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MULTIPLE SOLUTIONS FOR SEMILINEAR SCHRÖDINGER EQUATIONS WITH ELECTROMAGNETIC POTENTIAL

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Abstract. In this article, we consider the existence of infinitely many nontrivial solutions for the following semilinear Schrödinger equation with electromagnetic potential

$$\big(-i\nabla + A(x)\big)^2 u + V(x)u = f(x,|u|)u, \quad \text{in } \mathbb{R}^N$$

where i is the imaginary unit, V is the scalar (or electric) potential, A is the vector (or magnetic) potential. We establish the existence of infinitely many solutions via variational methods.

1. Introduction

This article concerns the following semilinear stationary Schrödinger equation with electromagnetic potential

$$\big(-i\nabla + A(x)\big)^2 u + V(x)u = f(x,|u|)u, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $u : \mathbb{R}^N \to \mathbb{C}$ and $N \geq 2$, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar (or electric) potential and $A = (A_1, \dots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ is a vector (or magnetic) potential. This equation arises in quantum mechanics and provides a description of the dynamics of the particle in a non-relativistic setting.

There have been lots of studies on the existence and multiplicity of solutions for nonlinear Schrödinger type equations without the presence of a magnetic potential, see [3, 4, 5, 8, 9, 19, 25, 30]. Compared with results of this case, the appearance of the magnetic potential brings in additional difficulties to the problems such as the effects of the magnetic potential on the linear spectral sets and on the solution structure. Thus, for equations with magnetic potential, it has been studied much less than for equations with magnetic potential, see [1, 6, 12, 17, 22, 13, 23]. It seems that the first work was studied in [12], the authors found the existence of solutions for problem (1.1) by solving an appropriate minimization problem for the corresponding energy functional in the case of $N = 2$ and 3. Later, the existence and multiplicity of solutions of problem (1.1) were obtained in [17] under certain assumptions that $\sigma(-(-i\nabla + A) + V)$ is discrete. In [1], the authors obtained multiplicity of solutions under the assumptions that V, f and $B := curl A$ depend

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periodically on $x \in \mathbb{R}^N$. For singular perturbation problem and concentration phenomenon of semi-classical states, we refer the readers to [2, 7, 10, 11, 13, 23] and the references therein.

It is worth pointing out that the aforementioned authors always assumed the potential $V(x)$ is positive. However, to the best of our knowledge, for the sign-changing potential case, there are not many results for problem (1.1). In the case of zero magnetic field (i.e. $A_i = 0$, $i = 1, 2, \dots, N$), there have been some works focused on the study of the sign-changing potential, we refer the readers to [8, 9, 18, 14, 15, 19, 20, 24, 26, 27, 28, 30] and the references therein.

Motivated by the above references, we consider problem (1.1) with sign-changing potential, and establish the existence of infinitely many solutions by symmetric Mountain Pass Theorem in [21]. More precisely, we make the following assumptions:

- (A1) $A \in C(\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > -\infty$;
- (A2) There exists a constant $d_0 > 0$ such that
$$\lim_{|y|\to\infty} \text{meas}\big(\{x \in \mathbb{R}^N : |x - y| \le d_0, V(x) \le M\}\big) = 0, \quad \forall M > 0,$$
where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;
- (A3) $f(x, |u|) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $p \in (2, 2^*)$ such that
$$|f(x, |u|)| \le c_1 + c_2|u|^{p-2}, \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{C};$$
where $2^* = +\infty$ if $N \le 2$ and $2_* = \frac{2N}{N-2}$ if $N > 2$;
- (A4) $\lim_{|u|\to\infty} F(x, |u|)/|u|^2 = \infty$, a. e. $x \in \mathbb{R}^N$, and there exists $r_0 \ge 0$ such that
$$F(x, |u|) \ge 0, \quad \text{for } |u| \ge r_0, \tag{1.2}$$
where $F(x, |u|) = \int_0^{|u|} f(x, |t|)t dt$;
- (A5) $\mathcal{F}(x, |u|) = \frac{1}{2} f(x, |u|)|u|^2 - F(x, |u|) \ge 0$, and there exist $c_3 > 0$ and $\kappa > \max\{1, N/2\}$ such that
$$|F(x, |u|)|^\kappa \le c_3|u|^{2\kappa}\mathcal{F}(x, |u|), \quad \text{for } |u| \ge r_0;$$
- (A6) There exist $\mu > 2$ and $\varrho > 0$ such that
$$\mu F(x, |u|) \le |u|^2 f(x, |u|) + \varrho|u|^2 \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{C}.$$

