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POSITIVE LEAST ENERGY SOLUTIONS OF FRACTIONAL LAPLACIAN SYSTEMS WITH CRITICAL EXPONENT

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Abstract. We study the fractional Laplacian system with critical exponent

$$
\begin{gathered}
(-\Delta)^s u+\lambda_1 u=\mu_1|u|^{2_s^*-2}u+\beta|u|^{\frac{2_s^*}{2}-2}u|v|^{\frac{2_s^*}{2}}, \quad x\in\Omega,\\
(-\Delta)^s v+\lambda_2 v=\mu_2|v|^{2_s^*-2}v+\beta|v|^{\frac{2_s^*}{2}-2}v|u|^{\frac{2_s^*}{2}}, \quad x\in\Omega,\\
u=v=0\,, \quad x\in\partial\Omega,
\end{gathered}
$$

where $\Omega\subset\mathbb{R}^N$ $(N>2s)$ is a smooth bounded domain, $s\in(0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian, $2_s^*:=\frac{2N}{N-2s}$ is the critical Sobolev exponent, $-\lambda_1(\Omega)<\lambda_1,\lambda_2<0$, and $\mu_1,\mu_2>0$, here $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta)^s$ with Dirichlet boundary condition. For each fixed $\beta\geq\frac{2s}{N-2s}\max\{\mu_1,\mu_2\}$, we show that this system has a positive least energy solution.

1. Introduction

In this article, we consider the coupled system

$$
\begin{gathered}
(-\Delta)^s u+\lambda_1 u=\mu_1|u|^{2_s^*-2}u+\beta|u|^{\frac{2_s^*}{2}-2}u|v|^{\frac{2_s^*}{2}}, \quad x\in\Omega,\\
(-\Delta)^s v+\lambda_2 v=\mu_2|v|^{2_s^*-2}v+\beta|v|^{\frac{2_s^*}{2}-2}v|u|^{\frac{2_s^*}{2}}, \quad x\in\Omega,\\
u=v=0\,, \quad x\in\partial\Omega,
\end{gathered}
\tag{1.1}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N>2s)$, $2_s^*:=\frac{2N}{N-2s}$ is the critical Sobolev exponent, $\mu_1,\mu_2>0$, $-\lambda_1(\Omega)<\lambda_1,\lambda_2<0$, $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta)^s$ with Dirichlet boundary condition, and $\beta\neq 0$ is a coupling constant.

In recent years, considerable attention has been given to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplace operator, see [2, 9, 16, 23, 29] and the references therein. The fractional power of the Laplacian $(-\Delta)^s$ is the infinitesimal generator of Lévy stable diffusion processes. It arises in several areas such as plasmas (see [6]), flames propagation, chemical reactions in liquids, population dynamics, geophysical fluid in dynamics, crystal dislocation and so on.

Recently, the solutions of the fractional Laplacian attract more attention of researchers in nonlinear analysis. Chang and González [15] studied this operator in conformal geometry. Caffarelli et al [11, 12] investigated free boundary problems of

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the fractional Laplacian. Silvestre [27] obtained some regularity results of the obstacle problem of fractional Laplacian. Maximum principle has been researched in the fractional Laplacian operator in [9, 14]. Caffarelli and Silvestre [13], introduced the s-harmonic extension to define the fractional Laplacian operator, and gave a new formulation of the fractional Laplacian through Dirichlet-to-Neumann maps. This is commonly used in the recent literature since it allows us to write nonlocal problems in a local way. Also this permits us to use the variational methods to solve those kinds of problems. Several results of the fractional version of the classical elliptic problems were obtained, we would like to mention [3, 10, 28] and the references therein.

In [10], Cabré and Tan combined the spectral decomposition of the Laplacian operator with zero Dirichlet boundary conditions defined the operator of the square root of Laplacian. Using classical local techniques, they established existence, regularity and an L^∞-estimate of Brezis-Kato type for weak solutions. In particular, Tan [28] studied the problem

$$\begin{gathered} (-\Delta)^{1/2}u = u^{\frac{n+1}{n-1}} + \mu u, \quad u > 0\,, \quad x \in \Omega, \\ u = 0\,, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

where $\mu > 0$. He employed the Brezis-Nirenberg technique to build an analogue result to the problem in [8], but with the square root of the Laplacian instead of the Laplacian. Barrios et al [3] studied the nonlinear problem

$$\begin{gathered} (-\Delta)^{\frac{\alpha}{2}}u = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 0\,, \quad x \in \Omega, \\ u = 0\,, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where $0 < q < \frac{N+\alpha}{N-\alpha}$, $0 < \alpha < 2$ and $N > \alpha$, they obtain the existence of positive solutions under certain conditions.

Problem (1.1) can be seen as a counterpart of the system

$$\begin{gathered} -\Delta u + \lambda_1 u = \mu_1 |u|^{2^*-2}u + \beta |u|^{\frac{2^*}{2}-2}u|v|^{\frac{2^*}{2}}, \quad x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{2^*-2}v + \beta |u|^{\frac{2^*}{2}}|v|^{\frac{2^*}{2}-2}v, \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In [17, 19], Chen and Zou studied problem (1.4) for $N \geq 4$, by variational arguments, they showed the existence of positive least energy solutions in some ranges of $\lambda_1, \lambda_2, \beta$.

Problem (1.4) is closely related to the following system, which arises in the Hartree-Fock theory for a double condensate

$$\begin{gathered} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, \quad x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \quad x \in \Omega, \\ u,\, v \in H_0^1(\Omega), \end{gathered} \tag{1.5}$$

here Ω is a domain in $\mathbb{R}^N (N \leq 3)$, possibly unbounded, with empty or smooth boundary. In the past decades, there has been increasing interest in studying problem (1.5), especially on the existence of positive solutions, multiple solutions, sign-changing solutions and ground states, see for example [1, 4, 5, 17, 18, 19, 22, 24, 25, 26, 30].

Motivated by the works just described as above, it is natural and significant to investigate the solutions of the system (1.1). Since the fractional Laplacian operator $(-\Delta)^s$ is involved, problem (1.1) is a nonlocal problem, which implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.1) particularly interesting. In this article, we are mainly study the existence of positive least energy solutions of equation (1.1).

Before present our main results, we would like to mention that the following fractional Brezis-Nirenberg problem

$$\begin{gathered} (-\Delta)^s u + \lambda u = |u|^{2_s^*-2}u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.6}$$

has a positive least energy solution u_λ for $-\lambda_1(\Omega) < \lambda < 0$ (see [3]).

Our first result deals with the special case $\lambda_1 = \lambda_2$.

Theorem 1.1. *Assume that $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0$ and $N \geq 4s$. If $\beta \geq \frac{2s}{N-2s}\max\{\mu_1, \mu_2\}$, then $(\sqrt{k_0}u_\lambda, \sqrt{l_0}u_\lambda)$ is a positive least energy solution of system* (1.1)*, where the positive pair (k_0, l_0) will be given in* (3.1)*.*

Moreover, there exists $\beta_0 \geq \frac{2s}{N-2s}\max\{\mu_1, \mu_2\}$ determined by (μ_1, μ_2), such that, if $\beta > \beta_0$ and (u, v) is any a positive least energy solution of (1.1)*, then $(u, v) = (\sqrt{k_0}u_\lambda, \sqrt{l_0}u_\lambda)$ where u_λ is a positive least energy solution of the Brezis-Nirenberg problem* (1.6)*.*

Now, we consider the general case $-\lambda_1(\Omega) < \lambda_1,\ \lambda_2 < 0$. Without loss of generality, we may assume that $\lambda_1 \leq \lambda_2$. Our following result deal with the case $\lambda_1 < \lambda_2$.

