$. Our results for $d(\mathbf{a}) \neq 0$ allow to partially describe the set $\mathcal{X}$ for Kawahara equations (1.4) with $\beta = 1$ and $\kappa \neq 0$ or $\beta = 0$ and $\kappa < 0$. For Kawahara equations (1.4) with $\beta = 0$, $\kappa > 0$, as a consequence of Theorem 1.1, we obtain that $\mathcal{X}$ is empty.

Now we summarize the results obtained in this article in the following theorem guided by the roots of polynomial p, as we will shortly see.

Theorem 1.1. *Let $r \in \mathbb{R}$, a non null $\mathbf{a} \in \mathbb{C}^4$ and $L > 0$, and consider the function $f_{\mathbf{a}}(\cdot, L)$ defined by the product*

$$N_{\mathbf{a}}(\xi, L) = f_{\mathbf{a}}(\xi, L)\, p(\xi) \tag{1.10}$$

in its maximal domain. Let us suppose that $N_{\mathbf{a}}(\xi, L)$ and $p(\xi)$ are as in (1.7). Let $\mathcal{X}_{\mathbf{a}r}$ be the set of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ defined by (1.10) is entire for $L = L_0$.

(I) If $L_0 > 0$ is such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire for some non null $\mathbf{a}_0 \in \mathbb{C}^4$, then $d(\mathbf{a}_0) \neq 0$. In other words, for any non null $\mathbf{a}$, if $d(\mathbf{a}) = 0$, we obtain $\mathcal{X}_{\mathbf{a}r} = \emptyset$, for any $r \in \mathbb{R}$. The reciprocal, however, is false.

(II) If $\mathbf{a}$ is such that $d(\mathbf{a}) \neq 0$ and one of following three items occurs:

- (a) *$\beta = 1$ and $|r| > z - \kappa z^3 + \eta z^5$, where $z = \sqrt{\frac{3\kappa - \sqrt{9\kappa^2 - 20\eta}}{10\eta}}$;*
- (b) *$\beta = 0$, $\kappa > 0$ and $r \in \mathbb{R}$;*
- (c) *$\beta = 0$, $\kappa < 0$ and $|r| > -\kappa z^3 + \eta z^5$, where $z = \sqrt{\frac{3\kappa}{5\eta}}$.*

Then there is no $L > 0$ that renders the function $f_{\mathbf{a}}(\cdot, L)$ entire. Therefore, $\mathcal{X}_{\mathbf{a}r} = \emptyset$.

(III)

- (a) *If $\beta = 1$ and $r = 0$, then there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*

$$L_0 \in \Big\{L \in \mathbb{R},\ k \operatorname{cotanh}\Big(\frac{Lk}{2}\Big) = -\rho \cot\Big(\frac{L\rho}{2}\Big)\Big\}$$

where

$$\rho = \sqrt{\frac{\kappa - \sqrt{\kappa^2 - 4\eta}}{2\eta}} \quad \text{and} \quad k = \sqrt{\Big|\frac{\kappa + \sqrt{\kappa^2 - 4\eta}}{2\eta}\Big|}.$$

- (b) *If $\beta = 0$, $\kappa < 0$ and $r = 0$, then there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*

$$L_0 \in \Big\{L > 0,\ \tan\frac{\rho L}{2} = \frac{\rho L}{2}\Big\},$$

where $\rho = \sqrt{\kappa/\eta}$.

The sets in (a) and (b) are enumerable.

The knowledge of the zeros of $g_{\mathbf{k}}(\xi, l)$ in (1.9) plays a key role in the Rosier's analysis of the existence of non-trivial solutions for the Kortweg de Vries equation, whose energies do not decay over time. The function $f_{\mathbf{a}}(\xi, L)$ related to Kawahara equation does not resemble this fact and its structure together with the order of the polynomial turn the analysis of the Kawahara case into a hard problem. Many other authors have made efforts to tackle this problem (see for instance, Glass and Guerrero [7], for a particular case of (III)(b); Araruna, Capistrano-Filho and Doronin [1], for an example of a critical set). Our results take their contributions into account. We show they can be presented and obtained in a systematic way and we go a step further.