The main results of this article are the following theorems.

Theorem 1.1. *Suppose that* (A1)–(A5) *are satisfied. Then problem* (1.1) *has infinitely many solutions.*

Theorem 1.2. *Suppose that* (A1)–(A4), (A6) *are satisfied. Then problem* (1.1) *has infinitely many solutions.*

2. Variational setting and proof of the main results

Before establishing the variational setting for problem (1.1), we have the following Remark

Remark 2.1. From (A1), we know that there exists a constant $V_0 > 0$ such that $\bar{V}(x) := V(x) + V_0$ for all $x \in \mathbb{R}^N$. Let $\bar{f}(x, |u|)u := f(x, |u|)u + V_0 u$ and consider the new equation
$$\big(-i\nabla + A(x)\big)^2 u + \bar{V}(x)u = \bar{f}(x, |u|)u, \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Then problem (2.1) is equivalent to problem (1.1). It is easy to check that the hypotheses (A1)–(A6) hold for $\bar{V}$ and $\bar{f}$ provided that those hold for V and f.

In view of Remark 2.1, now we will study the equivalent problem (2.1). Throughout the following sections, we make the following assumption, instead of (A1),

(A1') $A \in C(\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.

For convenience, write $\nabla_A u = (\nabla + iA)u$. Let

$$H^1_A(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\}.$$

Hence, $H^1_A(\mathbb{R}^N)$ is the Hilbert space under the scalar product

$$(u,v) = \int_{\mathbb{R}^N} (\nabla_A u \overline{\nabla_A v} + u\bar{v}) dx,$$

and the norm induced by the above product is

$$\|u\|_{H^1_A(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + |u|^2) dx\Big)^{1/2}.$$

Let

$$E = \{u \in H^1_A(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < +\infty\},$$

and the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V(x)|u|^2\big)\, dx\Big)^{1/2}.$$

The well-known diamagnetic inequality [16, Theorem 7.21],

$$|\nabla |u|(x)| \le |\nabla u(x) + iA(x)u(x)|\,, \quad \text{for a.e. } x \in \mathbb{R}^N$$

implies that for any $u \in E$, we can get that $|u|$ belongs to $H^1(\mathbb{R}^N)$, which embeds continuously into $L^s(\mathbb{R}^N)$, $s \in [2, 2^*]$. And therefore $u \in L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*]$. It is thus clear that for any $s \in [2, 2^*]$, there exists γ_s such that

$$\|u\|_s \le \gamma_s \|u\|, \quad \forall u \in E. \tag{2.2}$$

Combining with the assumption (A2), we have the following Lemma (see [3, 29])

Lemma 2.2. *Under assumptions* (A1') *and* (A2)*, the embedding* $E \hookrightarrow L^s(\mathbb{R}^N)$ *is compact for any* $s \in [2, 2^*)$.

For each $u \in E$, we define

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V(x)|u|^2\big)\, dx - \int_{\mathbb{R}^N} F(x, |u|) dx. \tag{2.3}$$

From assumptions (A1'), (A2) and (A3), we can easily get that $\Phi \in C^1(E, \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} \big(\nabla_A u \overline{\nabla_A v} + V(x) u\bar{v}\big)\, dx - \int_{\mathbb{R}^N} f(x, |u|) u\bar{v} dx, \tag{2.4}$$

for all $u, v \in E$.