Theorem 1.2. *Assume that $-\lambda_1(\Omega) < \lambda_1 < \lambda_2 < 0$ and $N \geq 4s$. Then system* (1.1) *has a positive least energy solution (u, v) for $\beta \geq \frac{2s}{N-2s}\max\{\mu_1, \mu_2\}$.*

Theorems 1.1 and 1.2 partially generalized the results in [17, 19], which deals with Schrödinger system (1.4), to the fractional Schrödinger problem. The main difficulty in proving our main results is the nonlocal operator $(-\Delta)^s$. We apply the s-harmonic extension and Dirichlet-to-Neumann maps, which developed by Caffarelli and Silvestre [13], to transform the nonlocal problem (1.1) into a local problem (2.9). Thus, the usual variational methods can be used to solve problem (2.9), and then the problem (1.1).

This paper is organized as follows. In Section 2, we first present some variational framework of problem (1.1). In Section 3, we give the proof of Theorem 1.1 and study the limit problem of (1.1). Finally, via energy comparison, we prove Theorem 1.2 in Section 4.

2. Preliminaries and functional setting

Denote the upper half space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}_+^{N+1} = \big\{(x, y) : x \in \mathbb{R}^N, y > 0\big\},$$

the half cylinder standing on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ by $\mathcal{C} = \Omega \times (0, +\infty)$, the points in $\mathcal{C}$ are denoted by (x, y) for $x \in \Omega$, $y \in (0, +\infty)$, and its lateral boundary by $\partial_L\mathcal{C} = \partial\Omega \times (0, +\infty)$.

Let $\{\varphi_k\}$ be an orthonnormal basis of $L^2(\Omega)$ with $\|\varphi_k\|_2 = 1$ forming a spectral decomposition of $-\Delta$ in Ω with zero Dirichlet boundary conditions and λ_k be the corresponding eigenvalues. Let

$$H_0^s(\Omega)=\Big\{u=\sum_{k=0}^{\infty}a_k\varphi_k\in L^2(\Omega):\|u\|_{H_0^s(\Omega)}=(\sum_{k=0}^{\infty}a_k^2\lambda_k^s)^{1/2}<+\infty\Big\}.$$

For $u\in H_0^s(\Omega)$, $u=\sum_{k=0}^{\infty}a_k\varphi_k$ with $a_k=\int_\Omega u\varphi_k dx$, the fractional power of Dirichlet Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u=\sum_{k=0}^{\infty}a_k\lambda_k^s\varphi_k\in H^{-s}(\Omega),$$

where $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$. We say that $\{(\varphi_k,\lambda_k^s)\}$ are the eigenfunctions and eigenvalues of $(-\Delta)^s$ in Ω with zero Dirichlet boundary condition.

It is easy to see that $H_0^s(\Omega)$ is a Hilbert space equipped with the following inner product and norm

$$\langle u,v\rangle_{H_0^s(\Omega)}=\int_\Omega(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v dx,\quad \|u\|=\langle u,u\rangle_{H_0^s(\Omega)}^{1/2}.$$

For a given regular function $u\in H_0^s(\Omega)$, we define its s-harmonic extension $w=E_s(u)$ to $\mathcal{C}$ as the solution of the problem

$$\begin{gathered}-\operatorname{div} y^{1-2s}\nabla w)=0,\quad \text{in } \mathcal{C},\\ w=0,\quad \text{on } \partial_L\mathcal{C},\\ w(x,0)=u,\quad \text{on } \Omega.\end{gathered}$$

For any regular function u, the fractional Laplacian $(-\Delta)^s$ acting on u is defined by

$$(-\Delta)^s u(x)=-\frac{1}{\kappa_s}\lim_{y\to0^+}y^{1-2s}\frac{\partial w}{\partial y}(x,y),\quad \forall\, x\in\Omega,$$

where $w=E_s(u)$ and $\kappa_s=\frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

Define $H_{0,L}^s(\mathcal{C})$ as the closure of $C_0^\infty(\mathcal{C})$ under the norm

$$\|w\|_{H_{0,L}^s(\mathcal{C})}=\Big(\kappa_s\int_{\mathcal{C}}y^{1-2s}|\nabla w|^2\,dx\,dy\Big)^{1/2}.$$

We will use the following notation:

$$\begin{gathered}\widetilde{H}:=H_{0,L}^s(\mathcal{C})\times H_{0,L}^s(\mathcal{C}),\\ L_s w:=-\operatorname{div} y^{1-2s}\nabla w),\quad \frac{\partial w}{\partial\nu^s}:=-\kappa_s\lim_{y\to0^+}y^{1-2s}\frac{\partial w}{\partial y}.\end{gathered}$$

For simplicity, we assume throughout this paper that $\kappa_s=1$. The following lemma is due to [7], which reflect the relationship between the spaces $H_0^s(\Omega)$ and $H_{0,L}^s(\mathcal{C})$.

Lemma 2.1. *(i)*

$$\|(-\Delta)^s u\|_{H^{-s}(\Omega)}=\|u\|_{H_0^s(\Omega)}=\|E_s(u)\|_{H_{0,L}^s(\mathcal{C})}.$$

(ii) For any $w\in H_{0,L}^s(\mathcal{C})$, there exist a constant C independent of w such that

$$\|tr_\Omega w\|_{L^r(\Omega)}\le C\|w\|_{H_{0,L}^s(\mathcal{C})}$$

holds for every $r\in[2,\frac{2N}{N-2s}]$, where $tr_\Omega w(x,y)=w(x,0)$. Moreover, $H_{0,L}^s(\mathcal{C})$ is compactly embedded into $L^r(\Omega)$ for $r\in[2,\frac{2N}{N-2s})$.

Another useful tool is the following trace inequality

$$\int_{\mathcal{C}} y^{1-2s}|\nabla w(x,y)|^2\,dx\,dy \geq C\Big(\int_\Omega |w(x,0)|^r dx\Big)^{2/r}, \tag{2.1}$$

for any $1 \leq r \leq \frac{2N}{N-2s}$, $N > 2s$, $w \in H^s_{0,L}(\mathcal{C})$. In fact, inequality (2.1) is equivalent to the following inequality for any $v \in H^s_0(\Omega)$ (see [20, 21]),

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 dx \geq C\Big(\int_\Omega |v|^r dx\Big)^{2/r}, \tag{2.2}$$

where $1 \leq r \leq \frac{2N}{N-2s}$, $N > 2s$.

When $r = 2^*_s$, the best constant in (2.1) will be denote by $S(s,N)$. So we have

$$\int_{\mathcal{C}} y^{1-2s}|\nabla w(x,y)|^2\,dx\,dy \geq S(s,N)\Big(\int_\Omega |w(x,0)|^{\frac{2N}{N-2s}} dx\Big)^{\frac{N-2s}{N}}.$$

However, $S(s,N)$ is indeed achieved for the case $\Omega = \mathbb{R}^N$ when

$$w(x,y) := w_\varepsilon(x,y) = E_s(u_\varepsilon), \tag{2.3}$$

where u_ε takes the form

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(|x|^2+\varepsilon^2)^{\frac{N-2s}{2}}},$$

with $\varepsilon > 0$ arbitrary. Moreover, the following critical problem

$$(-\Delta)^s u = |u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^N,$$

has positive solutions (see [20, 21])

$$U_\varepsilon(x) = \frac{C_{N,s}\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{N-2s}{2}}}, \quad C_{N,s} := 2^{(N-2s)/2}\Big(\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N-2s}{2})}\Big)^{\frac{N-2s}{4s}}, \tag{2.4}$$

for any $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$. Furthermore,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*_s} dx = S(s,N)^{\frac{N}{2s}}.$$