2. Proof of main results

First we establish some lemmas need for proving the three parts of Theorem 1.1.

Part (I). The main idea behind Part (I) of Theorem 1.1 is to find out whether there is at least one zero of polynomial p that is not a zero of $N_{\mathbf{a}}(\cdot, L)$. The following lemma is a decisive factor in obtaining this result.

Lemma 2.1. *Let non null $\mathbf{a} \in \mathbb{C}^4$ with $d(\mathbf{a}) = 0$ and $L > 0$. Then the set of the imaginary parts of the zeros of $N_{\mathbf{a}}(\cdot, L)$ has at most two elements.*

Proof. Fix an arbitrarily $L > 0$ and a non null $\mathbf{a}$ such that $d(\mathbf{a}) = 0$. The nullity of the discriminant of $\mathbf{a}$ guarantees that the vectors (a_1, a_3) and (a_2, a_4) are linearly dependent. We can then assume that there exists some complex number λ such that $(a_1, a_3) = \lambda(a_2, a_4)$. Thus (a_2, a_4) cannot be zero and, if (a_1, a_3) is zero, then $\lambda = 0$. Therefore, we can write

$$N_{\mathbf{a}}(\xi, L) = (a_2 i\xi + a_4)\,(\lambda - e^{-iL\xi}) \tag{2.1}$$

Finally, we see that in one of the factors of (2.1) there is at most one zero and, in the other, an infinite number of zeros all with the same imaginary part. Thus, the set of the imaginary parts of the zeros of $N_{\mathbf{a}}(\cdot, L)$ has no more than two elements. If we assume that $(a_2, a_4) = \lambda(a_1, a_3)$, the same conclusion about the zeros of $N_{\mathbf{a}}(\cdot, L)$ is valid, proving the result. □

By quickly verifying the result shown in Lemma 2.6, we conclude the set of the imaginary parts of the polynomial p has at least three elements, except for $r = \beta = 0$ and $\kappa < 0$, when all the roots are real. With the aim of proving part (I) of Theorem 1.1, we fix an arbitrary $L > 0$ and a non null $\mathbf{a}$ such that $d(\mathbf{a}) = 0$. We initially consider any complementar case to $r = \beta = 0$ and $\kappa < 0$. The Lemmas 2.6 and 2.1 combined guarantee that there will always be a zero of the polynomial p that is not a zero of $N_{\mathbf{a}}(\cdot, L)$. Consequently, the function $f_{\mathbf{a}}(\cdot, L)$ is not entire. Finally, we suppose that $r = \beta = 0$ and $\kappa < 0$ and assume that $f_{\mathbf{a}}(\cdot, L)$ is entire. Since 0 is a root of multiplicity three, given the equivalence between (S1) and (S2), we have $N_{\mathbf{a}}(0, L) = N'_{\mathbf{a}}(0, L) = N''_{\mathbf{a}}(0, L) = 0$ (differentiating with respect to ξ). These three equations imply that $\mathbf{a} = \mu(-\frac{L}{2}, \frac{L}{2}, 1, 1)$ for a complex number $\mu \neq 0$. Here, we obtain $d(\mathbf{a}) = -\mu L \neq 0$, which contradicts the hypothesis of the nullity of the discriminant of $\mathbf{a}$. Therefore, $f_{\mathbf{a}}(\cdot, L)$ is not entire for any value of $L > 0$ and non null $\mathbf{a}$ such that $d(\mathbf{a}) = 0$. This proves part (I) of Theorem 1.1.

Part (II). The following lemma essentially states that if the polynomial p has "too many" complex roots, equation (1.8) cannot be satisfied.