We say that $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$I(u_n) \to c, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0 \tag{2.5}$$

has a convergent subsequence.

Lemma 2.3 ([21]). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- (A7) *$I(0) = 0$, $I(-u) = I(u)$ for all $u \in X$;*
- (A8) *there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;*
- (A9) *for any finite dimensional subspace $\tilde{X} \subset X$, there exists $R = R(\tilde{X}) > 0$ such that $I(u) \leq 0$ on $\tilde{X} \setminus B_R$,*

then I possesses an unbounded sequence of critical values.

Lemma 2.4. *Under assumptions* (A1'), (A2)–(A5), *any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c > 0, \quad \langle \Phi'(u_n), u_n \rangle \to 0 \tag{2.6}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_s \leq \gamma_s \|v_n\| = \gamma_s$ for $2 \leq s \leq 2^*$. For n large enough, we have

$$c + 1 \geq \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, |u_n|)dx. \tag{2.7}$$

It follows from (2.3)*and*(2.6) that

$$\frac{1}{2} \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{|F(x, |u_n|)|}{\|u_n\|^2} dx. \tag{2.8}$$

For $0 \leq a < b$, let

$$\Omega_n(a, b) = \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}. \tag{2.9}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E, then by Lemma 2.2, $v_n \to v_1$ in $L^s(\mathbb{R}^N)$ for all $s \in [2, 2^*)$, and $v_n(x) \to v_1(x)$ a. e. in $\mathbb{R}^N$.

If $v_1 = 0$, then $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in [2, 2^*)$, and $v_n \to 0$ a. e. in $\mathbb{R}^N$. From (A3), we know that

$$|F(x, |u|)| \leq \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p, \tag{2.10}$$

then

$$\begin{aligned} \int_{\Omega_n(0,r_0)} \frac{|F(x, |u_n|)|}{|u_n|^2} |v_n|^2 dx &\leq (\frac{c_1}{2} + \frac{c_2 r_0^{p-2}}{p}) \int_{\Omega_n(0,r_0)} |v_n|^2 dx \\ &\leq (\frac{c_1}{2} + \frac{c_2 r_0^{p-2}}{p}) \int_{\mathbb{R}^N} |v_n|^2 dx \to 0. \end{aligned} \tag{2.11}$$

Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > \max\{1, N/2\}$, we obtain $2\kappa' \in (2, 2^*)$. Hence, from (A5) and (2.7), we have

$$
\begin{aligned}
&\int_{\Omega_n(r_0,\infty)} \frac{|F(x,|u_n|)|}{|u_n|^2}|v_n|^2dx \\
&\le \Big(\int_{\Omega_n(r_0,\infty)} (\frac{|F(x,|u_n|)|}{|u_n|^2})^\kappa dx\Big)^{1/\kappa}\Big(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}dx\Big)^{1/\kappa'} \\
&\le c_3^{1/\kappa}\Big(\int_{\Omega_n(r_0,\infty)} \mathcal{F}(x,|u_n|)dx\Big)^{1/\kappa}\Big(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}dx\Big)^{1/\kappa'} \\
&\le [c_3(c+1)]^{1/\kappa}\Big(\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}dx\Big)^{1/\kappa'} \to 0.
\end{aligned}
\tag{2.12}
$$

Combining (2.11) with (2.12), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|F(x,|u_n|)|}{\|u_n\|^2}dx \\
&= \int_{\Omega_n(0,r_0)} \frac{|F(x,|u_n|)|}{|u_n|^2}|v_n|^2dx + \int_{\Omega_n(r_0,\infty)} \frac{|F(x,|u_n|)|}{|u_n|^2}|v_n|^2dx \to 0,
\end{aligned}
$$

which contradicts (2.8).