Recall that the equations

$$(-\Delta)^s u + \lambda_i u = \mu_i |u|^{2^*_s-2}u, \quad u \in H^s_0(\Omega),\ i = 1,2, \tag{2.5}$$

have positive least energy solutions $u_{\mu_i} \in H^s_0(\Omega)$ for $-\lambda_1(\Omega) < \lambda_i < 0$. We denote

$$B_{\mu_i} = \frac{s}{N}\Big(\int_{\mathcal{C}} y^{1-2s}|\nabla w_i|^2\,dx\,dy + \lambda\int_\Omega u^2_{\mu_i} dx\Big) = \frac{s}{N}\mu_i\int_\Omega u^{2^*_s}_{\mu_i} dx, \tag{2.6}$$

where $w_i = E_s(u_{\mu_i})$. Moreover,

$$\int_{\mathcal{C}} y^{1-2s}|\nabla w_i|^2\,dx\,dy + \lambda\int_\Omega u^2_{\mu_i} dx \geq (\frac{N}{s}B_{\mu_i})^{2s/N}\Big(\mu_i\int_\Omega u^{2^*_s}_{\mu_i} dx\Big)^{\frac{2}{2^*_s}}, \tag{2.7}$$

Similarly, we denote B_1 the energy of u_λ, that is

$$B_1 = \frac{s}{N}\Big(\int_{\mathcal{C}} y^{1-2s}|\nabla E_s(u_\lambda)|^2\,dx\,dy + \lambda\int_\Omega u^2_\lambda dx\Big) = \frac{s}{N}\int_\Omega u^{2^*_s}_\lambda dx, \tag{2.8}$$

where u_λ is the positive least energy of (1.6).

Now, by s-harmonic extension, we can transform the nonlocal problem (1.1) into the local problem

$$\begin{gathered} L_s w_1 = 0, \quad L_s w_2 = 0 \quad \text{in } \mathcal{C}, \\ w_1 = 0, \quad w_2 = 0 \quad \text{on } \partial_L \mathcal{C}, \\ \frac{\partial w_1}{\partial \nu^s} = -\lambda_1 u + \mu_1 |u|^{2_s^*-2}u + \beta |u|^{\frac{2_s^*}{2}-2} u |v|^{\frac{2_s^*}{2}} \quad \text{on } \Omega, \\ \frac{\partial w_2}{\partial \nu^s} = -\lambda_2 v + \mu_2 |v|^{2_s^*-2}v + \beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}-2} v \quad \text{on } \Omega. \end{gathered} \tag{2.9}$$

Definition 2.2. We say that $(u,v) \in H$ is a weak solution of (1.1) if $(w_1, w_2) \in \widetilde{H}$ is weak solution of (2.9), i.e.,

$$\begin{aligned} &\int_{\mathcal{C}} y^{1-2s} \nabla w_1 \nabla E_s(\varphi)\, dx\, dy + \int_{\mathcal{C}} y^{1-2s} \nabla w_2 \nabla E_s(\psi)\, dx\, dy + \lambda_1 \int_\Omega u\varphi dx \\ &+ \lambda_2 \int_\Omega v\psi dx - \int_\Omega \big(\mu_1 |u|^{2_s^*-2} u\varphi + \beta |u|^{\frac{2_s^*}{2}-2} u |v|^{\frac{2_s^*}{2}} \varphi \\ &\quad + \beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}-2} v\psi + \mu_2 |v|^{2_s^*-2} v\psi \big) dx = 0, \end{aligned} \tag{2.10}$$

for any $(\varphi, \psi) \in H$, where $(w_1, w_2) = (E_s(u), E_s(v))$.

Define the energy functional $E : H \to \mathbb{R}$ corresponding to problem (1.1) by

$$\begin{aligned} E(u,v) = &\frac{1}{2}\Big(\int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2\, dx\, dy + \int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2\, dx\, dy \Big) \\ &+ \frac{1}{2} \int_\Omega (\lambda_1 u^2 + \lambda_2 v^2) dx - \frac{1}{2^*} \int_\Omega \big(\mu_1 |u|^{2_s^*} + \mu_2 |v|^{2_s^*} \\ &+ 2\beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} \big) dx, \quad \forall (u,v) \in H, \end{aligned} \tag{2.11}$$

where $(w_1, w_2) = (E_s(u), E_s(v))$. As in [19], we also define

$$\begin{aligned} \mathcal{M} = \Big\{ &(u,v) \in H,\ u \neq 0,\ v \neq 0,\ (w_1, w_2) = (E_s(u), E_s(v)), \\ &\int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2\, dx\, dy + \lambda_1 \int_\Omega u^2 dx = \mu_1 \int_\Omega |u|^{2_s^*} dx + \beta \int_\Omega |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} dx, \\ &\int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2\, dx\, dy + \lambda_2 \int_\Omega v^2 dx = \mu_2 \int_\Omega |v|^{2_s^*} dx + \beta \int_\Omega |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} dx \Big\}. \end{aligned} \tag{2.12}$$

Then any nontrivial solutions of (1.1) must belong to $\mathcal{M}$. Take $\varphi, \psi \in C_0^\infty(\Omega)$ with $\varphi, \psi \not\equiv 0$ and $supp(\varphi) \bigcap supp(\psi) = \emptyset$, then there exist $t_1, t_2 > 0$ such that $(t_1\varphi, t_2\psi) \in \mathcal{M}$, thus $\mathcal{M} \neq \emptyset$. We define

$$\begin{aligned} B := \inf_{(u,v)\in\mathcal{M}} E(u,v) = \inf_{(u,v)\in\mathcal{M}} \frac{s}{N} \Big\{ &\Big(\int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2\, dx\, dy + \lambda_1 \int_\Omega u^2 dx \Big) \\ &+ \Big(\int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2\, dx\, dy + \lambda_2 \int_\Omega v^2 dx \Big) \Big\}. \end{aligned} \tag{2.13}$$

Since the nonlinearity and coupling term are both critical in (1.1), the existence of nontrivial solutions of (2.9) depends heavily on the existence of the least energy solution of the following limit problem

$$
\begin{gathered}
L_s w_1 = 0, \quad L_s w_2 = 0, \quad \text{in } \mathbb{R}^{N+1}_+, \\
\frac{\partial w_1}{\partial \nu^s} = \mu_1 |u|^{2^*_s - 2} u + \beta |u|^{\frac{2^*_s}{2} - 2} u |v|^{\frac{2^*_s}{2}}, \quad x \in \mathbb{R}^N, \\
\frac{\partial w_2}{\partial \nu^s} = \mu_2 |v|^{2^*_s - 2} v + \beta |u|^{\frac{2^*_s}{2}} |v|^{\frac{2^*_s}{2} - 2} v, \quad x \in \mathbb{R}^N, \\
w_1, w_2 \in D,
\end{gathered}
\tag{2.14}
$$

where $D := D^s(\mathbb{R}^{N+1}_+) \times D^s(\mathbb{R}^{N+1}_+)$, $D^s(\mathbb{R}^{N+1}_+)$ is defined as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ with respect to the norm $\|U\|_{\mathbb{R}^{N+1}_+} = (\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla U|^2)^{1/2}$. And the C^1 function $I : D \to \mathbb{R}$ given by

$$
\begin{aligned}
I(u,v) = & \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \big(y^{1-2s} |\nabla w_1|^2 + y^{1-2s} |\nabla w_2|^2 \big)\, dx\, dy \\
& - \frac{1}{2^*_s} \int_{\mathbb{R}^N} \big(\mu_1 |u|^{2^*_s} + \mu_2 |v|^{2^*_s} + 2\beta |u|^{\frac{2^*_s}{2}} |v|^{\frac{2^*_s}{2}} \big) dx,
\end{aligned}
$$