Lemma 2.2. *For any $L > 0$, there is no Möbius transformation M such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}\}$$

with $\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}$ all distinct in $\mathbb{C}$.

Proof. Write $\xi_j = x_j + iy_j$ and $\omega_j = M(\xi_j)$ for $j = 1, 2$ and note that $M(\overline{\xi_j}) = \frac{1}{\overline{\omega_j}}$. The Möbius transformation M that satisfies

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}\}$$

exists if, and only if, the equality

$$(\xi_1, \xi_2; \overline{\xi_1}, \overline{\xi_2}) = \Big(\omega_1, \omega_2; \frac{1}{\overline{\omega_1}}, \frac{1}{\overline{\omega_2}}\Big)$$

is valid [18, Theorem VI,p. 178]. (Note that $(\xi_1, \xi_2; \xi_3, \xi_4)$ stands for the cross ratio of pairs (ξ_1, ξ_2) and (ξ_3, ξ_4)) However, this does not occur, as

$$\Big(\omega_1, \omega_2; \frac{1}{\overline{\omega_1}}, \frac{1}{\overline{\omega_2}}\Big) = \frac{4\sinh Ly_1 \sinh Ly_2}{2\cosh L(y_1 + y_2) - 2\cos L(x_2 - x_1)} > \frac{4y_1y_2}{K} = (\xi_1, \xi_2; \overline{\xi_1}, \overline{\xi_2}),$$

where $K = |\xi_1 - \overline{\xi_2}|$. To prove this statement, let us assume, without loss of generality, that $y_1, y_2 > 0$ and define the function:

$$F(t) = K\sinh(y_1t)\sinh(y_2t) - 2y_1y_2[\cosh((y_1 + y_2)t) - \cos((x_2 - x_1)t)]$$

for $t \in \mathbb{R}$. A direct calculation (see the Appendix) shows that $F(0) = F'(0) = 0$ and $F''(t) > 0$ for all $t > 0$. Using a second order Taylor expansion for the function F, we conclude that $F(t) > 0$ for all $t > 0$. In particular, this means that for $L > 0$, the inequality

$$4K\sinh(Ly_1)\sinh(Ly_2) > 4y_1y_2(2\cosh(L(y_1 + y_2)) - 2\cos(L(x_2 - x_1)))$$

is valid. This completes the proof. $\square$

Thus, now we can prove part (II) of Theorem 1.1. Let us suppose there exists $\mathbf{a}$ such that $d(\mathbf{a}) \neq 0$ and that one of (a), (b) with $r \neq 0$, or (c) of part (II) of the theorem occurs (case (b) with $r = 0$ is proven in Lemma 2.5).

Here, Lemma 2.6 guarantees that polynomial p has a single real root, whose multiplicity is equal to 1. This means that this polynomial has two pairs of complex conjugate roots. Let us assume, in contradiction, that there exists $L > 0$ such that the function $f_{\mathbf{a}}(\cdot, L)$ is entire. Then, all roots of polynomial p must satisfy (1.8); i.e., there exists a Möbius transformation that takes each root ξ_0 of p into $e^{-iL\xi_0}$. However, this contradicts Lemma 2.2 and proves part (II) of the theorem except for the case where $\beta = 0$, $\kappa > 0$ and $r = 0$ that will be shown in Lemma 2.5.

Part (III). Lemma 2.3 below, unlike Lemma 2.2, guarantees the existence of a Möbius transformation in a case when the polynomial p has exactly three real roots whose multiplicities are equal to 1. Lemma 2.5 below guarantees the existence of a Möbius transformation when all roots of polynomial p are real.