Next we consider the case that $v_1 \neq 0$. Set $H := \{x \in \mathbb{R}^N : v_1(x) \neq 0\}$, then $\text{meas}(H) > 0$. For $x \in H$, we have $|u_n(x)| \to \infty$ as $n \to \infty$. Hence, $x \in \Omega_n(r_0,\infty)$ for large $n \in \mathbb{N}$, which implies that $\chi_{\Omega_n(r_0,\infty)}(x) = 1$ for large n, where χ_{Ω_n} denotes the characteristic function on Ω. Since $v_n \to v_1$ a.e. in $\mathbb{R}^N$, we have $\chi_{\Omega_n(r_0,\infty)}(x)v_n \to v_1$ a.e. in H. It follows from (2.3), (2.10), (A4) and Fatou's Lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \Big(\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2dx\Big) \\
&= \lim_{n\to\infty} \Big(\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2dx\Big) \\
&\le \limsup_{n\to\infty} \Big(\frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p}r_0^{p-2}) \int_{\mathbb{R}^N} |v_n|^2dx \\
&\quad - \int_{\Omega_n(r_0,\infty)} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2dx\Big) \\
&\le \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p}r_0^{p-2})\gamma_2^2 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{F(x,|u_n|)}{|u_n|^2}|v_n|^2dx \\
&= \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p}r_0^{p-2})\gamma_2^2 - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,|u_n|)}{|u_n|^2}[\chi_{\Omega_n(r_0,\infty)}(x)]|v_n|^2dx \\
&\le \frac{1}{2} + (\frac{c_1}{2} + \frac{c_2}{p}r_0^{p-2})\gamma_2^2 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,|u_n|)}{|u_n|^2}[\chi_{\Omega_n(r_0,\infty)}(x)]|v_n|^2dx \\
&= -\infty,
\end{aligned}
\tag{2.13}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. □

Lemma 2.5. *Under assumptions* (A1'), (A2)–(A5)*, any sequence* $\{u_n\} \subset E$ *satisfying* (2.6) *has a convergent subsequence in* E.

Proof. From Lemma 2.4, we know that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. By Lemma 2.2, $u_n \to u$ in $L^s(\mathbb{R}^N)$ for all $2 \le s < 2^*$, thus

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x,|u_n|)u_n - f(x,|u|)u| |\overline{u_n - u}| dx \\ &\le \int_{\mathbb{R}^N} \left[(c_1|u_n| + c_2|u_n|^{p-1}) + (c_1|u| + c_2|u|^{p-1})\right] |u_n - u| dx \\ &\le c_1\Big(\int_{\mathbb{R}^N} (|u_n| + |u|)^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} |u_n - u|^2 dx\Big)^{1/2} \\ &\quad + c_2\Big(\int_{\mathbb{R}^N} |u_n|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N} |u_n - u|^p dx\Big)^{1/p} \\ &\quad + c_2\Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N} |u_n - u|^p dx\Big)^{1/p} \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.14}$$

Observe that

$$\begin{aligned} \|u_n - u\|^2 &= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\ &\quad + \int_{\mathbb{R}^N} [f(x,|u_n|)u_n - f(x,|u|)u](\overline{u_n - u}) dx. \end{aligned} \tag{2.15}$$

It is clear that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

From (2.14), (2.15) and (2.16), we obtain $\|u_n - u\| \to 0$ as $n \to \infty$. □

Lemma 2.6. *Under assumptions* (A1'), (A2)–(A4), (A6)*, any sequence* $\{u_n\} \subset E$ *satisfying* (2.6) *has a convergent subsequence in* E.

Proof. First, we prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\|_s \le \gamma_s \|v_n\| = \gamma_s$ for all $2 \le s < 2^*$. By (2.3), (2.4), (2.6) and (A6), we have

$$\begin{aligned} c + 1 &\ge \Phi(u_n) - \frac{1}{\mu}\langle \Phi'(u_n), u_n\rangle \\ &= \frac{\mu - 2}{2\mu}\|u_n\|^2 - \int_{\mathbb{R}^N} [F(x,|u_n|) - \frac{1}{\mu} f(x,|u_n|)|u_n|^2] dx \\ &\ge \frac{\mu - 2}{2\mu}\|u_n\|^2 - \frac{\varrho}{\mu}\|u_n\|_2^2 \quad \text{for large } n \in \mathbb{N}, \end{aligned} \tag{2.17}$$