Similarly, we consider the set

$$
\begin{aligned}
\mathcal{N} = \Big\{ & (u,v) \in D,\ u \not\equiv 0,\ v \not\equiv 0, \\
& \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_1|^2 \, dx\, dy = \int_{\mathbb{R}^N} \big(\mu_1 |u|^{2^*_s} + \beta |u|^{\frac{2^*_s}{2}} |v|^{\frac{2^*_s}{2}} \big) dx, \\
& \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_2|^2 \, dx\, dy = \int_{\mathbb{R}^N} \big(\mu_2 |u|^{2^*_s} + \beta |u|^{\frac{2^*_s}{2}} |v|^{\frac{2^*_s}{2}} \big) dx \Big\}.
\end{aligned}
$$

Then any nontrivial solution of (2.14) belongs to $\mathcal{N}$. We can easily prove $\mathcal{N} \neq \emptyset$ by the same method as we used for $\mathcal{M} \neq \emptyset$. We set

$$
A := \inf_{(u,v) \in \mathcal{N}} I(u,v) = \inf_{(u,v) \in \mathcal{N}} \frac{s}{N} \int_{\mathbb{R}^{N+1}_+} \big(y^{1-2s} |\nabla w_1|^2 + y^{1-2s} |\nabla w_2|^2 \big)\, dx\, dy.
$$

We have the following theorem, which plays an important role in the proof of Theorem 1.2.

Theorem 2.3. *Suppose that $N \geq 4s$ and $\beta \geq \frac{2s}{N-2s} \max\{\mu_1, \mu_2\}$. Then $I(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon) = A$, where (k_0, l_0) is given in (3.1). That is $(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon)$ is a positive least energy solution of (2.14), where U_ε is given in (2.4).*

3. Proof of Theorems 1.1 and 2.3

The following system is crucial in proving our main results,

$$
\begin{gathered}
\mu_1 k^{\frac{2^*_s}{2} - 1} + \beta k^{\frac{2^*_s}{4} - 1} l^{\frac{2^*_s}{4}} = 1, \\
\mu_2 l^{\frac{2^*_s}{2} - 1} + \beta l^{\frac{2^*_s}{4} - 1} k^{\frac{2^*_s}{4}} = 1, \\
k \geq 0, \quad l \geq 0.
\end{gathered}
\tag{3.1}
$$

We will prove in Lemma 3.2 that there exist (k_0, l_0), such that (k_0, l_0) satisfies (3.1) and $k_0 = \min \big\{ k : (k,l) \text{ is a solution of (3.1)} \big\}$.

Remark 3.1. For $N = 4s$, we have $2_s^* = 4$, then (3.1) becomes

$$\begin{gathered}\mu_1 k + \beta l = 1,\\ \mu_2 l + \beta k = 1,\\ k \geq 0, \quad l \geq 0.\end{gathered} \tag{3.2}$$

Since $\beta \geq \max\{\mu_1, \mu_2\}$, it follows that $\beta^2 \geq \mu_1\mu_2$. Thus, (3.2) has a unique solution (k_0, l_0).

In the following, we only consider the case $N > 4s$. Define

$$\begin{gathered}\alpha_1(k,l) = \mu_1 k^{\frac{2_s^*}{2}-1} + \beta k^{\frac{2_s^*}{4}-1} l^{\frac{2_s^*}{4}} - 1, \quad k > 0,\ l \geq 0;\\ \alpha_2(k,l) = \mu_2 l^{\frac{2_s^*}{2}-1} + \beta l^{\frac{2_s^*}{4}-1} k^{\frac{2_s^*}{4}} - 1, \quad l > 0,\ k \geq 0;\\ h_1(k) = \beta^{-\frac{4}{2_s^*}} (k^{1-\frac{2_s^*}{4}} - \mu_1 k^{\frac{2_s^*}{4}})^{\frac{4}{2_s^*}}, \quad 0 < k \leq \mu_1^{-\frac{N-2s}{2s}};\\ h_2(l) = \beta^{-\frac{4}{2_s^*}} (l^{1-\frac{2_s^*}{4}} - \mu_2 l^{\frac{2_s^*}{4}})^{\frac{4}{2_s^*}}, \quad 0 < l \leq \mu_2^{-\frac{N-2s}{2s}}.\end{gathered}$$

Then we have $\alpha_1(k, h_1(k)) \equiv 0$ and $\alpha_2(h_2(l), l) \equiv 0$.

Lemma 3.2. *Assume that $\beta > 0$, then the equation*

$$\alpha_1(k,l) = 0, \quad \alpha_2(k,l) = 0,\ k,\ l > 0 \tag{3.3}$$

has a solution (k_0, l_0), which satisfies

$$\alpha_2(k, h_1(k)) < 0, \quad \text{for } 0 < k < k_0. \tag{3.4}$$

Similarly, (3.3) *has a solution (k_1, l_1) that satisfies*

$$\alpha_1(h_2(l), l) < 0, \quad \text{for } 0 < l < l_1. \tag{3.5}$$

Proof. First we denote $p = \frac{2_s^*}{2} = \frac{N}{N-2s}$. Equation $\alpha_1(k,l) = 0$, k, $l > 0$ imply that

$$l = h_1(k), \quad 0 < k < \mu_1^{-\frac{1}{p-1}}.$$

While $\alpha_2(k,l) = 0$ implies that $\mu_2 l^{\frac{p}{2}} + \beta k^{\frac{p}{2}} = l^{1-\frac{p}{2}}$. Therefore, we turn to prove that

$$\mu_2 \frac{1-\mu_1 k^{p-1}}{\beta k^{\frac{p}{2}-1}} + \beta k^{\frac{p}{2}} = \Big(\frac{1-\mu_1 k^{p-1}}{\beta k^{\frac{p}{2}-1}}\Big)^{\frac{2-p}{p}}, \tag{3.6}$$

has a solution in $(0, \mu_1^{-\frac{1}{p-1}})$. Note that (3.6) is equivalent to

$$f(k) = \Big(\frac{1}{\beta k^{p-1}} - \frac{\mu_1}{\beta}\Big)^{\frac{2-p}{p}} - \beta k^{p-1} - \mu_2 \frac{1-\mu_1 k^{p-1}}{\beta} = 0, \quad 0 < k^{p-1} < \frac{1}{\mu_1}. \tag{3.7}$$

Recall that $p = \frac{2_s^*}{2} = \frac{N}{N-2s}$, since $N > 4s$, we have $p < 2$, and so

$$\lim_{k\to 0^+} f(k) = +\infty,\ f(\mu_1^{-\frac{1}{p-1}}) = -\frac{\beta}{\mu_1} < 0.$$

Therefore, there exists $k_0 \in (0, \mu_1^{-\frac{1}{p-1}})$ such that $f(k_0) = 0$ and $f(k) > 0$ for $k \in (0, k_0)$. Let $l_0 = h_1(k_0)$, then (k_0, l_0) is a solution of (3.3). Moreover (3.4) follows directly from $f(k) > 0$ for $k \in (0, k_0)$. The proof of existence of (k_1, l_1) that satisfy (3.5) is similar. Then the proof of Lemma 3.2 is completed. ☐

Lemma 3.3. *Assume that $\beta \geq (\frac{2s}{N-2s}) \max\{\mu_1, \mu_2\}$, then $h_1(k) + k$ is strictly increasing for $k \in [0, \mu_1^{-\frac{N-2s}{2s}}]$, and $h_2(l) + l$ is strictly increasing for $l \in [0, \mu_2^{-\frac{N-2s}{2s}}]$.*