Lemma 2.3. *Let L, k and ρ be real numbers with $L > 0$ and $k \neq 0$. There is a unique Möbius transformation M which satisfies $M(0) = 1$, $M(\pm\rho) = e^{\mp iL\rho}$ and $M(\pm ik) = e^{\pm Lk}$ if and only if the following equality occurs*

$$k \operatorname{cotanh}\Big(\frac{Lk}{2}\Big) = -\rho \cot\Big(\frac{L\rho}{2}\Big).$$

Proof. First we prove the necessity part. Assume that exists a unique Möbius transformation M that satisfies $M(0) = 1$, $M(\pm\rho) = e^{\mp iL\rho}$ and $M(\pm ik) = e^{\pm Lk}$. We know [18, Theorem V, p. 176] that the pairs $(0, 1)$, (ik, e^{Lk}) and $(-ik, e^{-Lk})$ unambiguously determine the Möbius transformation of M_1, whose expression is

$$M_1(z) = -\frac{z + ik \operatorname{cotanh}\big(\frac{Lk}{2}\big)}{z - ik \operatorname{cotanh}\big(\frac{Lk}{2}\big)} \tag{2.2}$$

and can be obtained by (5.1). Thus, we must obtain $M = M_1$. (For the coefficients $M_1(z)$ and $M_2(z)$, see the Appendix)

Similarly, the injectivity of M implies that L is not an integer multiple of $\frac{\pi}{\rho}$. Thus the pairs $(0, 1)$, $(\rho, e^{-iL\rho})$ and $(-\rho, e^{iL\rho})$ determine a unique Möbius transformation M_2. The formula (5.1) guarantees that M_2 is expressed by

$$M_2(z) = -\frac{z - i\rho \cot\big(\frac{L\rho}{2}\big)}{z + i\rho \cot\big(\frac{L\rho}{2}\big)}. \tag{2.3}$$

Consequently, we obtain $M = M_2$. Thus, $M_1 = M_2$. We know [14, Theorem 3, p. 26] that if M_1 and M_2 coincide, their coefficients must be such that

$$\begin{bmatrix} -1 & -ik \operatorname{cotanh}\big(\frac{Lk}{2}\big) \\ 1 & -ik \operatorname{cotanh}\big(\frac{Lk}{2}\big) \end{bmatrix} = \begin{bmatrix} -1 & i\rho \cot\big(\frac{L\rho}{2}\big) \\ 1 & i\rho \cot\big(\frac{L\rho}{2}\big) \end{bmatrix}.$$

The same result guarantees the sufficiency part. □

Lemma 2.4. *Let k and ρ be real numbers with $k \neq 0$. The positive solutions of the equation*

$$k \operatorname{cotanh}\Big(\frac{Lk}{2}\Big) = -\rho \cot\Big(\frac{L\rho}{2}\Big)$$

form a countable set.

Proof. If $\rho = 0$, there is nothing to prove. Thus, suppose that $\rho \neq 0$. For every $n \in \mathbb{N}$, let $J_n = ((n-1)\pi, n\pi)$ and $J = \cup_{n\in\mathbb{N}} J_n$. The continuous function $g : J \to \mathbb{R}$, defined by $g(t) = k \operatorname{cotanh}(\frac{kt}{2}) + \rho \cot(\frac{\rho t}{2})$, is strictly decreasing on each J_n and such that $g[J_n] = \mathbb{R}$. Therefore, there exists a unique L_n in each J_n such that $g(L_n) = 0$, proving that the solutions of the equation form a countable set. This completes the proof. □

We have all the ingredients to prove item (a) of part (III) of Theorem 1.1. If $\beta = 1$ and $r = 0$, the polynomial p has exactly three real roots, whose multiplicities are equal to 1. Now, it is sufficient to observe that Lemma 2.3 together with equation (1.8), as well as Lemma 2.4, conclude this. Finally, we prove the final lemma that supports item (b) of Theorem 1.1.