which implies

$$1 \le \frac{2\varrho}{\mu - 2} \limsup_{n\to\infty} \|v_n\|_2^2. \tag{2.18}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E, then by Lemma 2.2, $v_n \to v_1$ in $L^s(\mathbb{R}^N)$ for all $2 \le s < 2^*$, and $v_n(x) \to v_1(x)$ a. e. in $\mathbb{R}^N$. Hence, it follows from (2.18) that $v_1 \ne 0$. Similar to (2.13), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 2.5. □

Lemma 2.7. *Under assumptions* (A1'), (A2)–(A4)*, for any finite dimensional subspace* $\tilde{E} \subset E$*, there holds*

$$\Phi(u) \to -\infty, \quad \|u\| \to \infty, \quad u \in \tilde{E}. \tag{2.19}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \tilde{E}$ with $\|u_n\| \to \infty$, there exists $M_1 > 0$ such that $\Phi(u_n) \geq -M_1$ for all $n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v_1$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v_1 \in \tilde{E}$ in E, $v_n(x) \to v_1(x)$ a. e. in $\mathbb{R}^N$, and so $\|v_1\| = 1$. Hence, we can conclude a contradiction by a similar fashion as (2.13). □

Corollary 2.8. *Under assumptions* (A1'), (A2)–(A4)*, for any finite dimensional subspace* $\tilde{E} \subset E$*, there exists* $R = R(\tilde{E}) > 0$*, such that*

$$\Phi(u) \leq 0, \quad \forall u \in \tilde{E}, \ \|u\| \geq R. \tag{2.20}$$

Let $\{e_j\}$ be a total orthonormal basis of E and define

$$X_j = \mathbb{R}e_j, \quad Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, quad k \in \mathbb{Z}. \tag{2.21}$$

Similar to [28, Lemma 3.8], we have the following lemma.

Lemma 2.9. *Under assumptions* (A1') *and* (A2)*, for* $2 \leq s < 2^*$*,*

$$\beta_k(s) := \sup_{u \in Z_k, \|u\|=1} \|u\|_s \to 0, \quad k \to \infty. \tag{2.22}$$

By this lemma, we can choose an integer $m \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2c_1}\|u\|^2, \quad \|u\|_p^p \leq \frac{p}{4c_2}\|u\|^p, \quad \forall u \in Z_m. \tag{2.23}$$

Lemma 2.10. *Under assumptions* (A1'), (A2) *and* (A3)*, there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z_m} \geq \alpha$*.*

Proof. Combining (2.3), (2.10) with (2.23), for $u \in Z_m$, choosing $\rho := \|u\| = \frac{1}{2}$ we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, |u|)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|_2^2 - \frac{c_2}{p}\|u\|_p^p \\ &\geq \frac{1}{4}(\|u\|^2 - \|u\|^p) \\ &= \frac{2^{p-2}-1}{2^{p+2}} := \alpha > 0. \end{aligned} \tag{2.24}$$

Thus, the proof is complete. □

Proof of Theorem 1.1. Let $X = E, Y = Y_m$ and $Z = Z_m$. Obviously, $\bar{f}$ satisfies (A3)–(A5), and $\Phi(u)$ is even. By Lemmas 2.4, 2.5, 2.10 and Corollary 2.8, all conditions of Lemma 2.3 are satisfied. Thus, problem (2.1) possesses infinitely many nontrivial solutions. By Remark 2.1, problem (1.1) also possesses infinitely many nontrivial solutions. □

Proof of Theorem 1.2. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $\bar{f}$ satisfies (A3), (A4), (A6) and $\Phi(u)$ is even. The rest proof is the same as that of Theorem 1.1, but using Lemma 2.6 instead of Lemmas 2.4 and 2.5. □

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