Proof. We also denote $p = \frac{2_s^*}{2} = \frac{N}{N-2s}$. Since for $k > 0$, we have

$$h_1'(k) = \frac{2}{p}\beta^{-\frac{2}{p}}(k^{1-\frac{p}{2}} - \mu_1 k^{\frac{p}{2}})^{\frac{2-p}{p}}\big(\frac{2-p}{2}k^{-\frac{p}{2}} - \frac{p}{2}\mu_1 k^{\frac{p-2}{2}}\big).$$

We see that $h_1'(k) \ge 0$ for $0 < \mu_1 k^{p-1} \le \frac{2-p}{p}$ or $\mu_1 k^{p-1} = 1$, and $h_1'(k) < 0$ for $\frac{2-p}{p} < \mu_1 k^{p-1} < 1$. By direct computations, we deduce from $h_1''(k) = 0$, $\frac{2-p}{p} < \mu_1 k^{p-1} < 1$ that $k = (\mu_1 p)^{-\frac{1}{p-1}}$. Since $\beta \ge (p-1)\max\{\mu_1, \mu_2\}$, we have

$$\min_{0<a<\mu_1^{-\frac{1}{p-1}}} h_1'(k) = h_1'\big((\mu_1 p)^{-\frac{1}{p-1}}\big) = -\beta^{-\frac{2}{p}}\mu_1^{2/p}(p-1)^{2/p} \ge -1.$$

and $h_1'(k) > -1$ for $0 < k \le \mu_1^{-\frac{1}{p-1}}$ with $k \ne (\mu_1 p)^{-\frac{1}{p-1}}$. This implies that $h_1(k)+k$ is strictly increasing for $k \in [0, \mu_1^{-\frac{1}{p-1}}]$. Similarly, $h_2(l) + l$ is strictly increasing for $l \in [0, \mu_2^{-\frac{1}{p-1}}]$. □

Lemma 3.4. *Assume that $\beta \ge (\frac{2s}{N-2s})\max\{\mu_1, \mu_2\}$. Let (k_0, l_0) be the solution in Lemma 3.2, then $\max\{\mu_1(k_0 + l_0)^{p-1}, \mu_2(k_0 + l_0)^{\frac{2s}{N-2s}}\} < 1$ and*

$$\alpha_2(k, h_1(k)) < 0, \forall\, 0 < k < k_0; \;\; \alpha_1(h_2(l), l) < 0, \quad \text{for } 0 < l < l_0.$$

Proof. By Lemma 3.3, we obtain

$$h_1(\mu_1^{-\frac{1}{p-1}}) + \mu_1^{-\frac{1}{p-1}} = \mu_1^{-\frac{1}{p-1}} > h_1(k_0) + k_0 = k_0 + l_0,$$

where $p = \frac{2_s^*}{2}$, that is, $\mu_1(k_0 + l_0)^{p-1} < 1$. Similarly, $\mu_2(k_0 + l_0)^{p-1} < 1$. By Lemma 3.2, it suffices to prove that $(k_0, l_0) = (k_1, l_1)$. By (3.4)-(3.5), we see that $k_1 \ge k_0$, $l_0 \ge l_1$. If $k_1 > k_0$, we have $k_1 + h_1(k_1) > k_0 + h_1(k_0)$, that is, $h_2(l_1) + l_1 = k_1 + l_1 > k_0 + l_0 = h_2(l_0) + l_0$, and so $l_1 > l_0$, then we obtain contradiction. Therefore, $k_1 = k_0$ and $l_0 = l_1$. This completes the proof. □

Lemma 3.5. *Assume that $\beta \ge (\frac{2s}{N-2s})\max\{\mu_1, \mu_2\}$. Then*

$$\begin{gathered} k + l \le k_0 + l_0, \\ \alpha_1(k, l) \ge 0, \quad \alpha_2(k, l) \ge 0, \\ k, l \ge 0, \quad (k, l) \ne (0, 0), \end{gathered} \tag{3.8}$$

has a unique solution.

Proof. Note that (k_0, l_0) satisfies (3.8). Let $(\tilde{k}, \tilde{l})$ be any a solution of (3.8). Without loss of generality, we assume that $\tilde{k} > 0$. If $\tilde{l} = 0$, then by $\tilde{k} \le k_0 + l_0$ and $\alpha_1(\tilde{k}, 0) \ge 0$, we obtain that

$$1 \le \mu_1 \tilde{k}^{p-1} \le \mu_1(k_0 + l_0)^{p-1}$$

which contradicts with Lemma 3.4. Therefore $\tilde{l} > 0$.

Assume by contradiction that $\tilde{k} < k_0$. Similar as the proof of Lemma 3.3, it is easy to see that $h_2(l)$ is strictly increasing for $0 < \mu_2 l^{p-1} \le \frac{2-p}{p}$, and strictly decreasing for $\frac{2-p}{p} \le \mu_2 l^{p-1} \le 1$. Moreover, $h_2(0) = h_2(\mu_2^{-\frac{1}{p-1}}) = 0$. Since $0 < \tilde{k} < k_0 = h_2(l_0)$, there exists $0 < l_2 < l_3 < \mu_2^{-\frac{1}{p-1}}$ such that $h_2(l_2) = h_2(l_3) = \tilde{k}$ and

$$\alpha_2(\tilde{k}, l) < 0 \;\Leftrightarrow\; h_2(l) > \tilde{k} \;\Leftrightarrow\; l_2 < l < l_3. \tag{3.9}$$

Because $\alpha_2(\tilde{k},\tilde{l}) \geq 0$, we have $\tilde{l} \leq l_2$ or $\tilde{l} \geq l_3$. Since $\alpha_1(\tilde{k},\tilde{l}) \geq 0$, we have $\tilde{l} > h_1(\tilde{k})$. By Lemma 3.4, we have $\alpha_2(\tilde{k}, h_1(\tilde{k})) < 0$, and so $l_2 < h_1(\tilde{k}) < l_3$. These imply that

$$\tilde{l} \geq l_3. \tag{3.10}$$

On the other hand, since $l_1 := k_0 + l_0 - \tilde{k} > l_0$, we obtain

$$h_2(l_1) + k_0 + l_0 - \tilde{k} = h_2(l_1) + l_1 > h_2(l_0) + l_0 = k_0 + l_0,$$

that is, $h_2(l_1) > \tilde{k}$. By (3.9), we obtain $l_2 < l_1 < l_3$. Since $\tilde{k} + \tilde{l} \leq k_0 + l_0$, we have

$$\tilde{l} \leq l_1 < l_3,$$

which contradicts with (3.10). Therefore, $\tilde{k} \geq k_0$. By a similar argument, we also have $\tilde{l} \geq l_0$. Therefore, $(\tilde{k},\tilde{l}) = (k_0, l_0)$. This completes the proof. $\square$

Proof of Theorem 1.1. Since $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0$, then, it is obviously that $(\sqrt{k_0}u_\lambda, \sqrt{l_0}u_\lambda)$ is a nontrivial solution of (1.1) and

$$0 < B \leq E(\sqrt{k_0}u_\lambda, \sqrt{l_0}u_\lambda) = (k_0 + l_0)B_1. \tag{3.11}$$

We now prove that $B = E(\sqrt{k_0}w_\lambda, \sqrt{l_0}w_\lambda)$ when $\beta \geq \frac{2s}{N-2s}\max\{\mu_1, \mu_2\}$. Let $\{(u_n, v_n)\} \subset M$ be a minimizing sequence for B. Define

$$c_n = \Big(\int_\Omega |u_n|^{2p}dx\Big)^{1/p}, \quad d_n = \Big(\int_\Omega |v_n|^{2p}dx\Big)^{1/p}.$$