Lemma 2.5. *Consider the family of functions $f_{\mathbf{a}}(\cdot, L)$ defined by (1.7) and let us suppose that $r = \beta = 0$. Let $\mathcal{X}_{\mathbf{ar}}$ be the set of values $L_0 > 0$ for which the function $f_{\mathbf{a}}(\cdot, L)$ is entire for $L = L_0$.*

(1) *If $\kappa < 0$, there exist $L_0 > 0$ and non null $\mathbf{a}_0$ such that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire if and only if*
$$L_0 \in \Big\{L > 0, \tan\frac{\rho L}{2} = \frac{\rho L}{2}\Big\}$$
where $\rho = \sqrt{\kappa/\eta}$.

(2) *If $\kappa > 0$, then $f_{\mathbf{a}}(\cdot, L)$ is not entire for any non null $\mathbf{a}$ and $L > 0$. That is, $\mathcal{X}_{\mathbf{a}0} = \emptyset$.*

Proof. In the case $\kappa < 0$, the polynomial p allows the decomposition $p(\xi) = \eta\xi^3(\xi - \rho)(\xi + \rho)$ where $\rho = \sqrt{\kappa/\eta}$. Assume L_0 is in
$$\mathcal{P} = \Big\{L > 0,\ \tan\frac{\rho L}{2} = \frac{\rho L}{2}\Big\}$$
and set $\mathbf{a}_0 = (-\frac{L_0}{2}, \frac{L_0}{2}, 1, 1)$. We obtain
$$N_{\mathbf{a}_0}(0, L_0) = N'_{\mathbf{a}_0}(0, L_0) = N''_{\mathbf{a}_0}(0, L_0) = 0,$$
i.e., 0 is the root of $N_{\mathbf{a}_0}(0, L_0)$ with multiplicity at least equal to 3. Now, if we prove that $N_{\mathbf{a}_0}(\pm\rho, L_0) = 0$, then we will be done. To this end, observe that $N_{\mathbf{a}_0}(\rho, L_0) = 0$ if and only if
$$\Big(\frac{i\rho L_0}{2} + 1\Big)e^{(\frac{i\rho L_0}{2}+1)} = \overline{\Big(\frac{i\rho L_0}{2} + 1\Big)e^{(\frac{i\rho L_0}{2}+1)}} \tag{2.4}$$
This equality, in turn, is equivalent to $\sin\frac{\rho L_0}{2} = \frac{\rho L_0}{2}\cos\frac{\rho L_0}{2}$, which is valid, since we chose L_0 in $\mathcal{P}$. The same argument proves that $N_{\mathbf{a}_0}(-\rho, L_0) = 0$. Then, the function $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire, proving that the condition is sufficient. Now, we will prove that it is necessary. Suppose that $f_{\mathbf{a}_0}(\cdot, L_0)$ is entire for a certain pair $(\mathbf{a}_0, L_0)$. This tells us that
$$N_{\mathbf{a}_0}(0, L_0) = N'_{\mathbf{a}_0}(0, L_0) = N''_{\mathbf{a}_0}(0, L_0) = N_{\mathbf{a}_0}(\pm\rho, L_0) = 0. \tag{2.5}$$
The three first equalities in (2.5) can be used to explicitly determine $\mathbf{a}_0$, i.e., $\mathbf{a}_0 = \mathbf{a}(L_0) = \gamma(-\frac{L_0}{2}, \frac{L_0}{2}, 1, 1)$ for $\gamma \neq 0$. As $N_{\mathbf{a}(L_0)}(\pm\rho, L_0) = 0$, we conclude from argument (2.4), that L_0 must be in $\mathcal{P}$, proving that the condition is also necessary.