On the other hand, by (2.7), we have

$$\begin{aligned}
(\frac{N}{s}B_1)^{2s/N}c_n &\leq \int_{\mathcal{C}} y^{1-2s}|\nabla w_{1,n}|^2\,dx\,dy + \lambda\int_\Omega u_n^2dx \\
&= \mu_1\int_\Omega u_n^{2_s^*}dx + \beta\int_\Omega u_n^{\frac{2_s^*}{2}}v_n^{\frac{2_s^*}{2}}\,dx \\
&\leq \mu_1 c_n^p + \beta c_n^{\frac{p}{2}}d_n^{\frac{p}{2}},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
(\frac{N}{s}B_1)^{2s/N}d_n &\leq \int_{\mathcal{C}} y^{1-2s}|\nabla w_{2,n}|^2\,dx\,dy + \lambda\int_\Omega v_n^2dx \\
&= \mu_2\int_\Omega v_n^{2_s^*}dx + \beta\int_\Omega u_n^{\frac{2_s^*}{2}}v_n^{\frac{2_s^*}{2}}\,dx \\
&\leq \mu_2 d_n^p + \beta c_n^{\frac{p}{2}}d_n^{\frac{p}{2}},
\end{aligned} \tag{3.13}$$

where $(w_{1,n},\, w_{2,n}) = \big(E_s(u_n),\, E_s(v_n)\big)$. Using the fact that

$$\begin{aligned}
E(u_n, v_n) = \frac{s}{N}\Big\{&\int_{\mathcal{C}} y^{1-2s}|\nabla w_{1,n}|^2\,dx\,dy + \int_{\mathcal{C}} y^{1-2s}|\nabla w_{2,n}|^2\,dx\,dy \\
&+ \lambda\int_\Omega u_n^2dx + \lambda\int_\Omega v_n^2dx\Big\},
\end{aligned}$$

and by (3.11), we obtain

$$(\frac{N}{s}B_1)^{2s/N}(c_n + d_n) \leq \frac{N}{s}E(u_n, v_n) \leq \frac{N}{s}(k_0 + l_0)B_1, \tag{3.14}$$

$$\mu_1 c_n^{p-1} + \beta c_n^{\frac{p}{2}-1}d_n^{\frac{p}{2}} \geq (\frac{N}{s}B_1)^{2s/N}, \tag{3.15}$$

$$\mu_2 d_n^{p-1} + \beta d_n^{\frac{p}{2}-1}c_n^{\frac{p}{2}} \geq (\frac{N}{s}B_1)^{2s/N}. \tag{3.16}$$

First, this means c_n, d_n are uniformly bounded. Passing to a subsequence, we assume that $c_n \to c, d_n \to d$. Combining (3.12) and (3.13), we have $\mu_1 c^p + 2\beta c^{\frac{p}{2}} d^{\frac{p}{2}} + \mu_2 d^{\frac{p}{2}} \geq \frac{N}{s} B > 0$. Hence, without loss of generality, we assume that $c > 0$. If $d = 0$, then (3.14) implies $c < (\frac{N}{s} B_1)^{1-\frac{2s}{N}}(k_0 + l_0)$. Then by (3.15) and Lemma 3.4, we obtain

$$\big(\frac{N}{s} B_1\big)^{2s/N} \leq \mu_1 c^{p-1} \leq \mu_1 (k_0 + l_0)^{p-1} (\frac{N}{s} B_1)^{2s/N} < \big(\frac{N}{s} B_1\big)^{2s/N},$$

which is a contradiction. Therefore, $c > 0$ and $d > 0$. Let $k = \frac{c}{(\frac{N}{s} B_1)^{1-\frac{2s}{N}}}$, $l = \frac{d}{(\frac{N}{s} B_1)^{1-\frac{2s}{N}}}$, since (3.14)-(3.16), we see that (k, l) satisfies (3.8). By Lemma 3.5, we have $(k, l) = (k_0, l_0)$. It follows that $c_n \to k_0 (\frac{N}{s} B_1)^{1-\frac{2s}{N}}$ and $d_n \to l_0 (\frac{N}{s} B_1)^{1-\frac{2s}{N}}$ as $n \to +\infty$, and

$$\frac{N}{s} B = \lim_{n\to+\infty} \frac{N}{s} E(u_n, v_n) \geq \lim_{n\to+\infty} (\frac{N}{s} B_1)^{2s/N}(c_n + d_n) = \frac{N}{s}(k_0 + l_0) B_1.$$

Combining this with (3.11), one has that

$$B = (k_0 + l_0) B_1 = E(\sqrt{k_0} u_\lambda, \sqrt{l_0} u_\lambda),$$

and so $(\sqrt{k_0} u_\lambda, \sqrt{l_0} u_\lambda)$ is a positive least energy solution of (1.1).

The proof of the second part of Theorem 1.1 is similar to the proof of [19, Theorem 1.2], which only need a slight modification. So we omit it here. □

Proof of Theorem 2.3. By lemma 3.2, equation (3.1) has a solution (k_0, l_0). Then $(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon)$ is a nontrivial solution of (2.14) and

$$A \leq I(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon) = \frac{s}{N}(k_0 + l_0) S(s, N)^{\frac{N}{2s}}. \tag{3.17}$$

Since $\beta \geq (p-1)\max\{\mu_1, \mu_2\}$. Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequence for A, that is $I(u_n, v_n) \to A$ as $n \to \infty$. Denote $c_n = \big(\int_{\mathbb{R}^N} |u_n|^{2p}\big)^{1/p}$, $d_n = \big(\int_{\mathbb{R}^N} |v_n|^{2p}\big)^{1/p}$, then we have

$$\begin{aligned} S(s, N) c_n &\leq \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_{1,n}|^2 \,dx\,dy \\ &= \mu_1 \int_{\mathbb{R}^N} u_n^{2^*_s} dx + \beta \int_{\mathbb{R}^N} u_n^{\frac{2^*_s}{2}} v_n^{\frac{2^*_s}{2}} dx \\ &\leq \mu_1 c_n^p + \beta c_n^{\frac{p}{2}} d_n^{\frac{p}{2}}, \end{aligned}$$

and

$$\begin{aligned} S(s, N) d_n &\leq \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_{2,n}|^2 \,dx\,dy \\ &= \mu_2 \int_{\mathbb{R}^N} v_n^{2^*_s} dx + \beta \int_{\mathbb{R}^N} u_n^{\frac{2^*_s}{2}} v_n^{\frac{2^*_s}{2}} dx \\ &\leq \mu_2 d_n^p + \beta c_n^{\frac{p}{2}} d_n^{\frac{p}{2}}, \end{aligned}$$

where $E_s(u_n) = (w_{1,n}, w_{2,n}) = \big(E_s(u_n), E_s(v_n)\big)$. This means

$$\begin{gathered} S(s, N)(c_n + d_n) \leq \frac{N}{s} I(u_n, v_n) \leq (k_0 + l_0) S(s, N)^{\frac{N}{2s}} + o(1), \\ \mu_1 c_n^{p-1} + \beta c_n^{\frac{p}{2}-1} d_n^{\frac{p}{2}} \geq S(s, N), \quad \mu_2 d_n^{p-1} + \beta c_n^{\frac{p}{2}} d_n^{\frac{p}{2}-1} \geq S(s, N). \end{gathered}$$

Similarly as the proof of Theorem 1.1, we see that $c_n \to k_0 S(s,N)^{\frac{N}{2s}-1}$, $d_n \to l_0 S(s,N)^{\frac{N}{2s}-1}$ as $n \to +\infty$, and

$$\frac{NA}{s} = \lim_{n\to+\infty} \frac{N}{s} I(u_n, v_n) \geq \lim_{n\to+\infty} S(s,N)(c_n + d_n) = (k_0 + l_0) S(s,N)^{\frac{N}{2s}}.$$

This implies

$$A = \frac{s}{N}(k_0 + l_0) S(s,N)^{\frac{N}{2s}} = I(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon)$$

and so $(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon)$ is a positive least energy solution of (2.14). $\square$

4. Proof of Theorem 1.2

In this section, by showing that the mountain pass energy level $\mathcal{B}$ is strictly less than B_{μ_1}, B_{μ_2} and A, we then complete the proof of Theorem 1.2.