Finally, we assume $\kappa > 0$. Here, the polynomial p admits the factorization $p(\xi) = \eta\xi^3(\xi - \widetilde{\rho})(\xi + \widetilde{\rho})$ where $\widetilde{\rho} = i\rho$ and $\rho = \sqrt{-\kappa/\eta}$. Suppose that function $f_{\mathbf{a}}(\cdot, L)$ is entire for the pair $(\mathbf{a}, L)$. This implies
$$N_{\mathbf{a}}(0, L) = N'_{\mathbf{a}}(0, L) = N''_{\mathbf{a}}(0, L) = N_{\mathbf{a}}(\pm\widetilde{\rho}, L) = 0. \tag{2.6}$$
As in the first part, we can obtain $\mathbf{a}(L) = \gamma(-\frac{iL}{2}, \frac{iL}{2}, 1, 1)$ for $\gamma \neq 0$. Thus
$$N_{\mathbf{a}(L)}(\xi, L) = \gamma\Big[-\frac{iL}{2}\xi + 1 - \Big(\frac{iL}{2}\xi + 1\Big)e^{-i\xi L}\Big].$$
On the other hand, as $N_{\mathbf{a}(L)}(\pm\widetilde{\rho}, L) = 0$, the following equalities must be valid
$$\frac{\pm\rho L}{2} + 1 - \Big(\frac{\mp\rho L}{2} + 1\Big)e^{\pm\rho L} = 0 \tag{2.7}$$
However, this is absurd. To prove this, note that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = e^t(2 - t) - (2 + t)$ is such that $h(0) = 0$ and $h'(t) < 0$ for all $t \in \mathbb{R}$. This completes the proof. □

Note that the sets obtained from the final lemma are clearly countable. Thus Theorem 1.1 is proved.

2.1. **Describing the roots.** The following lemma separates the roots into groups according to their algebraic structure. To characterize a group, we consider the quantity of real roots and their respective multiplicities. It is worth noting that for each polynomial p, the relation of its roots with these groups determines whether or not a solution exists for the problem of determining an entire member of family $f_{\mathbf{a}}(\cdot, L)$.

Lemma 2.6. *Consider the polynomial* $p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5$ *where* r, β, κ *and* η *are real such that* $\beta \in \{0, 1\}$, $\kappa \neq 0$ *and* $\eta < 0$.

(1) *If* $\beta = 0$,
- (a) $\kappa > 0$ *and*
  - (i) $r = 0$, *then* 0 *is the only real root of* p *with multiplicity* 3.
  - (ii) $r \neq 0$, *then* p *has a single real root, with multiplicity 1.*
- (b) $\kappa < 0$ *and*
  - (i) $r = 0$, *then* 0 *and* $\pm\rho$ *are roots of* p, *where* $\rho = \sqrt{\frac{\kappa}{\eta}}$, *being* 0 *the root of multiplicity* 3.
  - (ii) $0 < |r| < -\kappa z^3 + \eta z^5$, *with* $z = \sqrt{\frac{3\kappa}{5\eta}}$, *then* p *has exactly three real roots, all of which are of multiplicity* 1.
  - (iii) $|r| = -\kappa z^3 + \eta z^5$, *with* $z = \sqrt{\frac{3\kappa}{5\eta}}$, *then* p *has exactly three real roots, one of which has multiplicity* 2.
  - (iv) $|r| > -\kappa z^3 + \eta z^5$, *with* $z = \sqrt{\frac{3\kappa}{5\eta}}$, *then* p *has exactly one real root, with multiplicity* 1.

(2) *If* $\beta = 1$ *and*
- (a) $r = 0$, *then the roots of* p *are* $0, \pm\rho, \pm ik$ *where*
$$\rho = \sqrt{\frac{\kappa - \sqrt{\kappa^2 - 4\eta}}{2\eta}} \quad \text{and} \quad k = \sqrt{\left|\frac{\kappa + \sqrt{\kappa^2 - 4\eta}}{2\eta}\right|}.$$
- (b) $0 < |r| < z - \kappa z^3 + \eta z^5$, *where* $z = \sqrt{\frac{3\kappa - \sqrt{9\kappa^2 - 20\eta}}{10\eta}}$, *then* p *has exactly three non null real roots, with multiplicity* 1.
- (c) $|r| = z - \kappa z^3 + \eta z^5$, *where* $z = \sqrt{\frac{3\kappa - \sqrt{9\kappa^2 - 20\eta}}{10\eta}}$, *then* p *has exactly three real roots, one of which has multiplicity* 2.
- (d) $|r| > z - \kappa z^3 + \eta z^5$, *where* $z = \sqrt{\frac{3\kappa - \sqrt{9\kappa^2 - 20\eta}}{10\eta}}$, *then* p *has exactly one real root with multiplicity* 1.