The mountain pass energy level $\mathcal{B}$ is given by

$$\mathcal{B} := \inf_{h\in\Gamma} \max_{t\in[0,1]} E(h(t)), \tag{4.1}$$

where $\Gamma = \{h \in C([0,1], H) : h(0) = (0,0), E(h(1)) < 0\}$. For any $(u,v) \in H$ with $(u,v) \neq (0,0)$, similarly, set $(w_1, w_2) = (E_s(u), E_s(v))$, we then have

$$\begin{aligned} \max_{t>0} E(tu, tv) &= E(t_0 u, t_0 v) \\ &= \frac{s}{N} t_0^2 \Big(\int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2 \, dx\, dy + \int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2 \, dx\, dy \\ &\quad + \lambda_1 \int_\Omega u^2 dx + \lambda_2 \int_\Omega v^2 dx \Big) \\ &= \frac{s}{N} t_0^{2_s^*} \int_\Omega \big(\mu_1 |u|^{2_s^*} + 2\beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} + \mu_2 |v|^{2_s^*} \big) dx, \end{aligned} \tag{4.2}$$

where $t_0 > 0$ satisfies

$$t_0^{2_s^*-2} = \frac{\int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2 \, dx\, dy + \int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2 \, dx\, dy + \lambda_1 \int_\Omega u^2 dx + \lambda_2 \int_\Omega v^2 dx}{\int_\Omega \big(\mu_1 |u|^{2_s^*} + 2\beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} + \mu_2 |v|^{2_s^*} \big) dx}. \tag{4.3}$$

It is obvious that $(t_0 u, t_0 v) \in \mathcal{M}'$, where

$$\begin{aligned} \mathcal{M}' := \Big\{ &(u,v) \in H \setminus \{(0,0)\}, \, G(u,v) := \int_{\mathcal{C}} y^{1-2s} |\nabla w_1|^2 \, dx\, dy \\ &+ \int_{\mathcal{C}} y^{1-2s} |\nabla w_2|^2 \, dx\, dy + \lambda_1 \int_\Omega u^2 dx + \lambda_2 \int_\Omega v^2 dx \\ &- \int_\Omega \big(\mu_1 |u|^{2_s^*} + 2\beta |u|^{\frac{2_s^*}{2}} |v|^{\frac{2_s^*}{2}} + \mu_2 |v|^{2_s^*} \big) dx = 0 \Big\}. \end{aligned} \tag{4.4}$$

It is easy to check that

$$\mathcal{B} = \inf_{(u,v)\neq(0,0)} \max_{t>0} E(tu, tv) = \inf_{(u,v)\in\mathcal{M}'} E(u,v). \tag{4.5}$$

Lemma 4.1. *Let $\beta \geq (\frac{2s}{N-2s}) \max\{\mu_1, \mu_2\}$ and $N \geq 4s$, then*

$$\mathcal{B} < \min\{B_{\mu_1}, B_{\mu_2}, A\}$$

Proof. We first prove $\mathcal{B} < A$. Define

$$u_\varepsilon(x) = \sqrt{k_0}\overline{u}_\varepsilon(x) = \sqrt{k_0}\eta(x)U_\varepsilon(x), \quad v_\varepsilon(x) = \sqrt{l_0}\overline{u}_\varepsilon(x) = \sqrt{k_0}\eta(x)U_\varepsilon(x),$$

where $\eta \in C_0^\infty\big(B_{2\delta}(x_0)\big)$, $\delta > 0$ is a constant and $B_{2\delta}(x_0) \subset \Omega$. Set

$$\overline{w}_\varepsilon(x,y) = E_s(\overline{u}_\varepsilon), \quad w_{1,\varepsilon}(x,y) = \sqrt{k_0}\overline{w}_\varepsilon(x,y), \quad w_{2,\varepsilon}(x,y) = \sqrt{l_0}\overline{w}_\varepsilon(x,y).$$

Then, we have the following estimations (see [3]):

$$\|\overline{w}_\varepsilon\|^2_{H^s_{0,L}(\mathcal{C})} = \|E_s(U_\varepsilon)\|^2_{H^s_{0,L}(\mathbb{R}^{N+1}_+)} + O(\varepsilon^{N-2s}) = S(s,N)^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \tag{4.6}$$

$$\int_\Omega |\overline{u}_\varepsilon(x)|^{2^*_s} dx = \int_{\mathbb{R}^N} U_\varepsilon^{2^*_s}(x) dx + O(\varepsilon^{N-2s}) = S(s,N)^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \tag{4.7}$$

$$\int_\Omega |\overline{u}_\varepsilon(x)|^2 dx \geq \begin{cases} C_s\varepsilon^{2s} + O(\varepsilon^{N-2s}), & \text{if } N > 4s, \\ C_s\varepsilon^{2s}\log\frac{1}{\varepsilon} + O(\varepsilon^{2s}), & \text{if } N = 4s, \\ C_s\varepsilon^{N-2s} + O(\varepsilon^{2s}), & \text{if } N < 4s. \end{cases} \tag{4.8}$$

Thus, we deduce that

$$\begin{aligned}
&E(tu_\varepsilon, tv_\varepsilon) \\
&= \frac{t^2}{2}\Big\{\int_{\mathcal{C}} y^{1-2s}\big(|\nabla w_{1,\varepsilon}|^2 + |\nabla w_{2,\varepsilon}|^2\big)\,dx\,dy + \lambda_1\int_\Omega u_\varepsilon^2 dx + \lambda_2\int_\Omega v_\varepsilon^2 dx\Big\} \\
&\quad - \frac{t^{2^*_s}}{2^*_s}\int_\Omega \big(\mu_1 u_\varepsilon^{2^*_s} + 2\beta u_\varepsilon^{\frac{2^*_s}{2}} v_\varepsilon^{\frac{2^*_s}{2}} + \mu_2 v_\varepsilon^{2^*_s}\big)dx \\
&\leq \frac{t^2}{2}\Big\{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\big(k_0|\nabla U_\varepsilon|^2 + l_0|\nabla U_\varepsilon|^2\big)\,dx\,dy + O(\varepsilon^{N-2s}) - C\varepsilon^{2s}\Big\} \\
&\quad - \frac{t^{2^*_s}}{2^*_s}\Big\{\int_{\mathbb{R}^N} \big(\mu_1 k_0^{\frac{2^*_s}{2}} U_\varepsilon^{2^*_s} + 2\beta k_0^{\frac{2^*_s}{4}} l_0^{\frac{2^*_s}{4}} U_\varepsilon^{2^*_s} + \mu_2 l_0^{\frac{2^*_s}{2}} U_\varepsilon^{2^*_s}\big)dx + O(\varepsilon^{N-2s})\Big\} \\
&= \frac{t^2}{2}\big(\frac{N}{s}A + O(\varepsilon^{N-2s}) - C\varepsilon^{2s}\big) - \frac{t^{2^*_s}}{2^*_s}\big(\frac{N}{s}A + O(\varepsilon^{N-2s})\big) \\
&\leq \frac{s}{N}\Big(\frac{N}{s}A + O(\varepsilon^{N-2s}) - C\varepsilon^{2s}\Big)\Big(\frac{\frac{N}{s}A + O(\varepsilon^{N-2s}) - C\varepsilon^{2s}}{\frac{N}{s}A + O(\varepsilon^{N-2s})}\Big)^{\frac{N-2s}{2s}} \\
&< A \qquad \text{for } \varepsilon \text{ small enough and } N \geq 4s.
\end{aligned} \tag{4.9}$$