3. Final Remarks

It is worth noting that when $d(\mathbf{a}) = 0$, the sets $\mathcal{X}_{\mathbf{a}r}$ were completely characterized for any $r \in \mathbb{R}$. The same happens when we consider $d(\mathbf{a}) \neq 0$, $\beta = 0$, $\kappa > 0$ and $r \in \mathbb{R}$; i.e., when $p(\xi) = r - \kappa\xi^3 + \eta\xi^5$. In particular, in this case, Theorem 1.1 tells us that the sets $\mathcal{X}_{\mathbf{a}r}$ are empty for all non null $\mathbf{a} \in \mathbb{C}^4$ and $r \in \mathbb{R}$. Thus the set $\mathcal{X}$ is empty and the problem of the initial and boundary value, analyzed by Vasconcellos and Silva [19] and associated with the linear Kawahara equation $u_t + \kappa u_{xxx} + \eta u_{xxxxx} = 0$, does not admit non-trivial solutions whose energies do not decay over time. Note that for $p(\xi) = r + \beta\xi - \kappa\xi^3 + \eta\xi^5$, the case $d(\mathbf{a}) \neq 0$

remains to be solved in two situations: (a) when $r \neq 0$ and p has exactly three real roots, with all the multiplicities being equal to 1 and (b) p has exactly three real roots with one of them having a multiplicity of 2.

4. Appendix: Derivatives of $F(t)$

Consider the function

$$F(t) = K \sinh y_1 t \sinh y_2 t - 2y_1 y_2(\cosh((y_1+y_2)t) - \cos((x_2-x_1)t))$$

for $t \in \mathbb{R}$. Note that $F(0) = 0$ and that

$$\begin{aligned} F'(t) &= K(y_1 \cosh y_1 t \sinh y_2 t + y_2 \sinh y_1 t \cosh y_2 t) \\ &\quad - 2y_1 y_2((y_1+y_2)\sinh((y_1+y_2)t) + (x_2-x_1)\sin((x_2-x_1)t)) \\ F''(t) &= (x_2-x_1)^2(y_1^2+y_2^2)\sinh y_1 t \sinh y_2 t \\ &\quad + (y_1+y_2)^2 (y_1-y_2)^2 \sinh y_1 t \sinh y_2 t + 2(x_2-x_1)^2 y_1 y_2 (\cosh y_1 t \cosh y_2 t \\ &\quad - \cos((x_2-x_1)t)) \end{aligned}$$

Note that $F'(0) = 0$ and that, for $y_1, y_2 > 0$, $F''(t) > 0$ for all $t > 0$.

5. Appendix: Coefficients of M_1 and M_2

Let ξ_1, ξ_2 and ξ_3 be three distinct points that are mapped by M into three distinct points w_1, w_2 and w_3. Since there is always one and only one linear fractional transformation that transforms any three distinct points into three given distinct points (see, for instance, [18, Theorem V p. 176], if we take

$$\begin{gathered} a = \begin{vmatrix} w_1\xi_1 & w_1 & 1 \\ w_2\xi_2 & w_2 & 1 \\ w_3\xi_3 & w_3 & 1 \end{vmatrix}, \quad b = \begin{vmatrix} w_1\xi_1 & \xi_1 & w_1 \\ w_2\xi_2 & \xi_2 & w_2 \\ w_3\xi_3 & \xi_3 & w_3 \end{vmatrix}, \\ c = \begin{vmatrix} \xi_1 & w_1 & 1 \\ \xi_2 & w_2 & 1 \\ \xi_3 & w_3 & 1 \end{vmatrix}, \quad d = \begin{vmatrix} w_1\xi_1 & \xi_1 & 1 \\ w_2\xi_2 & \xi_2 & 1 \\ w_3\xi_3 & \xi_3 & 1 \end{vmatrix}, \end{gathered} \tag{5.1}$$

then $M(\xi) = \frac{a\xi+b}{c\xi+d}$ is the Möbius transformation $M(\xi)$ such that $M(\xi_j) = w_j$ $(j = 1,2,3)$.

5.1. **Coefficients of M_1.**

Let $M_1(\xi)$ be the linear fractional transformation such that $M_1(0) = 1$, $M_1(ik) = e^{Lk}$, $M_1(-ik) = e^{-Lk}$. By (5.1), the coefficients of M_1 may be determined by

$$\begin{gathered} a_I = -ik\big(w_I + \frac{1}{w_I} - 2\big) = -2ik(\cosh(Lk)-1) \\ b_I = k^2\big(w_I - \frac{1}{w_I}\big) = d_I = 2k^2\sinh(Lk) \\ c_I = ik\big(w_I + \frac{1}{w_I} - 2\big) = 2ik(\cosh(Lk)-1) \end{gathered}$$

Therefore,

$$M_1(\xi) = \frac{-2ik(\cosh(Lk)-1)\xi + 2k^2\sinh(Lk)}{2ik(\cosh(Lk)-1)\xi + 2k^2\sinh(Lk)} = -\frac{\xi + ik \operatorname{cotanh}\left(\frac{Lk}{2}\right)}{\xi - ik \operatorname{cotanh}\left(\frac{Lk}{2}\right)}.$$

5.2. **Coefficients of M_2.** Let $M_2(\xi)$ be the linear fractional transformation such that $M_2(0) = 1$, $w_\rho = M_2(\rho) = e^{-iL\rho}$, $M_2(-\rho) = e^{iL\rho}$. By (5.1), the coefficients of M_2 may be determined by

$$\begin{aligned} a_\rho &= -\rho(w_\rho + \overline{w_\rho} - 2) = -2\rho(\cos(L\rho) - 1) \\ b_\rho &= -\rho^2 \left(w_\rho - \overline{w_\rho}\right) = d_\rho = -2i\rho^2 \sin(L\rho) \\ c_\rho &= \rho \left(w_\rho + \overline{w_\rho} - 2\right) = 2\rho(\cos(L\rho) - 1)\,. \end{aligned}$$

Therefore,

$$M_2(\xi) = \frac{-2\rho(\cos(L\rho) - 1)\xi - 2i\rho^2 \sin(L\rho)}{2\rho(\cos(L\rho) - 1)\xi - 2i\rho^2 \sin(L\rho)} = -\frac{\xi - i\rho \cot\left(\frac{L\rho}{2}\right)}{\xi + i\rho \cot\left(\frac{L\rho}{2}\right)}.$$

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André Luiz C. dos Santos
DEMAT/CEFET/Maracanã; PPG-EM/UERJ - Rua Fonseca Teles, 121, 1o. andar, São Cristóvão, Rio de Janeiro, RJ. CEP: 20940-903, Brazil
E-mail address: andreluiz.cordeiro@gmail.com

Patrícia N. da Silva
IME/UERJ, Rua São Francisco Xavier, 524, 6o. andar, Rio de Janeiro, RJ, CEP 20550-900, Brazil
E-mail address: nunes@ime.uerj.br

Carlos Frederico Vasconcellos
IME/UERJ, Rua São Francisco Xavier, 524, 6o. andar, Rio de Janeiro, RJ, CEP 20550-900, Brazil
E-mail address: cfredvasc@ime.uerj.br