Hence, for $\varepsilon > 0$ small enough, it holds

$$\mathcal{B} \leq \max_{t>0} E(tu_\varepsilon, tv_\varepsilon) < A. \tag{4.10}$$

We now prove that $\mathcal{B} < B_{\mu_1}$. Define

$$t(a)^{2^*_s-2} := \frac{\frac{N}{s}B_{\mu_1} + a^2\frac{N}{s}B_{\mu_2}}{\frac{N}{s}B_{\mu_1} + \frac{N}{s}|a|^{2^*_s}B_{\mu_2} + |a|^{\frac{2^*_s}{2}}\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx}.$$

Note that $t(0) = 1$ and recall that $1 < p = \frac{N}{N-2s} < 2$, by direct computation, we obtain

$$\lim_{a\to 0}\frac{t'(a)}{|a|^{p-2}a} = -\frac{p\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx}{(2p-2)\frac{NB_{\mu_1}}{s}};$$

that is,

$$t'(a) = -\frac{p\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx}{(2p-2)\frac{NB_{\mu_1}}{s}}|a|^{p-2}a\big(1+o(1)\big), \quad \text{as } a \to 0,$$

and so

$$t(a) = 1 - \frac{\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx}{(2p-2)\frac{NB_{\mu_1}}{s}}|a|^p\big(1+o(1)\big), \quad \text{as } a \to 0.$$

This implies

$$t(a)^{2p} = 1 - \frac{2p\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx}{(2p-2)\frac{NB_{\mu_1}}{s}}|a|^p(1+o(1)), \quad \text{as } a \to 0.$$

Therefore, from (4.2) and $\frac{2p}{2p-2} = \frac{N}{2s}$ we deduce that

$$\begin{aligned}
\mathcal{B} &\le E\big(t(a)u_{\mu_1}, t(a)au_{\mu_2}\big) \\
&= \frac{s}{N}t(a)^{2p}\Big(\frac{NB_{\mu_1}}{s} + |a|^{2p}\frac{NB_{\mu_2}}{s} + |a|^p\int_\Omega 2\beta|u_{\mu_1}|^p|u_{\mu_2}|^p dx\Big) \\
&= B_{\mu_1} - \frac{s}{N}(\frac{1}{p-1})|a|^p\int_\Omega 2\beta|u_{\mu_1}|^p||u_{\mu_2}|^p dx + o(|a|^p) \\
&< B_{\mu_1} \quad \text{for } |a|>0 \text{ small enough.}
\end{aligned} \tag{4.11}$$

Similarly, we can proof $\mathcal{B} < B_{\mu_2}$. The proof is complete. □

Proof of Theorem 1.2. Since the functional E has a mountain pass structure, by the Mountain Pass Theorem, there exists $\{(u_n, v_n)\} \subset H$ such that

$$\lim_{n\to+\infty} E(u_n, v_n) = \mathcal{B}, \quad \lim_{n\to+\infty} E'(u_n, v_n) = 0.$$

It is standard to see that $\{(u_n, v_n)\}$ is bounded in H, and so we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in H. Set $\tau_n = u_n - u$, $\sigma_n = v_n - v$. Thus, by Brezis-Lieb Lemma and [19, Lemma 3.3], we obtain

$$\int_{\mathcal{C}} y^{1-2s}|\nabla\rho_{1,n}|^2\,dx\,dy - \int_\Omega(\mu_1\tau_n^{2_s^*} + \beta\tau_n^{\frac{2_s^*}{2}}\sigma_n^{\frac{2_s^*}{2}})dx = o_n(1), \tag{4.12}$$

$$\int_{\mathcal{C}} y^{1-2s}|\nabla\rho_{2,n}|^2\,dx\,dy - \int_\Omega(\mu_2\sigma_n^{2_s^*} + \beta\tau_n^{\frac{2_s^*}{2}}\sigma_n^{\frac{2_s^*}{2}})dx = o_n(1), \tag{4.13}$$

where $\rho_{1,n} = E_s(\tau_n)$, $\rho_{2,n} = E_s(\sigma_n)$. Then

$$E(u_n, v_n) = E(u, v) + I(\tau_n, \sigma_n) + o_n(1). \tag{4.14}$$

Suppose

$$\lim_{n\to+\infty}\int_{\mathcal{C}} y^{1-2s}|\nabla\rho_{1n}|^2\,dx\,dy = b_1, \quad \lim_{n\to+\infty}\int_{\mathcal{C}} y^{1-2s}|\nabla\rho_{2n}|^2\,dx\,dy = b_2, \tag{4.15}$$

then, for n large enough, we obtain

$$I(\tau_n, \sigma_n) = \frac{s}{N}(b_1 + b_2) + o_n(1). \tag{4.16}$$

Moreover,

$$0 \le E(u, v) \le E(u, v) + \frac{s}{N}(b_1 + b_2) = \lim_{n\to+\infty} E(u_n, v_n) = \mathcal{B}. \tag{4.17}$$

Case 1. $u \equiv 0, v \equiv 0$. By (4.17), we have $b_1 + b_2 > 0$, then we may assume that $(\tau_n, \sigma_n) \neq (0, 0)$ for n large enough. It is easy to check that there exists $t_n > 0$ such that $(t_n\tau_n, t_n\sigma_n) \in \mathcal{N}$ and $t_n \to 1$ as $n \to \infty$. Then, we have

$$\mathcal{B} = \frac{s}{N}(b_1 + b_2) = \lim_{n\to+\infty} I(\tau_n, \sigma_n) = \lim_{n\to+\infty} I(t_n\tau_n, t_n\sigma_n) \geq A. \tag{4.18}$$

This is a contradiction with Lemma 4.1. Therefore, Case 1 is impossible.

Case 2. $u \not\equiv 0$, $v \equiv 0$ or $u \equiv 0$, $v \not\equiv 0$. Without loss of generality, we may assume that $u \not\equiv 0$, $v \equiv 0$. Then u is a nontrivial solution of $(-\Delta)^s u + \lambda_1 u = \mu_1|u|^{2_s^*-2}u$, by (4.17) again, $\mathcal{B} \geq E(u, 0) \geq B_{\mu_1}$, a contradiction with Lemma 4.1. Therefore, Case 2 is also impossible.

Since Cases 1 and 2 are both impossible, we have that $u \not\equiv 0$, $v \not\equiv 0$. Since $E'(u, v) = 0$, we have $(u, v) \in \mathcal{M}$. By $\mathcal{B} \leq B$ and (4.17), we have $E(u, v) = \mathcal{B} = B$. This means $(|u|, |v|) \in \mathcal{M} \subset \mathcal{M}'$ and $E(|u|, |v|) = \mathcal{B} = B$. By (4.4) and (4.5), there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that

$$E'(|u|, |v|) - \gamma G'(|u|, |v|) = 0,$$

where G is given in (4.4). Since $E'(|u|, |v|)(|u|, |v|) = G(|u|, |v|) = 0$ and

$$G'(|u|, |v|)(|u|, |v|) = -(2p-2)\int_\Omega \left(\mu_1|u|^{2p} + 2\beta|u|^p|v|^p + \mu_2|v|^{2p}\right)dx \neq 0,$$

we obtain that $\gamma = 0$ and so $E'(|u|, |v|) = 0$. This means that $(|u|, |v|)$ is a least energy solution of (1.1). By the maximum principle [14], we see that $|u|, |v| > 0$ in Ω. Therefore, $(|u|, |v|)$ is a positive least energy solution of (1.1). $\square$

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